ON \( p \)-RADICAL BLOCKS OF FINITE GROUPS

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(Communicated by Warren J. Wong)

Abstract. We give a sufficient condition on a \( p \)-block of a finite group under which the block is \( p \)-radical.

Let \( k[G] \) be the group algebra of a finite group \( G \) over an algebraically closed field \( k \) of characteristic \( p > 0 \). We call \( G \) \( p \)-radical if the induced module \( (k_P)^G = k_P \otimes_{k[P]} k[G] \) from the trivial \( k[P] \)-module \( k_P \) is semisimple as a right \( k[G] \)-module where \( P \) is a Sylow \( p \)-subgroup of \( G \) (see [T; F, VI, 6]). Koshitani [Ko] showed that if the vertex \( v_x(V) \) of \( V \) is contained in \( \text{Ker} \; V \) (the kernel of \( V \)) for any simple \( k[G] \)-module \( V \) then \( G \) is \( p \)-radical.

We generalize this to a \( p \)-block form. Let \( B \) be a \( p \)-block of \( G \) and \( e_B \) be the block idempotent in \( k[G] \) corresponding to \( B \). We call \( B \) \( p \)-radical if \( (k_P)^G \cdot e_B \) is a semisimple \( k[G] \)-module following Tsushima [T].

Theorem. If the vertex \( v_x(V) \) of \( V \) is contained in \( \text{Ker} \; V \) for any simple \( k[G] \)-module \( V \) in a \( p \)-block \( B \) of \( G \), then \( B \) is \( p \)-radical.

Throughout this paper we keep the notation as in the theorem. See the book of Feit [F] for the notion of vertices and other terminology.

Lemma 1. Assume that \( v_x(V) \subseteq \text{Ker} \; V \) for any simple \( k[G] \)-module \( V \) in \( B \).

(a) [HM, Lemma 1.3; Ko, Lemma 1]. If \( H \) is a normal \( p \)- or \( p' \)-subgroup of \( G \) and \( B \) is a \( p \)-block of \( G/H \) such that \( B = B \), then \( v_x(V) \subseteq \text{Ker} \; V \) for any simple \( k[G/H] \)-module \( V \) in \( B \).

(b) [Kn2, 3.7 Corollary; Ko, Lemma 2]. If \( B \) is the principal block of \( G \), then \( G \) is \( p \)-solvable.

Lemma 2. Let \( H \) be a normal subgroup of \( G \) containing a defect group of \( B \). If every block of \( H \) covered by \( B \) is \( p \)-radical then \( B \) is \( p \)-radical.

Proof. By [Kn1, Theorem 2.9] it suffices to show that \( (k_P)^G \cdot e_B \) is semisimple as a \( k[H] \)-module. Let \( \{b_i\} \) be the set of all blocks of \( H \) covered by \( B \). By Mackey decomposition \( (k_P)^G \cdot e_B \) is a direct summand of

\[
\bigoplus_{i} \bigoplus_{t \in P \setminus G/H} (k_P \cap H)^H \cdot e_{b_i}
\]
as a $k[H]$-module. Since $P^t \cap H$ is a Sylow $p$-subgroup of $H$ for all $t$ and $b_i$ is $p$-radical for all $i$, $(k_F)^G \cdot e_B$ is a semisimple $k[H]$-module.

**Proof of the theorem.** First, assume that $B$ is not the principal block. There is a simple $k[G]$-module $V$ in $B$ such that $\text{vx}(V)$ is equal to the defect group of $B$. Let $K = \text{Ker} V$, and let $b$ be any block of $K$ covered by $B$. For any simple $k[K]$-module $W$ in $b$, there is a simple $k[G]$-module $U$ in $B$ such that $W$ is isomorphic to a direct summand of $U$ as a $k[K]$-module. We may assume $\text{vx}(W) \subseteq \text{vx}(U)$. Hence, $\text{vx}(W) \subseteq \text{vx}(U) \cap K \subseteq \text{Ker} U \cap K \subseteq \text{Ker} W$. Since $K \neq G$, $b$ is $p$-radical by induction on $|G|$. Hence $B$ is $p$-radical by Lemma 2. So we may assume that $B$ is the principal block. By Lemma 1(b), $G$ is $p$-solvable. If $0_p, (G) = 1$, then $B$ is the unique block of $G$ by Fong's Theorem [F, X Theorem 1.5]. By Lemma 1(a) and induction, $G/0_p(G)$ is $p$-radical. Hence $G$ is $p$-radical by [T, Proposition 1]. Let $H = 0_p, (G)$. If $H \neq 1$ then $G/H$ is $p$-radical by Lemma 1(a) and induction. Hence $B$ is $p$-radical since $(k_F)^G \cdot e_B \cong (k_H)^G$.

**Corollary** [Ko, Theorem]. If the vertex of $V$ is contained in $\text{Ker} V$ for any simple $k[G]$-module $V$, then $G$ is $p$-radical.

**ACKNOWLEDGMENTS**

I would like to thank Doctor S. Koshitani for suggesting this problem.

**REFERENCES**


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