ON THE IDEAL STRUCTURE OF THE NEVANLINNA CLASS

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Abstract. Let $N$ denote the Nevanlinna class, i.e. the algebra of holomorphic functions of bounded characteristic in the open unit disc. We study analytic conditions for a finitely generated ideal to be equal to the whole algebra $N$. Then we characterize the finitely generated prime ideals containing a nontangential interpolating Blaschke product. Further, we give an example of an ideal of $N$ whose closure in the natural metric on $N$ is not an ideal.

1. Introduction

Let $D$ denote the open unit disc $\{|z| < 1\}$ in the complex plane $\mathbb{C}$ and let $T$ denote the boundary of $D$. The Nevanlinna class $N$ is the set of the holomorphic functions on $D$ with

$$\lim_{r \to 1^-} \frac{1}{2\pi} \int_{-\pi}^{\pi} \log^+ |f(re^{it})| \, dt < \infty,$$

i.e. with bounded characteristic. Here $\log^+ t = \max\{0, \log t\}$.

$N$ is an algebra under the usual pointwise algebraic operations and a complete metric space under the metric defined by $d(f, g) = \|f - g\|$, where $\| \cdot \|$ is the quasinorm on $N$ given by

$$\|f\| = \lim_{r \to 1^-} \frac{1}{2\pi} \int_{-\pi}^{\pi} \log^+ |f(re^{it})| \, dt.$$ 

It is remarkable that the scalar multiplication in $N$ is not continuous [9].

We write $S_\mu$ for the singular inner function associated with the (finite positive) singular (Borel) measure $\mu$ (on $T$). (From now on we will omit the words put in parentheses.) Every function $f \in N \setminus \{0\}$ can be factored uniquely, up to a constant of modulus 1, as

$$f = B_f S_{\nu_f} F_f,$$

where $B_f$ is the Blaschke product with respect to the zeros of $f$, $F_f$ is an outer function, and $\nu_f$ and $\mu_f$ are singular and mutually singular measures [2, Chapter II, Theorem 5.5].

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It is interesting that
\[(1) \|f\| = \|F_f\| + \mu_f(T)\]
for every \( f \) in \( N \) [10, Lemma 4.5].

The set \( N^+ = \{ f \in N : \mu_f = 0 \} \) is called the Smirnov class.

2. The Corona Property

We say an algebra \( A \) with unit of holomorphic functions on \( \mathbb{D} \) has the Corona Property if the ideal generated by \( f_1, \ldots, f_n \in A \) is equal to \( A \) if and only if there is an invertible element \( f \) of \( A \) with
\[|f(z)| \leq \sum_{i=1}^{n} |f_i(z)| \quad (z \in \mathbb{D}).\]

This definition is motivated by the famous Corona Theorem of Carleson (see for example [2, Chapter VIII]), which states that the algebra \( H^\infty \) of all bounded analytic functions on \( \mathbb{D} \) has the Corona Property.

Mortini noted [5, Satz 4] that the following theorem is an easy consequence of a result of Wolff [2, Chapter VIII, Theorem 2.3].

Theorem 1. The Nevanlinna class has the Corona Property.

We remark that an element of \( N \) is invertible if and only if it is zero free.

Since an analytic function \( f \) on \( \mathbb{D} \) is in \( N \) if and only if \( \log |f| \) has a positive harmonic majorant on \( \mathbb{D} \) [2, Chapter II, §5], we conclude:

Corollary 1. Let \( f_1, \ldots, f_n \in N \). Then \( (f_1, \ldots, f_n) = N \) if and only if \( -\log \sum_{i=1}^{n} |f_i| \) has a positive harmonic majorant on \( \mathbb{D} \).

We write \( (f_1, \ldots, f_n) \) for the ideal of \( N \) generated by \( f_1, \ldots, f_n \in N \).

For further investigations, for example, for the characterization of finitely generated prime ideals of \( N \), it would be useful to have more simple "analytic" conditions for \( (f_1, \ldots, f_n) = N \). In the next section we solve this problem under an additional assumption.

The following theorem is also a consequence of Wolff's result.

Theorem 2. If \( g \in (f_1, \ldots, f_n) \), then there exists a function \( f \) invertible in \( N \) with
\[|g(z)f(z)| \leq \sum_{i=1}^{n} |f_i(z)| \quad (z \in \mathbb{D}).\]

The converse does not hold. The counterexample can be constructed similar to the analogous counterexample for \( H^\infty \) (see [6]).

