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CENTRALIZERS OF EXPANDING MAPS ON THE CIRCLE

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ABSTRACT. We prove here that the elements of an open and dense subset of expanding maps on the circle have trivial centralizers; i.e., the maps commute only with their own powers. Using a theorem proved in [1], we deduce that the result is also true for an open and dense subset of immersions of S^1.

1. INTRODUCTION

Let $\mathrm{Imm}^n(S^1)$ be the space of C^n immersions of the circle S^1 (i.e., mappings f for which $f'(x) \neq 0$) endowed with the C^n topology. An immersion $f\colon S^1 \to S^1$ is *expanding* iff $|f'(x)| > 1$ for all $x \in S^1$. Let $\mathrm{Exp}^n(S^1)$ denote the set of C^n expanding maps of S^1.

We continue here to study, initiated in [1], of centralizers of immersions of the circle. Recall that for $f \in \mathrm{Imm}^n(S^1)$, its *centralizer* $Z(f)$ in $\mathrm{Imm}^n(S^1)$ is defined as the set of elements that commute with f. We say that f has *trivial centralizer* if $Z(f)$ is reduced to the iterates $\{f^n, n \in \mathbf{N}\}$ of f.

The purpose of this paper is to prove the following result.

Theorem. *For an open and dense subset of* $\mathrm{Exp}^n(S^1)$ $(n > 1)$, *the centralizer is trivial.*

It follows from [1] that a similar result is true for an open and dense subset of $\mathrm{Imm}^\infty(S^1) - \overline{\mathrm{Exp}^\infty(S^1)}$. Hence we have proved

Corollary. *There is an open and dense subset of* $\mathrm{Imm}^\infty(S^1)$ *whose elements have trivial centralizers.*

This result is an extension to immersions of a theorem of Kopell [2, Theorem 3], who showed the triviality of the centralizer for an open and dense subset of diffeomorphisms of the circle.

A fundamental tool for the proof of the theorem is [1, Lemma 2.1] which is an extension to expanding maps on the circle of a result of Kopell [2, Theorem 6].

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2. Proof of the theorem

We begin by recalling some basic concepts and establishing preliminary results.

Formally, we will think of the circle as $\mathbf{R}/\mathbf{Z}$ and use π to denote the canonical projection. Thus every continuous map f of the circle has countably many *lifts*, i.e., continuous maps $F: S^1 \to S^1$ satisfying

$$f \circ \pi = \pi \circ F.$$

Any two such lifts differ by an integer, and the unique integer d satisfying

$$F(x+1) = F(x) + d$$

for all lifts F and all x is called the degree of f.

Let f_d denote the immersion of S^1 given by $f_d(z) = z^d$.

Lemma 2.1. *Let f be an expanding map of the circle, and suppose that $f'(x) \neq f'(y)$ for all different fixed points of f. Then*

(a) *If $g \in Z(f)$, then g fixes the fixed points of f.*
(b) *Let $h: S^1 \to S^1$ be a homeomorphism of the circle such that $h \circ f = f_n \circ h$, where $n = \text{degree}\, f$. If $g \in Z(f)$, then $h \circ g = f_m \circ h$, where $m = \text{degree}\, g$.*

Proof. (a) Let x be a fixed point of f. Then

$$f \circ g(x) = g \circ f(x) = g(x),$$

and

$$f'(g(x)) = f'(x).$$

These properties and the hypothesis imply that $g(x) = x$.

(b) We have that

$$f_n \circ (h \circ g \circ h^{-1}) = h \circ f \circ g \circ h^{-1} = h \circ g \circ f \circ h^{-1} = (h \circ g \circ h^{-1}) \circ f_n.$$

Hence, by [1, Lemma 2.1], $h \circ g \circ h^{-1} = \omega f_m$, where ω is an $(n-1)$th root of unity and $m = \text{degree}\, g$. By (a), $h^{-1}(1)$ is a fixed point of g. Thus

$$1 = h \circ g \circ h^{-1}(1) = \omega f_m(1) = \omega.$$

Therefore, $h \circ g = f_m \circ h$, and the lemma is proved.

