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ON THE IDEAL STRUCTURE OF THE NEVANLINNA CLASS

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ABSTRACT. Let N denote the Nevanlinna class, i.e. the algebra of holomorphic functions of bounded characteristic in the open unit disc. We study analytic conditions for a finitely generated ideal to be equal to the whole algebra N. Then we characterize the finitely generated prime ideals containing a nontangential interpolating Blaschke product. Further, we give an example of an ideal of N whose closure in the natural metric on N is not an ideal.

1. INTRODUCTION

Let $\mathbb{D}$ denote the open unit disc $\{|z| < 1\}$ in the complex plane $\mathbb{C}$ and let T denote the boundary of $\mathbb{D}$. The Nevanlinna class N is the set of the holomorphic functions on $\mathbb{D}$ with

$$\lim_{r\to 1-} \frac{1}{2\pi} \int_{-\pi}^{\pi} \log^+ |f(re^{it})|\, dt < \infty,$$

i.e. with bounded characteristic. Here $\log^+ t = \max\{0, \log t\}$.

N is an algebra under the usual pointwise algebraic operations and a complete metric space under the metric defined by $d(f, g) = \|f - g\|$, where $\|\cdot\|$ is the quasinorm on N given by

$$\|f\| = \lim_{r\to 1-} \frac{1}{2\pi} \int_{-\pi}^{\pi} \log^+ |f(re^{it})|\, dt.$$

It is remarkable that the scalar multiplication in N is not continuous [9].

We write S_μ for the singular inner function associated with the (finite positive) singular (Borel) measure μ (on T). (From now on we will omit the words put in parentheses.) Every function $f \in N\backslash\{0\}$ can be factored uniquely, up to a constant of modulus 1, as

$$f = B_f \frac{S_{\nu_f}}{S_{\mu_f}} F_f,$$

where B_f is the Blaschke product with respect to the zeros of f, F_f is an outer function, and ν_f and μ_f are singular and mutually singular measures [2, Chapter II, Theorem 5.5].

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It is interesting that

$$\|f\| = \|F_f\| + \mu_f(T) \tag{1}$$

for every f in N [10, Lemma 4.5].

The set $N^+ = \{f \in N : \mu_f = 0\}$ is called the Smirnov class.

2. The Corona Property

We say an algebra A with unit of holomorphic functions on $\mathbb{D}$ has the *Corona Property* if the ideal generated by $f_1, \dots, f_n \in A$ is equal to A if and only if there is an invertible element f of A with

$$|f(z)| \le \sum_{i=1}^{n} |f_i(z)| \qquad (z \in \mathbb{D}).$$

This definition is motivated by the famous Corona Theorem of Carleson (see for example [2, Chapter VIII]), which states that the algebra H^∞ of all bounded analytic functions on $\mathbb{D}$ has the Corona Property.

Mortini noted [5, Satz 4] that the following theorem is an easy consequence of a result of Wolff [2, Chapter VIII, Theorem 2.3].

Theorem 1. *The Nevanlinna class has the Corona Property.*

We remark that an element of N is invertible if and only if it is zero free.

Since an analytic function f on $\mathbb{D}$ is in N if and only if $\log|f|$ has a positive harmonic majorant on $\mathbb{D}$ [2, Chapter II, §5], we conclude:

Corollary 1. *Let $f_1, \dots, f_n \in N$. Then $(f_1, \dots, f_n) = N$ if and only if $-\log\sum_{i=1}^{n}|f_i|$ has a positive harmonic majorant on $\mathbb{D}$.*

We write $(f_1, \dots, f_n)$ for the ideal of N generated by $f_1, \dots, f_n \in N$.

For further investigations, for example, for the characterization of finitely generated prime ideals of N, it would be useful to have more simple "analytic" conditions for $(f_1, \dots, f_n) = N$. In the next section we solve this problem under an additional assumption.

The following theorem is also a consequence of Wolff's result.

Theorem 2. *If $g \in (f_1, \dots, f_n)$, then there exists a function f invertible in N with*

$$|g(z)f(z)| \le \sum_{i=1}^{n} |f_i(z)| \qquad (z \in \mathbb{D}).$$

The converse does not hold. The counterexample can be constructed similar to the analogous counterexample for H^∞ (see [6]).