Corollary 2. If \( g \in (f_1, \ldots, f_n) \), then \( \log |g| - \log \sum_{i=1}^{n} |f_i| \) has a positive harmonic majorant.

3. Finitely generated ideals
and interpolating Blaschke products

Using Harnack's inequality for positive harmonic functions on \( \mathbb{D} \), we notice that
\[ -\log \sum_{i=1}^{n} |f_i(z)| \leq \frac{c}{1-|z|} \quad (z \in \mathbb{D}), \]
with a constant $c$, is a necessary condition for $(f_1, \ldots, f_n) = N$. In this section we show that under an additional assumption this condition is also sufficient.

A sequence of points $z_1, z_2, \ldots$ in $\mathbb{D}$ is called an interpolating sequence if, for every bounded sequence of complex numbers $w_1, w_2, \ldots$, there exists a function $f$ in $H^\infty$ with $f(z_k) = w_k$ for every $k$. An interpolating Blaschke product is a Blaschke product whose (simple) zeros form an interpolation sequence.

A non-tangential approach region is a region of the form

\[ \Gamma_\alpha(\omega) = \left\{ z \in \mathbb{D} : \frac{|\omega - z|}{1 - |z|} < \alpha \right\} \]

with $\omega \in T$ and $\alpha \geq 1$. A sequence of points $z_1, z_2, \ldots$ in $\mathbb{D}$ is called non-tangential if the points are contained in a finite number of non-tangential approach regions. A non-tangential Blaschke product is a Blaschke product whose zeros form a non-tangential sequence.

Further, let

\[ \rho(z, w) = \frac{|z - w|}{1 - \overline{z}w} \quad (z, w \in \mathbb{D}) \]

denote the pseudohyperbolic distance, and let $\Delta(z, \eta) = \{ w \in \mathbb{D} : \rho(z, w) < \eta \}$ be the pseudohyperbolic disc with center $z \in \mathbb{D}$ and radius $\eta > 0$.

**Lemma 1.** Let $z, w \in \mathbb{D}$ and $0 < \eta < 1$ with $w \in \Delta(z, \eta)$. Then

\[ |w - z| < \frac{\eta(1 - |z|^2)(1 + \eta|z|)}{1 - \eta^2|z|^2} \]

and

\[ |w| < \frac{|z|(1 - \eta^2) + \eta(1 - |z|^2)}{1 - \eta^2|z|^2}. \]

**Proof.** This follows easily from [2, Chapter I, (1.6) and (1.7)]. □

**Lemma 2.** Let $\Gamma$ be a non-tangential approach region and $0 < \eta < 1$. Then there is a non-tangential approach region $\Gamma'$ with $\bigcup_{z \in \Gamma} \Delta(z, \eta) \subset \Gamma'$.

**Proof.** W.l.o.g. we assume $\Gamma = \Gamma_\alpha(1)$. Let $z \in \Gamma$ and $w \in \Delta(z, \eta)$. With the previous lemma we get

\[ \frac{|1 - w|}{1 - |w|} \leq \frac{|1 - z| + |z - w|}{1 - |w|} \leq \frac{\alpha(1 - |z|)(1 - \eta^2|z|^2) + \eta(1 - |z|^2)(1 + \eta|z|)}{1 - \eta^2|z|^2 - |z|(1 - \eta^2) - \eta(1 - |z|^2)} \leq \frac{\alpha + 4}{(1 - \eta)^2}. \]

So $w \in \Gamma' = \Gamma_\beta(1)$ with $\beta = (\alpha + 4)/(1 - \eta)^2$. □

**Theorem 3.** Assume that the ideal $(f_1, \ldots, f_n)$ of $N$ contains a non-tangential interpolating Blaschke product $B$. Then we have $(f_1, \ldots, f_n) = N$ if and only if

\[ -\log \sum_{i=1}^{n} |f_i(z)| \leq \frac{c}{1 - |z|} \quad (z \in \mathbb{D}) \]

for a constant $c$.

