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EXTREMAL COMPRESSIONS OF CLOSED OPERATORS

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ABSTRACT. Let X be a Banach space, A a closed linear operator on X, and $\lambda_1, \ldots, \lambda_n$ isolated eigenvalues of A of finite multiplicity. If P is a projection on X such that $\lambda_1, \ldots, \lambda_n$ belong to the resolvent of the compression of A on the range of P it is easy to see that

$$\dim N(P) \geq \max\{\dim N(\lambda_i - A) \colon 1 \leq i \leq n\}.$$

It is shown that there exist such projections where we have equality in this inequality.

Let X be a complex Banach space. For an operator A in X (not necessarily everywhere defined) we denote by $D(A)$, $N(A)$, and $R(A)$ the domain, the kernel, and the range of A, respectively. If P is a bounded linear projection on X with $P(D(A)) \subseteq D(A)$ the compression of A on $R(P)$ is the linear operator A_P defined in the closed subspace $R(P)$ by $D(A_P) = D(A) \cap R(P)$ and $A_P y = PAy$ for $y \in R(P)$. It is easy to see that A_P is injective (i.e., $N(A_P) = \{0\}$) iff $N(A) \cap R(P) = \{0\}$ and $R(AP) \cap N(P) = \{0\}$. Therefore $\dim N(P) \geq \dim N(A)$ if A_P is injective.

The collection of projections P on X with $P(D(A)) \subseteq D(A)$ will be denoted by $P_A(X)$. The main result of this note is the following:

Theorem. *Let A be a closed linear operator in a Banach space X and let $\lambda_1, \ldots, \lambda_n$ be isolated eigenvalues of A of finite (algebraic) multiplicity. Let Ω be a subset of $\mathbf{C}$ such that $\Omega \neq \mathbf{C}$ and $\Omega \cap \sigma(A) = \{\lambda_1, \ldots, \lambda_n\}$. If $\zeta \in \mathbf{C} \backslash \Omega$ then there is a projection $P \in P_A(X)$ such that $\Omega \subseteq \rho(A_P)$, $\sigma(A_P) = \sigma(A) \backslash \Omega \cup \{\zeta\}$ and*

$$\dim\, N(P) = \max_{\lambda \in \Omega} \dim\, N(\lambda - A).$$

In [1, 2] Islamov proved a similar result for finite-dimensional perturbations of A. It is well known that many results on compact or finite-dimensional perturbations have analogues in terms of compressions to subspaces of finite codimension, see [5, 6, 7]. The theorem above is another example of this relationship.

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The proof will be divided into two steps. The main step is the following lemma which is the theorem in the finite-dimensional case. For its proof we need some facts about λ-matrices (for details see [4]).

A λ-matrix is a matrix over the ring of polynomials in λ over $\mathbf{C}$. Let B be an $n \times n$ matrix over $\mathbf{C}$ and $L_B(\lambda) := \lambda I - B$. For $l = 1, \dots, n$ define the polynomial $d_l(\lambda)$ to be the greatest common divisor (gcd) of all minors of $L_B(\lambda)$ of order l and $d_0(\lambda) \equiv 1$.

$$i_l(\lambda) := d_l(\lambda)/d_{l-1}(\lambda) \qquad (l = 1, \dots, n)$$

are called the invariant polynomials of $L_B(\lambda)$. Following holds (see [4, 4.10]): Two $n \times n$ matrices A, B are similar iff L_A, L_B have the same invariant polynomials. Two λ-matrices $L_1(\lambda)$, $L_2(\lambda)$ are said to be equivalent if there are λ-matrices $P(\lambda)$, $Q(\lambda)$ with nonzero constant determinants such that $L_1(\lambda) = P(\lambda)L_2(\lambda)Q(\lambda)$.

Lemma. *Let B be an $n \times n$ matrix and $\zeta \in \mathbf{C}$. There is a subspace N of $\mathbf{C}^n$ and a projection P onto N such that for the compression B_P holds $\sigma(B_P) = \{\zeta\}$ and*

$$\dim N(P) = \max_{\lambda \in \mathbf{C}} \dim N(\lambda - B).$$

Proof. Let $i_{r+1}(\lambda), \dots, i_n(\lambda)$ be the nonconstant invariant polynomials of $L_B(\lambda)$. At first we show

$$(*) \qquad n - r = \max\{\dim N(\lambda - B) \colon \lambda \in \mathbf{C}\}.$$

It is well known (see e.g. [4, p. 143, 148]) that $i_l(\lambda)$ divides $i_{l'}(\lambda)$ for $l \le l'$. Thus for $\lambda_0 \in \mathbf{C}$ we get the implications $i_l(\lambda_0) = 0 \Rightarrow i_k(\lambda_0) = 0$ for $l \le k \le n$ and $i_l(\lambda_0) \ne 0 \Rightarrow i_k(\lambda_0) \ne 0$ for $1 \le k \le l$. By [4, Theorem 4.9.1] $L_B(\lambda)$ is equivalent to $\operatorname{diag}(i_1(\lambda), \dots, i_n(\lambda))$. So for every $\lambda_0 \in \mathbf{C}$ we have $\dim N(\lambda_0 - B) = \dim N(\operatorname{diag}(i_1(\lambda_0), \dots, i_n(\lambda_0))) = n - \min\{l \colon i_1(\lambda_0) = 0\} + 1$ (with $\min\{\} = n + 1$ if $i_n(\lambda_0) \ne 0$). Taking the maximum on each side, we get $(*)$.

