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ON p-RADICAL BLOCKS OF FINITE GROUPS

AKIHIKO HIDA

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ABSTRACT. We give a sufficient condition on a p-block of a finite group under which the block is p-radical.

Let $k[G]$ be the group algebra of a finite group G over an algebraically closed field k of characteristic $p > 0$. We call G *p-radical* if the induced module $(k_P)^G = k_P \otimes_{k[P]} k[G]$ from the trivial $k[P]$-module k_P is semisimple as a right $k[G]$-module where P is a Sylow p-subgroup of G (see [T; F, VI, 6]). Koshitani [Ko] showed that if the vertex $\mathrm{vx}(V)$ of V is contained in $\mathrm{Ker}\, V$ (the kernel of V) for any simple $k[G]$-module V then G is p-radical. We generalize this to a p-block form. Let B be a p-block of G and e_B be the block idempotent in $k[G]$ corresponding to B. We call B p-radical if $(k_P)^G \cdot e_B$ is a semisimple $k[G]$-module following Tsushima [T].

Theorem. *If the vertex $\mathrm{vx}(V)$ of V is contained in $\mathrm{Ker}\, V$ for any simple $k[G]$-module V in a p-block B of G, then B is p-radical.*

Throughout this paper we keep the notation as in the theorem. See the book of Feit [F] for the notion of vertices and other terminology.

Lemma 1. *Assume that $\mathrm{vx}(V) \subseteq \mathrm{Ker}\, V$ for any simple $k[G]$-module V in B.*
(a) [HM, Lemma 1.3; Ko, Lemma 1]. *If H is a normal p- or p'-subgroup of G and $\overline{B}$ is a p-block of G/H such that $B \supseteq \overline{B}$, then $\mathrm{vx}(\overline{V}) \subseteq \mathrm{Ker}\, \overline{V}$ for any simple $k[G/H]$-module $\overline{V}$ in $\overline{B}$.*
(b) [Kn2, 3.7 Corollary; Ko, Lemma 2]. *If B is the principal block of G, then G is p-solvable.*

Lemma 2. *Let H be a normal subgroup of G containing a defect group of B. If every block of H covered by B is p-radical then B is p-radical.*

Proof. By [Kn1, Theorem 2.9] it suffices to show that $(k_P)^G \cdot e_B$ is semisimple as a $k[H]$-module. Let $\{b_i\}$ be the set of all blocks of H covered by B. By Mackey decomposition $(k_P)^G \cdot e_B$ is a direct summand of

$$\bigoplus_i \bigoplus_{t \in P\backslash G/H} (k_{P^t \cap H})^H \cdot e_{b_i}$$

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as a $k[H]$-module. Since $P^t \cap H$ is a Sylow p-subgroup of H for all t and b_i is p-radical for all i, $(k_P)^G \cdot e_B$ is a semisimple $k[H]$-module.

Proof of the theorem. First, assume that B is not the principal block. There is a simple $k[G]$-module V in B such that $\mathrm{vx}(V)$ is equal to the defect group of B. Let $K = \mathrm{Ker}\, V$, and let b be any block of K covered by B. For any simple $k[K]$-module W in b, there is a simple $k[G]$-module U in B such that W is isomorphic to a direct summand of U as a $k[K]$-module. We may assume $\mathrm{vx}(W) \subseteq \mathrm{vx}(U)$. Hence, $\mathrm{vx}(W) \subseteq \mathrm{vx}(U) \cap K \subseteq \mathrm{Ker}\, U \cap K \subseteq \mathrm{Ker}\, W$. Since $K \neq G$, b is p-radical by induction on $|G|$. Hence B is p-radical by Lemma 2. So we may assume that B is the principal block. By Lemma 1(b), G is p-solvable. If $0_{p'}(G) = 1$, then B is the unique block of G by Fong's Theorem [F, X Theorem 1.5]. By Lemma 1(a) and induction, $G/0_p(G)$ is p-radical. Hence G is p-radical by [T, Proposition 1]. Let $H = 0_{p'}(G)$. If $H \neq 1$ then G/H is p-radical by Lemma 1(a) and induction. Hence B is p-radical since $(k_P)^G \cdot e_B \cong (k_{HP})^G$.

Corollary [Ko, Theorem]. *If the vertex of V is contained in* Ker V *for any simple $k[G]$-module V, then G is p-radical.*

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References

[F] W. Feit, *The representation theory of finite groups*, North-Holland, Amsterdam, 1982.

[HM] W. Hamernik and G. Michler, *On vertices of simple modules in p-solvable groups*, Mitteilungen aus dem Mathem. Seminar Giessen **121** (1976), 147–162.

[Kn1] R. Knörr, *Semisimplicity, induction and restriction for modular representations of finite groups*, J. Algebra **48** (1977), 347–367.

[Kn2] ——, *On the vertices of irreducible modules*, Ann. of Math. (2) **110** (1979), 487–499.

[Ko] S. Koshitani, *A remark on p-radical groups*, J. Algebra **134** (1990), 491–496.

[T] Y. Tsushima, *On p-radical groups*, J. Algebra **103** (1986), 80-86.

Department of Mathematics, Division of Mathematics and Physical Sciences, Graduate School of Science and Technology, Chiba University, Yayoi-cho, Chiba-city, 260, Japan