**Proof.** Because of the remark at the beginning of this section we need only to show the sufficiency. So assume $-\log \sum_{i=1}^{n} |f_i(z)| \leq c/(1 - |z|)$ for $z$ in $\mathbb{D}$.
Further, for the sake of simplicity, we assume that the zeros $a_1, a_2, \ldots$ of $B$ are contained in a single nontangential approach region $\Gamma = \Gamma_\alpha(1)$. Due to a lemma of Kerr-Lawson [4, Lemma 1] there is a $\delta > 0$ with $|B| > \delta$ on $\mathbb{D} \setminus \bigcup_{k=1}^\infty \Delta(a_k, 1/2)$. Therefore, by Corollary 2, there exists a positive harmonic function $u$ with
\[ -\log \sum_{i=1}^n |f_i(z)| \leq -\log \delta + u(z) \]
for $z$ in $\mathbb{D} \setminus \bigcup_{k=1}^\infty \Delta(a_k, 1/2)$.

Let $\Gamma_\beta(1)$ be the nontangential approach region covering every $\Delta(a_k, 1/2)$ according to Lemma 2. Then we have
\[ -\log \sum_{i=1}^n |f_i(z)| \leq \frac{c}{1 - |z|} \leq c\beta^2 \cdot \frac{1 - |z|^2}{|1 - z|^2} \]
for $z$ in $\Gamma_\beta(1)$.

Together we have
\[ -\log \sum_{i=1}^n |f_i(z)| \leq c\beta^2 \cdot \frac{1 - |z|^2}{|1 - z|^2} - \log \delta + u(z) \]
for every $z$ in $\mathbb{D}$. Since the right side is a positive harmonic function, Corollary 1 implies that $(f_1, \ldots, f_n) = N$. □

We call an ideal $I \neq N$ of $N$ free if the functions in $I$ have no common zero. Otherwise we call it fixed.

An inspection of the proof to Theorem 3 gives the following variant of this theorem, which we use in §5.

**Theorem 4.** Assume that the ideal $(f_1, \ldots, f_n)$ of $N$ is free and contains a nontangential interpolating Blaschke product $B$ with zeros $a_1, a_2, \ldots$. Further, let $0 < \eta < 1$. Then there is a sequence $b_1, b_2, \ldots$ of points in $\mathbb{D}$, so that $b_k \in \Delta(a_k, \eta)$ for all $k$ and
\[ \limsup_{k \to \infty} \left( -(1 - |b_k|) \log \sum_{i=1}^n |f_i(b_k)| \right) = +\infty. \]

**4. A Counterexample**

A natural question is whether the assumption made in Theorem 3 can be weakened. We show that in general the nontangential interpolating Blaschke product cannot be replaced by an interpolating Blaschke product.

First we prove a generalization of Theorem 9.2 in [1].

**Lemma 3.** Let $B$ be a Blaschke product whose zeros $a_1, a_2, \ldots$ form an exponential sequence, i.e. $1 - |a_{k+1}| \leq c \cdot (1 - |a_k|)$ for every $k$ and a constant $c < 1$. Further, let $A_k$ be the annulus
\[ \left\{ z : 1 - \frac{1 - |a_k|}{c_1} \leq |z| \leq 1 - \frac{1 - |a_{k+1}|}{c_2} \right\} \]
with constants $c < c_1, c_2 < 1$. Then there is a constant $\delta > 0$ with
\[ |B(z)| \geq \delta \cdot \frac{|a_n - z|}{|1 - a_n z|} \]
for every $n$.

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Proof. Let \( n \) be arbitrary and \( z \) be in \( K_n \). For \( k > n \) we have

\[
\left| \frac{a_k - z}{1 - \overline{a_k} z} \right| \geq \left| \frac{a_k - (1 - (1 - |a_{n+1}|)/c_2)}{1 - |a_k|(1 - (1 - |a_{n+1}|))/c_2} \right| \geq \frac{1 - c_2 k^{-n-1}}{1 + c_2 k^{-n-1}},
\]

since \( 1 - |a_k| \leq c^{k-n-1} (1 - |a_{n+1}|) \).

Analogously, we have for \( 1 \leq k < n \) the inequality

\[
\left| \frac{a_k - z}{1 - \overline{a_k} z} \right| \geq \frac{1 - (1 - |a_n|)/c_1 - |a_k|}{1 - |a_k|(1 - (1 - |a_n|))/c_1} \geq \frac{c_1 - c^{n-k}}{c_1 + c^{n-k}}.
\]

Together we have

\[
|B(z)| \geq \left| \frac{a_n - z}{1 - \overline{a_n} z} \right| \cdot \prod_{m=1}^{\infty} \frac{c_1 - c^m}{c_1 + c^m} \cdot \prod_{m=0}^{\infty} \frac{1 - c_2 c^m}{1 + c_2 c^m}.
\]

The standard test [8, Theorem 15.5], shows that the two infinite products converge to strictly positive numbers \( \delta_1 \) (resp. \( \delta_2 \)). With \( \delta = \delta_1 \delta_2 \) the assertion follows, since \( \delta \) depends only on \( c, c_1 \), and \( c_2 \).