Corollary 2. *If $g \in (f_1, \dots, f_n)$, then $\log|g| - \log\sum_{i=1}^{n}|f_i|$ has a positive harmonic majorant.*

3. Finitely generated ideals and interpolating Blaschke products

Using Harnack's inequality for positive harmonic functions on $\mathbb{D}$, we notice that

$$-\log\sum_{i=1}^{n} |f_i(z)| \le \frac{c}{1-|z|} \qquad (z \in \mathbb{D}),$$

with a constant c, is a necessary condition for $(f_1, \dots, f_n) = N$. In this section we show that under an additional assumption this condition is also sufficient.

A sequence of points $z_1, z_2, \dots$ in $\mathbb{D}$ is called an *interpolating sequence* if, for every bounded sequence of complex numbers $w_1, w_2, \dots$, there exists a function f in H^∞ with $f(z_k) = w_k$ for every k. An *interpolating Blaschke product* is a Blaschke product whose (simple) zeros form an interpolation sequence.

A *nontangential approach region* is a region of the form

$$\Gamma_\alpha(\omega) = \left\{ z \in \mathbb{D} : \frac{|\omega - z|}{1 - |z|} < \alpha \right\}$$

with $\omega \in T$ and $\alpha \geq 1$. A sequence of points $z_1, z_2, \dots$ in $\mathbb{D}$ is called *nontangential* if the points are contained in a finite number of nontangential approach regions. A *nontangential Blaschke product* is a Blaschke product whose zeros form a nontangential sequence.

Further, let

$$\rho(z, w) = \left| \frac{z - w}{1 - \overline{w} z} \right| \qquad (z, w \in \mathbb{D})$$

denote the *pseudohyperbolic distance*, and let $\Delta(z, \eta) = \{w \in \mathbb{D} : \rho(z, w) < \eta\}$ be the pseudohyperbolic disc with center $z \in \mathbb{D}$ and radius $\eta > 0$.

Lemma 1. *Let $z, w \in \mathbb{D}$ and $0 < \eta < 1$ with $w \in \Delta(z, \eta)$. Then*

$$|w - z| < \frac{\eta(1 - |z|^2)(1 + \eta|z|)}{1 - \eta^2|z|^2}$$

and

$$|w| < \frac{|z|(1 - \eta^2) + \eta(1 - |z|^2)}{1 - \eta^2|z|^2}.$$

Proof. This follows easily from [2, Chapter I, (1.6) and (1.7)]. □

Lemma 2. *Let Γ be a nontangential approach region and $0 < \eta < 1$. Then there is a nontangential approach region Γ' with $\bigcup_{z \in \Gamma} \Delta(z, \eta) \subset \Gamma'$.*

Proof. W.l.o.g. we assume $\Gamma = \Gamma_\alpha(1)$. Let $z \in \Gamma$ and $w \in \Delta(z, \eta)$. With the previous lemma we get

$$\begin{aligned} \frac{|1 - w|}{1 - |w|} &\leq \frac{|1 - z| + |z - w|}{1 - |w|} \\ &\leq \frac{\alpha(1 - |z|)(1 - \eta^2|z|^2) + \eta(1 - |z|^2)(1 + \eta|z|)}{1 - \eta^2|z|^2 - |z|(1 - \eta^2) - \eta(1 - |z|^2)} \leq \frac{\alpha + 4}{(1 - \eta)^2}. \end{aligned}$$

So $w \in \Gamma' = \Gamma_\beta(1)$ with $\beta = (\alpha + 4)/(1 - \eta)^2$. □

Theorem 3. *Assume that the ideal $(f_1, \dots, f_n)$ of N contains a nontangential interpolating Blaschke product B. Then we have $(f_1, \dots, f_n) = N$ if and only if*

$$-\log \sum_{i=1}^{n} |f_i(z)| \leq \frac{c}{1 - |z|} \qquad (z \in \mathbb{D})$$

for a constant c.

