ON p-RADICAL BLOCKS OF FINITE GROUPS

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Abstract. We give a sufficient condition on a p-block of a finite group under which the block is p-radical.

Let $k[G]$ be the group algebra of a finite group $G$ over an algebraically closed field $k$ of characteristic $p > 0$. We call $G$ p-radical if the induced module $(kP)^G = kP \otimes_{k[P]} k[G]$ from the trivial $k[P]$-module $kP$ is semisimple as a right $k[G]$-module where $P$ is a Sylow $p$-subgroup of $G$ (see [T; F, VI, 6]). Koshitani [Ko] showed that if the vertex $v x(V)$ of $V$ is contained in $\ker V$ (the kernel of $V$) for any simple $k[G]$-module $V$ then $G$ is $p$-radical. We generalize this to a $p$-block form. Let $B$ be a $p$-block of $G$ and $e_B$ be the block idempotent in $k[G]$ corresponding to $B$. We call $B$ p-radical if $(kP)^G \cdot e_B$ is a semisimple $k[G]$-module following Tsushima [T].

Theorem. If the vertex $v x(V)$ of $V$ is contained in $\ker V$ for any simple $k[G]$-module $V$ in a $p$-block $B$ of $G$, then $B$ is p-radical.

Throughout this paper we keep the notation as in the theorem. See the book of Feit [F] for the notion of vertices and other terminology.

Lemma 1. Assume that $v x(V) \subseteq \ker V$ for any simple $k[G]$-module $V$ in $B$.
(a) [HM, Lemma 1.3; Ko, Lemma 1]. If $H$ is a normal $p$- or $p'$-subgroup of $G$ and $B$ is a $p$-block of $G/H$ such that $B \supseteq \overline{B}$, then $v x(V) \subseteq \ker \overline{V}$ for any simple $k[G/H]$-module $\overline{V}$ in $\overline{B}$.
(b) [Kn2, 3.7 Corollary; Ko, Lemma 2]. If $B$ is the principal block of $G$, then $G$ is $p$-solvable.

Lemma 2. Let $H$ be a normal subgroup of $G$ containing a defect group of $B$. If every block of $H$ covered by $B$ is p-radical then $B$ is p-radical.

Proof. By [Kn1, Theorem 2.9] it suffices to show that $(kP)^G \cdot e_B$ is semisimple as a $k[H]$-module. Let $\{b_i\}$ be the set of all blocks of $H$ covered by $B$. By Mackey decomposition $(kP)^G \cdot e_B$ is a direct summand of

$$\bigoplus_{i \in P \setminus G/H} (kP \cap H)^H \cdot e_{b_i}$$

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as a $k[H]$-module. Since $P_t \cap H$ is a Sylow $p$-subgroup of $H$ for all $t$ and $b_i$ is $p$-radical for all $i$, $(k_P)^G \cdot e_B$ is a semisimple $k[H]$-module.

**Proof of the theorem.** First, assume that $B$ is not the principal block. There is a simple $k[G]$-module $V$ in $B$ such that $\nu x(V)$ is equal to the defect group of $B$. Let $K = \text{Ker} V$, and let $b$ be any block of $K$ covered by $B$. For any simple $k[K]$-module $W$ in $b$, there is a simple $k[G]$-module $U$ in $B$ such that $W$ is isomorphic to a direct summand of $U$ as a $k[K]$-module. We may assume $\nu x(W) \subseteq \nu x(U)$. Hence, $\nu x(W) \subseteq \nu x(U) \cap K \subseteq \text{Ker} U \cap K \subseteq \text{Ker} W$. Since $K \neq G$, $b$ is $p$-radical by induction on $|G|$. Hence $B$ is $p$-radical by Lemma 2. So we may assume that $B$ is the principal block. By Lemma 1(b), $G$ is $p$-solvable. If $0_p,(G) = 1$, then $B$ is the unique block of $G$ by Fong's Theorem [F, X Theorem 1.5]. By Lemma 1(a) and induction, $G/0_p(G)$ is $p$-radical. Hence $G$ is $p$-radical by [T, Proposition 1]. Let $H = 0_p,(G)$. If $H \neq 1$ then $G/H$ is $p$-radical by Lemma 1(a) and induction. Hence $B$ is $p$-radical since $(k_P)^G \cdot e_B \cong (k_H)^G$.

**Corollary [Ko, Theorem].** If the vertex of $V$ is contained in $\text{Ker} V$ for any simple $k[G]$-module $V$, then $G$ is $p$-radical.

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**REFERENCES**


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