Now consider a polynomial $p(\lambda) = \lambda^l + \alpha_{l-1}\lambda^{l-1} + \cdots + \alpha_0$ $(l > 0)$ and its companion matrix

$$L(p) = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ -\alpha_0 & \cdots & \cdots & -\alpha_{l-2} & -\alpha_{l-1} \end{pmatrix}.$$

Considering invariant polynomials we next show that it is possible to choose suitable $\beta_0, \dots, \beta_{l-1}$ such that $L(p)$ and

$$C_\zeta(p) = \begin{pmatrix} \zeta & 1 & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & & \zeta & 1 \\ \beta_0 & \cdots & \cdots & & \beta_{l-2} & \beta_{l-1} \end{pmatrix}$$

are similar.

By [4, 4.11] $\lambda - L(p)$ has the invariant polynomials $1, \ldots, 1, p$. $\lambda - C_\zeta(p)$ has constant nonzero minors of order $1, \ldots, l-1$ independently of the choice of $\beta_0, \ldots, \beta_{l-1}$. So the first $l-1$ invariant polynomials are identically 1. The last one is

$$\det[\lambda - C_\zeta(p)] = (\lambda-\zeta)^l - (\beta_{l-1} - \zeta)(\lambda-\zeta)^{l-1} - \beta_{l-2}(\lambda-\zeta)^{l-2} - \cdots - \beta_0$$

which induces the choice of $\beta_0, \ldots, \beta_{l-1}$ depending on p.

B is similar to $\operatorname{diag}(L(i_{r+1}), \ldots, L(i_n))$ [4, 4.11.1] and so similar to $D := \operatorname{diag}(C_\zeta(i_{r+1}), \ldots, C_\zeta(i_n))$. Let U be an invertible $n \times n$ matrix with $B = UDU^{-1}$ and $l_j = \deg i_j(\lambda)$. Define N_1 to be the subspace of $\mathbf{C}^n$ spanned by the canonical vectors e_j for $j \notin \{l_{r+1}, l_{r+1} + l_{r+2}, \ldots, l_{r+1} + l_{r+2} + \cdots + l_n\}$.

We set $N := UN_1$ and $P = UP_1U^{-1}$, where P_1 denotes the orthogonal projection onto N_1. Then P is a projection onto N, $\dim N(P) = \dim N(P_1) = n-r$. $B_P = VD_{P_1}V^{-1}$ where $V := U|_{N_1}: N_1 \to N$. B_P and D_{P_1} have the same eigenvalues. D_{P_1} is the matrix D where rows and columns with indices $l_{r+1}, l_{r+1}+l_{r+2}, \ldots, l_{r+1} + \cdots + l_n$ are deleted, so $\sigma(B_P) = \{\zeta\}$. The lemma is proved.

Proof of the theorem. $\Omega \cap \sigma(A) = \{\lambda_1, \ldots, \lambda_n\}$ is a bounded part of $\sigma(A)$, separated from $\sigma(A)\backslash\{\lambda_1, \ldots, \lambda_n\}$. By [3, Theorem 6.17] we have a decomposition of A according to a decomposition $X = M_1 \oplus M_2$ in such a way that we have for A_{P_1}, A_{P_2} (P_1 the projection onto M_1 along M_2, $P_2 = I - P_1$):

$$\dim P_1X = \dim M_1 < \infty, \qquad D(A_{P_1}) = M_1$$
$$\sigma(A_{P_1}) = \{\lambda_1, \ldots, \lambda_n\}, \sigma(A_{P_2}) = \sigma(A)\backslash\{\lambda_1, \ldots, \lambda_n\},$$

and

$$\dim N(\lambda - A) = \dim N(\lambda - A_{P_1}) \quad \text{for } \lambda \in \Omega.$$

By the preceding lemma there is a subspace N of M_1 and a projection $\widetilde{P}$ of M_1 onto N such that $\dim M_1/N = \max\{\dim N(\lambda - A_{P_1}) : \lambda \in \mathbf{C}\}$ and $\sigma((A_{P_1})_P = \{\zeta\}$.

Set $M := N \oplus M_2$ and $P := \widetilde{P}P_1 + P_2$. Then we have

$$\operatorname{codim}_X M = \operatorname{codim}_{M_1} N = \max_{\lambda \in \mathbf{C}} \dim N(\lambda - A_{P_1}) = \max_{\lambda \in \Omega} \dim N(\lambda - A),$$
$$P^2 = P, \qquad R(P) = M \quad \text{and} \quad \sigma(A_P) = \sigma(A)\backslash\Omega \cup \{\zeta\}.$$

The proof is complete.

With this result it is possible to show a corollary for the Browder spectrum analogously to [7].

Corollary. *Let A be a bounded operator on a complex Banach space X and $\varepsilon > 0$. Then there is an extremal compression A_P of A in such a way that $\sigma(A_P)$ is contained in the ε-neighborhood U of the Browder spectrum $\sigma_b(A)$ of A and*

$$\dim N(P) = \max_{\lambda \notin U} \dim N(\lambda - A).$$

Proof. Consider $\sigma(A)\backslash U$, with $U = \{\lambda \in \mathbf{C} : \operatorname{dist}(\lambda, \sigma_b(A)) < \varepsilon\}$. This is a finite set because $\sigma(A)\backslash U$ is a compact set of isolated points. Applying the theorem to A, $\Omega := \sigma(A)\backslash U$ and an arbitrary $\zeta \in \sigma_{\text{ess}}(A)$ shows the corollary.

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