Proof. Because of the remark at the beginning of this section we need only to show the sufficiency. So assume $-\log \sum |f_i(z)| \leq c/(1 - |z|)$ for z in $\mathbb{D}$.

Further, for the sake of simplicity, we assume that the zeros $a_1, a_2, \dots$ of B are contained in a single nontangential approach region $\Gamma = \Gamma_\alpha(1)$. Due to a lemma of Kerr-Lawson [4, Lemma 1] there is a $\delta > 0$ with $|B| > \delta$ on $\mathbb{D}\setminus\bigcup_{k=1}^{\infty}\Delta(a_k, 1/2)$. Therefore, by Corollary 2, there exists a positive harmonic function u with

$$-\log\sum_{i=1}^{n}|f_i(z)| \le -\log\delta + u(z)$$

for z in $\mathbb{D}\setminus\bigcup_{k=1}^{\infty}\Delta(a_k, 1/2)$.

Let $\Gamma_\beta(1)$ be the nontangential approach region covering every $\Delta(a_k, 1/2)$ according to Lemma 2. Then we have

$$-\log\sum_{i=1}^{n}|f_i(z)| \le \frac{c}{1-|z|} \le c\beta^2\cdot\frac{1-|z|^2}{|1-z|^2}$$

for z in $\Gamma_\beta(1)$.

Together we have

$$-\log\sum_{i=1}^{n}|f_i(z)| \le c\beta^2\cdot\frac{1-|z|^2}{|1-z|^2} - \log\delta + u(z)$$

for every z in $\mathbb{D}$. Since the right side is a positive harmonic function, Corollary 1 implies that $(f_1, \dots, f_n) = N$. □

We call an ideal $I \ne N$ of N *free* if the functions in I have no common zero. Otherwise we call it *fixed.*

An inspection of the proof to Theorem 3 gives the following variant of this theorem, which we use in §5.

Theorem 4. *Assume that the ideal $(f_1, \dots, f_n)$ of N is free and contains a nontangential interpolating Blaschke product B with zeros $a_1, a_2, \dots$. Further, let $0 < \eta < 1$. Then there is a sequence $b_1, b_2, \dots$ of points in $\mathbb{D}$, so that $b_k \in \Delta(a_k, \eta)$ for all k and*

$$\limsup_{k\to\infty}\left(-(1-|b_k|)\log\sum_{i=1}^{n}|f_i(b_k)|\right) = +\infty.$$

4. A counterexample

A natural question is whether the assumption made in Theorem 3 can be weakened. We show that in general the nontangential interpolating Blaschke product cannot be replaced by an interpolating Blaschke product.

First we prove a generalization of Theorem 9.2 in [1].

Lemma 3. *Let B be a Blaschke product whose zeros $a_1, a_2, \dots$ form an exponential sequence, i.e. $1-|a_{k+1}| \le c\cdot(1-|a_k|)$ for every k and a constant $c < 1$. Further, let A_k be the annulus*

$$\left\{z : 1 - \frac{1-|a_k|}{c_1} \le |z| \le 1 - \frac{1-|a_{k+1}|}{c_2}\right\}$$

with constants $c < c_1, c_2 < 1$. Then there is a constant $\delta > 0$ with

$$|B(z)| \ge \delta\cdot\left|\frac{a_n - z}{1-\overline{a}_n z}\right|$$

for every n.

Proof. Let n be arbitrary and z be in K_n. For $k > n$ we have

$$\left|\frac{a_k - z}{1 - \overline{a_k} z}\right| \geq \frac{|a_k| - (1 - (1 - |a_{n+1}|)/c_2)}{1 - |a_k|(1 - (1 - |a_{n+1}|)/c_2)} \geq \frac{1 - c_2 c^{k-n-1}}{1 + c_2 c^{k-n-1}},$$

since $1 - |a_k| \leq c^{k-n-1}(1 - |a_{n+1}|)$.