**Theorem 5.** There exist functions \( f_1 \) and \( f_2 \) in \( N \), so that

\[
- \log(|f_1(z)| + |f_2(z)|) \leq \frac{c}{1 - |z|} \quad (z \in \mathbb{D})
\]

for a constant \( c \), and \( (f_1, f_2) \) contains an interpolating Blaschke product, but \( (f_1, f_2) \) is not equal to \( N \).

**Proof.** For \( i = 1, 2 \), let \( f_i = B_i \) be the Blaschke product with respect to the zeros \( a_{i,0}, a_{i,1}, \ldots \). Here \( a_{i,n} = r_{i,n} e^{it_n} \), where \( t_n = \arccos(1 - 2^{-n}) \), \( r_{1,n} = 1 - 2^{-n} \), and \( r_{2,n} = 1 - 2^{-n} + 2^{-n}/\exp 2^n \). Both sequences are exponential sequences, so it follows that both Blaschke products are interpolating Blaschke products [1, Theorem 9.2].

Consider the annuli

\[
\mathcal{A}_{1,n} \cup \mathcal{A}_{2,n} = \{1 - \delta(1 - |a_{i,n}|) < |z| < 1 - \delta(1 - |a_{i,n+1}|)\}.
\]

We remark that \( \bigcup_{n=0,1,\ldots}(A_{1,n} \cup A_{2,n}) = \mathbb{D} \).

Lemma 3 implies the existence of \( \delta > 0 \), so that

\[
|B_i(z)| \geq \delta \cdot \left| \frac{a_{i,n} - z}{1 - \overline{a_{i,n}} z} \right|
\]

for every \( i \in \{1, 2\} \), every \( n \) and every \( z \) in \( A_{i,n} \). Now choose \( z \) arbitrary in \( \mathbb{D} \). Then there is an \( n \) with \( z \in A_{1,n} \cup A_{2,n} \). Therefore

\[
|B_1(z)| + |B_2(z)| \geq \delta \cdot \left( \left| \frac{a_{1,n} - z}{1 - \overline{a_{1,n}} z} \right| + \left| \frac{a_{2,n} - z}{1 - \overline{a_{2,n}} z} \right| \right) \geq \frac{\delta}{2^n \exp 2^n},
\]

and finally

\[
- \log(|B_1(z)| + |B_2(z)|) \leq (2 - \log \delta) 2^n \leq c \cdot \frac{3}{5} \cdot \frac{1}{1 - |a_{1,n}|} \leq \frac{c}{1 - |z|},
\]

where \( c = 5(2 - \log \delta)/3 \).

It remains to show that \( (B_1, B_2) \neq N \). By Theorem 1, it is enough to show that \( - \log(|B_1| + |B_2|) \) has no positive harmonic majorant. Now assume the
contrary and let \( u \) be a positive harmonic majorant. By Theorem 11.30(c) of [8], \( u \) is the Poisson integral of a measure \( \mu \). Let \( a = \mu(\{1\}) \) and \( \nu = \mu - a\delta_1 \), where \( \delta_1 \) is the unit mass at the point 1. Further, let \( v \) be the Poisson integral of \( \nu \). Then we have
\[
u(z) = v(z) + \Re \frac{1 + z}{1 - z} \quad (z \in \mathbb{D}).
\]
Since the measure \( \nu \) is continuous at the point 1, we conclude with a standard argument that \( \lim_{n \to \infty} (1 - |a_1, n|) v(a_1, n) = 0 \). Because of
\[
\Re \frac{1 + a_{1, n}}{1 - a_{1, n}} = \frac{1 - r_{1, n}^2}{1 - 2r_{1, n} \cos t_n + r_{1, n}^2} = 1,
\]
it follows
\[
\lim_{n \to \infty} (1 - |a_1, n|) u(a_1, n) = 0.
\]
But on the other hand,
\[
|B_1(a_1, n)| + |B_2(a_1, n)| = |B_2(a_1, n)| \leq \frac{|b_n - a_n|}{1 - b_n a_n} \leq \frac{1}{\exp 2^n},
\]
thus
\[
(1 - |a_1, n|) u(a_1, n) \geq (1 - r_{1, n}) \cdot 2^n = 1,
\]
a contradiction. \( \Box \)

5. Finitely generated prime ideals

It is easy to see that a fixed ideal of \( N \) is maximal if and only if it is of the form \( M_a = \{ f \in N : f(a) = 0 \} \), where \( a \in \mathbb{D} \). An ideal of the form \( M_a \) is a principal ideal, since it is generated by the function \( z - a \).