We will use the following simple fact.

Lemma 2.2. *Let $f: S^1 \to S^1$ be a C^n endomorphism of the circle, and let $\alpha: S^1 \to S^1$ be a C^n diffeomorphism. If $Z(\alpha^{-1} \circ f \circ \alpha)$ is trivial, then $Z(f)$ is trivial.*

Proof. Let $g \in Z(f)$. Then $\alpha^{-1} \circ g \circ \alpha \in Z(\alpha^{-1} \circ f \circ \alpha)$. Hence, by hypothesis, $\alpha^{-1} \circ g \circ \alpha = \alpha^{-1} \circ f^k \circ \alpha$ for some $k \in \mathbf{N}$. Thus, $g = f^k$ and so $Z(f)$ is trivial.

We are now ready to prove the theorem. Let

$$\mathscr{U} = \{f \in \text{Exp}^n(S^1) : f'(x) \neq f'(y) \text{ for different fixed points } x, y \text{ of } f\}.$$

This set is clearly an open and dense subset of $\operatorname{Exp}^n(S^1)$. Hence, it suffices to show that the elements of $\mathscr{U}$ have trivial centralizers. Let $f \in \mathscr{U}$. By a result of Shub [3], there exists an order-preserving homeomorphism $h\colon S^1 \to S^1$ such that

$$h \circ f = f_n \circ h, \quad \text{where } n = \text{degree } f.$$

By using a rotation of S^1, if necessary, we may assume by Lemma 2.2 that $f(1) = h(1) = 1$. Let $g \in Z(f)$. By Lemma 2.1(a), $g(1) = 1$. Hence, if F, G, H are the lifts of f, g, and h, respectively, then

$$F(0) = G(0) = H(0) = 0,$$
$$H \circ F(x) = nH(x) \quad \text{for all } x \in \mathbf{R},$$

and by Lemma 2.1(b), $H \circ G(x) = mH(x)$ where $m = \text{degree } g$. Let $l \in \mathbf{N}$ such that $n^l \le m \le n^{l+1}$. Then

$$m = n^l + p \quad \text{for some } 0 \le p \le n^l(n-1).$$

We claim that $G'(0) = (F'(0))^l$. In fact, let $t_k \in \mathbf{R}$ such that $H(t_k) = 1/n^k$. Let $\varepsilon > 0$ be given. Since H is an order-preserving homeomorphism, we can choose $k \in \mathbf{N}$ such that

$$0 \le H^{-1}\left(\frac{1}{n^k} + \frac{p}{n^{k+l}}\right) - t_k \le \varepsilon .$$

Since F^{-1} is a contraction, we have that

$$(*) \qquad \lim_{j\to\infty} F^{-j}\left(H^{-1}\left(\frac{1}{n^k} + \frac{p}{n^{k+l}}\right)\right) = 0$$

and

$$0 \le F^{-j}\left(H^{-1}\left(\frac{1}{n^k} + \frac{p}{n^{k+l}}\right)\right) - F^{-j}(t_k) \le \varepsilon$$

for all $j \in \mathbf{N}$. Thus

$$(**) \qquad 1 \le \frac{F^{-j}(H^{-1}(\frac{1}{n^k} + \frac{p}{n^{k+l}}))}{F^{-j}(t_k)} \le 1 + \varepsilon \quad \text{for all } j \in \mathbf{N}.$$

Let $y_i = F^{-j}(t_k)$. Then

$$\lim_{j\to\infty} y_j = 0$$

and

$$\frac{1}{n^k} = H(t_k) = H \circ F^j(y_j) = n^j H(y_j).$$

Hence, $H(y_j) = 1/n^{k+j}$ for all $j \in \mathbf{N}$. These properties, together with the fact that $F^{-1} \circ H^{-1}(nx) = H^{-1}(x)$ for all $x \in \mathbf{R}$, imply that