Analogously, we have for $1 \leq k < n$ the inequality

$$\left|\frac{a_k - z}{1 - \overline{a_k} z}\right| \geq \frac{1 - (1 - |a_n|)/c_1 - |a_k|}{1 - |a_k|(1 - (1 - |a_n|)/c_1)} \geq \frac{c_1 - c^{n-k}}{c_1 + c^{n-k}}.$$

Together we have

$$|B(z)| \geq \left|\frac{a_n - z}{1 - \overline{a_n} z}\right| \cdot \prod_{m=1}^{\infty} \frac{c_1 - c^m}{c_1 + c^m} \cdot \prod_{m=0}^{\infty} \frac{1 - c_2 c^m}{1 + c_2 c^m}.$$

The standard test [8, Theorem 15.5], shows that the two infinite products converge to strictly positive numbers δ_1 (resp. δ_2). With $\delta = \delta_1 \delta_2$ the assertion follows, since δ depends only on c, c_1, and c_2. □

Theorem 5. *There exist functions f_1 and f_2 in N, so that*

$$-\log(|f_1(z)| + |f_2(z)|) \leq \frac{c}{1 - |z|} \qquad (z \in \mathbb{D})$$

for a constant c, and (f_1, f_2) contains an interpolating Blaschke product, but (f_1, f_2) is not equal to N.

Proof. For $i = 1, 2$, let $f_i = B_i$ be the Blaschke product with respect to the zeros $a_{i,0}, a_{i,1}, \ldots$. Here $a_{i,n} = r_{i,n} e^{it_n}$, where $t_n = \arccos(1 - 2^{-n})$, $r_{1,n} = 1 - 2^{-n}$, and $r_{2,n} = 1 - 2^{-n} + 2^{-n}/\exp 2^n$. Both sequences are exponential sequences, so it follows that both Blaschke products are interpolating Blaschke products [1, Theorem 9.2].

Consider the annuli

$$A_{i,n} = \{1 - \tfrac{5}{3}(1 - |a_{i,n}|) \leq |z| \leq 1 - \tfrac{4}{3}(1 - |a_{i,n+1}|)\}.$$

We remark that $\bigcup_{n=0,1,\ldots}(A_{1,n} \cup A_{2,n}) = \mathbb{D}$.

Lemma 3 implies the existence of $\delta > 0$, so that

$$|B_i(z)| \geq \delta \cdot \left|\frac{a_{i,n} - z}{1 - \overline{a_{i,n}} z}\right|$$

for every $i \in \{1, 2\}$, every n and every z in $A_{i,n}$. Now choose z arbitrary in $\mathbb{D}$. Then there is an n with $z \in A_{1,n} \cup A_{2,n}$. Therefore

$$|B_1(z)| + |B_2(z)| \geq \delta \cdot \left(\left|\frac{a_{1,n} - z}{1 - \overline{a_{1,n}} z}\right| + \left|\frac{a_{2,n} - z}{1 - \overline{a_{2,n}} z}\right|\right) \geq \frac{\delta}{2^n \exp 2^n},$$

and finally

$$-\log(|B_1(z)| + |B_2(z)|) \leq (2 - \log\delta) 2^n \leq c \cdot \frac{3}{5} \cdot \frac{1}{1 - |a_{1,n}|} \leq \frac{c}{1 - |z|},$$

where $c = 5(2 - \log\delta)/3$.

It remains to show that $(B_1, B_2) \neq N$. By Theorem 1, it is enough to show that $-\log(|B_1| + |B_2|)$ has no positive harmonic majorant. Now assume the contrary and let u be a positive harmonic majorant. By Theorem 11.30(c) of [8], u is the Poisson integral of a measure μ. Let $a = \mu(\{1\})$ and $\nu = \mu - a\delta_1$, where δ_1 is the unit mass at the point 1. Further, let v be the Poisson integral of ν. Then we have

$$u(z) = v(z) + \operatorname{Re}\frac{1+z}{1-z} \qquad (z \in \mathbb{D}).$$

Since the measure ν is continuous at the point 1, we conclude with a standard argument that $\lim_{n\to\infty}(1-|a_{1,n}|)v(a_{1,n}) = 0$. Because of

$$\operatorname{Re}\frac{1+a_{1,n}}{1-a_{1,n}} = \frac{1-r_{1,n}^2}{1-2r_{1,n}\cos t_n + r_{1,n}^2} = 1,$$