**Theorem 6.** A prime ideal \( P \) of the Nevanlinna class \( N \) containing a nontangential interpolating Blaschke product \( B \) is finitely generated if and only if it is a fixed maximal ideal.

**Proof.** Assume \( P = \langle f_1, \ldots, f_n \rangle \). Let \( a_1, a_2, \ldots \) be the zeros of \( B \). As an interpolation sequence this sequence is separated, i.e. there is an \( \eta > 0 \) so that the pseudohyperbolic discs \( \Delta(a_k, \eta) \) are pairwise disjoint [2, p. 285]. Because of Theorem 4, there is a sequence \( b_1, b_2, \ldots \) of points in \( \mathbb{D} \), so that \( b_k \in \Delta(a_k, \eta) \) for all \( k \) and
\[
\limsup_{k \to \infty} \left( -(1 - |b_k|) \log \sum_{i=1}^n |f_i(b_k)| \right) = +\infty.
\]
Now we partition the sequence \( b_1, b_2, \ldots \) into two subsequences \( b_{p_1}, b_{p_2}, \ldots \) and \( b_{q_1}, b_{q_2}, \ldots \), so that both keep this property. Let \( B_1 \) and \( B_2 \) be the Blaschke product with respect to the sequences \( a_{p_1}, a_{p_2}, \ldots \) and \( a_{q_1}, a_{q_2}, \ldots \). Then we have \( B_1 B_2 = B \in P \). Since \( P \) is prime, we have \( B_1 \in P \) or \( B_2 \in P \). W.l.o.g. we assume \( B_1 \in P \). The lemma of Kerr-Lawson [4, Lemma 1] yields \( |B_1| \geq \delta > 0 \) on \( \mathbb{D} \setminus \bigcup_{k=1}^\infty \Delta(a_{p_k}, \eta) \). Especially, we have \( |B_1(b_{q_k})| \geq \delta \) for all
k. This implies
\[
\limsup_{k \to \infty} \left( \log |B_1(b_{q_k})| - (1 - |b_{q_k}|) \log \sum_{i=1}^{n} |f_i(b_{q_k})| \right) = +\infty.
\]
In view of Corollary 2, this is a contradiction to \( B_1 \in P \).

6. AN ALGEBRA WITHOUT THE CORONA PROPERTY

The algebras of holomorphic functions appearing in the literature usually have the Corona Property; for example, the Nevanlinna class \( N \), the Smirnov class \( N^+ \), the algebra \( H^\infty \), the disc algebra \( A(\mathbb{D}) \) of functions analytic in \( \mathbb{D} \) and continuous in \( \overline{\mathbb{D}} \), the Wiener algebra \( W^+ \) of analytic functions with absolute converging Taylor series, the algebras \( A^n(\mathbb{D}) \) of functions analytic in \( \mathbb{D} \) whose \( n \)th derivative extends continuously to \( \overline{\mathbb{D}} \), and others.

Therefore, we want to give an example for an algebra of analytic functions without the Corona Property.

Lemma 4. Let \( B \) be a Blaschke product with zeros \( a_1, a_2, \ldots \) in \([0, 1)\) and let \( \mu \) and \( \nu \) be singular measures with \( \nu(\{1\}) < \mu(\{1\}) \). Then the singular inner function \( S_\nu \) is not in the ideal \( (B, S_\nu)_{N^+} \) of \( N^+ \) generated by \( B \) and \( S_\nu \).

Proof. This is an easy consequence of the fact that \( N^+ \) has the Corona Property [5, Satz 4] and of equation (2.6) in [9].

Theorem 7. There exists a subalgebra of \( N \) containing \( N^+ \) without the Corona Property.