$$\begin{aligned}
G'(0) &= \lim_{j\to\infty} \frac{G(y_j) - G(0)}{y_j} = \lim_{j\to\infty} \frac{H^{-1}(mH(y_j))}{y_j} \\
&= \lim_{j\to\infty} \frac{H^{-1}(\frac{m}{n^{k+j}})}{y_j} = \lim_{j\to\infty} \frac{F^{-(j-l)} \circ H^{-1}(\frac{mn^{j-l}}{n^{k+j}})}{y_j} \\
&= \lim_{j\to\infty} \frac{F^{-(j-l)} \circ H^{-1}(\frac{n^j + pn^{j-l}}{n^{k+j}})}{y_j} \\
&= \lim_{j\to\infty} \frac{F^l(F^{-j}(H^{-1}(\frac{1}{n^k} + \frac{p}{n^{k+l}})))}{F^{-j}(H^{-1}(\frac{1}{n^k} + \frac{p}{n^{k+l}}))} \frac{F^{-j}(H^{-1}(\frac{1}{n^k} + \frac{p}{n^{k+l}}))}{F^{-j}(t_k)} .
\end{aligned}$$

But, by $(*)$,

$$\lim_{j\to\infty} \frac{F^l(F^{-j}(H^{-1}(\frac{1}{n^k}+\frac{p}{n^{k+l}})))}{F^{-j}(H^{-1}(\frac{1}{n^k}+\frac{p}{n^{k+l}}))} = (F^l)'(0).$$

So, by $(**)$,

$$(F^l)'(0) \le G'(0) \le (F^l)'(0) + \varepsilon.$$

Therefore $G'(0) = (F^l)'(0)$, and the claim is proved.

By Sternberg [4], there exists a C^{n-1}-diffeomorphism $\alpha\colon \mathbf{R}\to\mathbf{R}$ such that $\alpha(0)=0$ and $\alpha\circ F\circ\alpha^{-1}(x) = F'(0)x$ for all sufficiently small x. Let $L = \alpha\circ F\circ\alpha^{-1}$. Then $\alpha\circ G\circ\alpha^{-1}$ commutes with L. Thus, for all sufficiently small x, we have that

$$\begin{aligned}(\alpha\circ G\circ\alpha^{-1})(x)F'(0) &= (\alpha\circ G\circ\alpha^{-1})\left(L\left(\frac{x}{F'(0)}\right)\right)L'\left(\frac{x}{F'(0)}\right)\\ &= L'\left(\alpha\circ G\circ\alpha^{-1}\left(\frac{x}{F'(0)}\right)\right)(\alpha\circ G\circ\alpha^{-1})'\left(\frac{x}{F'(0)}\right)\\ &= F'(0)(\alpha\circ G\circ\alpha^{-1})'\left(\frac{x}{F'(0)}\right),\end{aligned}$$

so

$$(\alpha\circ G\circ\alpha^{-1})'(x) = (\alpha\circ G\alpha^{-1})'\left(\frac{x}{F'(0)}\right).$$

This and the fact that $|F'(0)|>1$ imply that

$$(\alpha\circ G\circ\alpha^{-1})'(x) = (\alpha\circ G\circ\alpha^{-1})'\left(\frac{x}{(F'(0))^q}\right) \quad \text{for all } q\in\mathbf{N}.$$

Since $(\alpha\circ G\circ\alpha^{-1})'$ is continuous, we have that

$$(\alpha\circ G\circ\alpha^{-1})'(x) = (\alpha\circ G\circ\alpha^{-1})'(0) = G'(0).$$

This and the claim above imply that, for all sufficiently small x,

$$\alpha\circ G\circ\alpha^{-1}(x) = G'(0)x = (F'(0))^l x = L^l(x),$$

so $G(x) = F^l(x)$ for all sufficiently small x. It follows from this that there exists an open interval I in S^1 such that $g(x) = f^l(x)$ for all $x\in I$. Since f is expanding, for all $y\in S^1$ there exist $x\in I$ and $q\in\mathbf{N}$ such that $f^q(x)=y$. Thus

$$g(y) = g\circ f^q(x) = f^q\circ g(x) = f^q\circ f^l(x) = f^l(y).$$

Therefore $g = f^l$, and the theorem is proved.

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