it follows

$$\lim_{n\to\infty}(1-|a_{1,n}|)u(a_{1,n}) = 0.$$

But on the other hand,

$$|B_1(a_{1,n})| + |B_2(a_{1,n})| = |B_2(a_{1,n})| \le \left|\frac{b_n - a_n}{1-\overline{b_n}a_n}\right| \le \frac{1}{\exp 2^n},$$

thus

$$(1-|a_{1,n}|)u(a_{1,n}) \ge (1-r_{1,n})\cdot 2^n = 1,$$

a contradiction. □

5. Finitely generated prime ideals

It is easy to see that a fixed ideal of N is maximal if and only if it is of the form $M_a = \{f \in N\colon f(a) = 0\}$, where $a \in \mathbb{D}$. An ideal of the form M_a is a principal ideal, since it is generated by the function $z - a$.

Theorem 6. *A prime ideal P of the Nevanlinna class N containing a nontangential interpolating Blaschke product B is finitely generated if and only if it is a fixed maximal ideal.*

Proof. Assume $P = (f_1, \dots, f_n)$. Let $a_1, a_2, \dots$ be the zeros of B. As an interpolation sequence this sequence is separated, i.e. there is an $\eta > 0$ so that the pseudohyperbolic discs $\Delta(a_k, \eta)$ are pairwise disjoint [2, p. 285]. Because of Theorem 4, there is a sequence $b_1, b_2, \dots$ of points in $\mathbb{D}$, so that $b_k \in \Delta(a_k, \eta)$ for all k and

$$\limsup_{k\to\infty}\left(-(1-|b_k|)\log\sum_{i=1}^{n}|f_i(b_k)|\right) = +\infty.$$

Now we partition the sequence $b_1, b_2, \dots$ into two subsequences $b_{p_1}, b_{p_2}, \dots$ and $b_{q_1}, b_{q_2}, \dots$, so that both keep this property. Let B_1 and B_2 be the Blaschke product with respect to the sequences $a_{p_1}, a_{p_2}, \dots$ and $a_{q_1}, a_{q_2}, \dots$. Then we have $B_1B_2 = B \in P$. Since P is prime, we have $B_1 \in P$ or $B_2 \in P$. W.l.o.g. we assume $B_1 \in P$. The lemma of Kerr-Lawson [4, Lemma 1] yields $|B_1| \ge \delta > 0$ on $\mathbb{D}\setminus\bigcup_{k=1}^{\infty}\Delta(a_{p_k}, \eta)$. Especially, we have $|B_1(b_{q_k})| \ge \delta$ for all

k. This implies

$$\limsup_{k\to\infty}\left(\log|B_1(b_{q_k})| - (1-|b_{q_k}|)\log\sum_{i=1}^{n}|f_i(b_{q_k})|\right) = +\infty.$$

In view of Corollary 2, this is a contradiction to $B_1 \in P$. □

6. An algebra without the Corona Property

The algebras of holomorphic functions appearing in the literature usually have the Corona Property; for example, the Nevanlinna class N, the Smirnov class N^+, the algebra H^∞, the disc algebra $A(\overline{\mathbb{D}})$ of functions analytic in $\mathbb{D}$ and continuous in $\overline{\mathbb{D}}$, the Wiener algebra W^+ of analytic functions with absolute converging Taylor series, the algebras $A^n(\overline{\mathbb{D}})$ of functions analytic in $\mathbb{D}$ whose nth derivative extends continuously to $\overline{\mathbb{D}}$, and others.

Therefore, we want to give an example for an algebra of analytic functions without the Corona Property.