Proof. Let \( B \) have a Blaschke product with zeros \( 1 - 1/4^k \) \((k = 1, 2, \ldots)\). For \( n = 0, 1, \ldots \), let \( I_n \) be the ideal of \( N^+ \) generated by the functions \( B^k S_{(n-k)\delta_1} \) \((k = 0, \ldots, n)\), where \( \delta_1 \) denotes the unit mass. We claim that
\[
A = \bigcup_{n=0}^{\infty} \left\{ \frac{h}{S_{n\delta_1}} : h \in I_n \right\}
\]
is an algebra without the Corona Property.

The verification that \( A \) is an algebra is straightforward.

A simple argument, using factorization and Lemma 4, shows that the invertible elements of \( A \) are exactly the invertible elements of \( N^+ \).

Now consider the Blaschke product \( B_0 \) with zeros \( 1 - 2/4^k \) \((k = 1, 2, \ldots)\). By Lemma 4 we have \( 1 \notin (S_{\delta_1}, B_0)_{N^+} \). So, since \( N^+ \) has the Corona Property, there is no function \( f \) invertible in \( N^+ \) (i.e. in \( A \)) with \( |f| \leq |B_0| + |S_{\delta_1}| \).

On the other hand, we have
\[
1 = g_1 B + g_2 B_0 = \frac{g_1 B}{S_{\delta_1}} \cdot S_{\delta_1} + g_2 \cdot B_0 \in (S_{\delta_1}, B_0)_A
\]
with \( g_1, g_2 \) in \( H^\infty \) [3, Chapter 10, Example 5]. Therefore \( A \) does not have the Corona Property.

7. THE TOPOLOGICAL CLOSURE OF IDEALS

In this section we pose the following question: Is the topological closure of an ideal in \( N \) again an ideal? This is true in every Banach algebra, because of the continuity of addition and scalar multiplication. But scalar multiplication in \( N \) is not continuous. We show that there is indeed an ideal of \( N \) whose closure is not an ideal.

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Let \( \lambda : N \to \mathbb{R} \) be the (nonlinear) functional with \( \lambda(f) = \mu_f(\{1\}) \). Then
\[
(2) \quad \lambda(f + g) = \max\{\lambda(f), \lambda(g)\}
\]
if \( \lambda(f) \neq \lambda(g) \) (see [9]).

**Theorem 8.** There exists an ideal of the Nevanlinna class \( N \) whose topological closure is not an ideal.

**Proof.** Let \( a_1, a_2, \ldots \) be a sequence of numbers in \([0, 1)\) with \( \sum_{n=1}^{\infty} (1 - |a_n|) < \infty \), and let \( B_n \) \( (n = 1, 2, \ldots) \) be the Blaschke product with zeros \( a_n, a_{n+1}, \ldots \). We consider the ideal \( I \) of \( N \) generated by \( B_1, B_2, \ldots \) and assume that the closure \( J = \text{cl}(I) \) is an ideal. A short calculation, using Schwarz inequality, Theorem 3.6 in Chapter II of [2], and the fact that \( \lim_{n \to \infty} \prod_{k=1}^{n} a_k = 1 \), shows \( \|B_n - 1\| \to 0 \) with \( n \to \infty \) (cf. [11, proof of Theorem 3.1]). So we have \( 1 \in J \), and therefore \( 1/S_{2\delta} \in J \). Thus, there is a function \( f \in I \) with \( \|f - 1/S_{2\delta}\| < 1 \), and (1) and (2) yield \( \mu_f(\{1\}) = 2 \). Let \( g = S_{\mu_f}(f - 1/S_{2\delta}) \). Then \( \|g/S_{\mu_f}\| < 1 \). By (1) we see that \( S_{\delta} \) must divide \( g \) in \( N^+ \), so there is an \( h \in N^+ \) with \( hS_{\delta} = fS_{\mu_f} - S_{\mu_f - 2\delta} \). Since \( B_n \) divides \( f \) for an \( n \), we conclude \( S_{\mu_f - 2\delta} \in (B_n, S_{\delta})_{N^+} \), a contradiction to Lemma 4. \( \square \)

In the Banach algebra \( H^\infty \) a complete description of the closed ideals is unknown. In contrast to this fact there is a simple characterization of the closed ideals in \( N \), which can be proved analogously to the related result for \( N^+ \) of Roberts and Stoll [7, Theorem 2], using Beurling’s famous invariant subspace theorem [8, Theorem 17.21].

**Theorem 9.** The topological closed ideals of the Nevanlinna class \( N \) are exactly the principal ideals.

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**References**


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