Lemma 4. *Let B be a Blaschke product with zeros $a_1, a_2, \ldots$ in $[0, 1)$ and let μ and ν be singular measures with $\nu(\{1\}) < \mu(\{1\})$. Then the singular inner function S_ν is not in the ideal $(B, S_\mu)_{N^+}$ of N^+ generated by B and S_μ.*

Proof. This is an easy consequence of the fact that N^+ has the Corona Property [5, Satz 4] and of equation (2.6) in [9]. □

Theorem 7. *There exists a subalgebra of N containing N^+ without the Corona Property.*

Proof. Let B have a Blaschke product with zeros $1-1/4^k$ $(k = 1, 2, \ldots)$. For $n = 0, 1, \ldots,$ let I_n be the ideal of N^+ generated by the functions $B^k S_{(n-k)\delta_1}$ $(k = 0, \ldots, n)$, where δ_1 denotes the unit mass. We claim that

$$A = \bigcup_{n=0}^{\infty}\left\{\frac{h}{S_{n\delta_1}} : h \in I_n\right\}$$

is an algebra without the Corona Property.

The verification that A is an algebra is straightforward.

A simple argument, using factorization and Lemma 4, shows that the invertible elements of A are exactly the invertible elements of N^+.

Now consider the Blaschke product B_0 with zeros $1-2/4^k$ $(k = 1, 2, \ldots)$. By Lemma 4 we have $1 \notin (S_{\delta_1}, B_0)_{N^+}$. So, since N^+ has the Corona Property, there is no function f invertible in N^+ (i.e. in A) with $|f| \le |B_0| + |S_{\delta_1}|$.

On the other hand, we have

$$1 = g_1 B + g_2 B_0 = \frac{g_1 B}{S_{\delta_1}} \cdot S_{\delta_1} + g_2 \cdot B_0 \in (S_{\delta_1}, B_0)_A$$

with g_1, g_2 in H^∞ [3, Chapter 10, Example 5]. Therefore A does not have the Corona Property. □

7. The topological closure of ideals

In this section we pose the following question: Is the topological closure of an ideal in N again an ideal? This is true in every Banach algebra, because of the continuity of addition and scalar multiplication. But scalar multiplication in N is not continuous. We show that there is indeed an ideal of N whose closure is not an ideal.

Let $\lambda: N \to \mathbb{R}$ be the (nonlinear) functional with $\lambda(f) = \mu_f(\{1\})$. Then

$$\lambda(f+g) = \max\{\lambda(f), \lambda(g)\} \tag{2}$$

if $\lambda(f) \neq \lambda(g)$ (see [9]).

Theorem 8. *There exists an ideal of the Nevanlinna class N whose topological closure is not an ideal.*

Proof. Let $a_1, a_2, \ldots$ be a sequence of numbers in $[0, 1)$ with $\sum_{n=1}^{\infty}(1-|a_n|) < \infty$, and let B_n $(n = 1, 2, \ldots)$ be the Blaschke product with zeros $a_n, a_{n+1}, \ldots$. We consider the ideal I of N generated by $B_1, B_2, \ldots$ and assume that the closure $J = \mathrm{cl}(I)$ is an ideal. A short calculation, using Schwarz inequality, Theorem 3.6 in Chapter II of [2], and the fact that $\lim_{n\to\infty}\prod_n^{\infty} a_k = 1$, shows $\|B_n - 1\| \to 0$ with $n \to \infty$ (cf. [11, proof of Theorem 3.1]). So we have $1 \in J$, and therefore $1/S_{2\delta_1} \in J$. Thus, there is a function $f \in I$ with $\|f - 1/S_{2\delta_1}\| < 1$. (1) and (2) yield $\mu_f(\{1\}) = 2$. Let $g = S_{\mu_f}(f - 1/S_{2\delta_1})$. Then $\|g/S_{\mu_f}\| < 1$. By (1) we see that S_{δ_1} must divide g in N^+, so there is an $h \in N^+$ with $hS_{\delta_1} = fS_{\mu_f} - S_{\mu_f - 2\delta_1}$. Since B_n divides f for an n, we conclude $S_{\mu_f - 2\delta_1} \in (B_n, S_{\delta_1})_{N^+}$, a contradiction to Lemma 4. □

In the Banach algebra H^∞ a complete description of the closed ideals is unknown. In contrast to this fact there is a simple characterization of the closed ideals in N, which can be proved analogously to the related result for N^+ of Roberts and Stoll [7, Theorem 2], using Beurling's famous invariant subspace theorem [8, Theorem 17.21].

Theorem 9. *The topological closed ideals of the Nevanlinna class N are exactly the principal ideals.*

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