

## REGULAR PI METRIC FLOWS ARE EQUICONTINUOUS

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**ABSTRACT.** Let  $(X, T)$  be a metrizable minimal flow. We show that a homomorphism  $X \xrightarrow{\pi} Y$ , which is regular, and PI can be decomposed as  $X \xrightarrow{\sigma} Z \xrightarrow{\rho} Y$ ,  $\pi = \rho \circ \sigma$ , where  $\rho$  is proximal and  $\sigma$  is a compact group extension. In particular, assuming further that  $T$  is abelian and taking  $Y$  to be the trivial one point flow, we find that a metric regular PI flow is a compact group rotation.

### 1. INTRODUCTION

By a theorem of W. H. Gottchalk [Go], a metrizable minimal distal regular flow  $(X, T)$ , where  $T$  is abelian, is necessarily a compact group rotation. This was generalized by J. Auslander in [A] to a general group  $T$  and by the author [G1], to point distal regular homomorphisms. Here we prove the following theorem:

**Theorem 1.** *Let  $(X, T)$  be a metric minimal flow,  $X \xrightarrow{\pi} Y$  a regular PI homomorphism. Then  $\pi = \mu \circ \nu$ , where  $X \xrightarrow{\nu} Z$  is a group homomorphism and  $Z \xrightarrow{\mu} Y$  is proximal. If in addition,  $\pi$  is RIC, then  $\pi$  is a group homomorphism.*

Taking  $Y$  as the trivial flow, we get

**Corollary 1.** *If  $(X, T)$  is a minimal regular PI metric flow, then  $(X, T)$  is a group extension of a proximal flow. If  $(X, T)$  is incontractible (as is always the case when  $T$  is abelian), then  $(X, T)$  is a compact group rotation.*

Theorem 4.7 of [G1], which concerns point distal regular open homomorphisms, is a corollary of Theorem 1. (Although in the proof of Theorem 1 we use Proposition 3.1 of [G1], which is about a distal regular homomorphism being a group homomorphism.) However, Theorem 4.5 of [G1], which states that a nontrivial factor of almost simple flow is up to almost 1-1 extension a group homomorphism, cannot be deduced from Theorem 1 because it is not known whether an almost simple flow is almost 1-1 equivalent to a metrizable regular flow.

Examples of regular PI (nonmetrizable) flows that are not equicontinuous are easy to find. One is the enveloping semigroup of the flow on the 2-torus  $\mathbb{R}^2/\mathbb{Z}^2$

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given by  $T(x, y) = (x + \alpha, x + y)$  [F, N]; another is the “two circle” minimal set [E].

In §2 we briefly review the various definitions of the notions mentioned above and describe the algebraic notations and techniques that are used in proving Theorem 1. We also prove some basic lemmas including the crucial Lemma 3 on the topological isomorphism of certain automorphism groups. Surprisingly, the proof of this lemma requires theorems of Souslin and Banach about measurable maps. Section 3 is devoted to the proof of Theorem 2, which asserts the regularity of each of the components of the canonical PI-tower corresponding to a regular homomorphism  $X \xrightarrow{\pi} Y$  over the base  $Y$ .

The last section uses Theorem 2 to reduce the proof of Theorem 1 to the case of a regular distal homomorphism, which by [G1], is necessarily a group homomorphism. Using essentially the same proof, we get Theorem 3, which says that when  $X$  is metric and  $X \xrightarrow{\pi} Y$  is RIC and regular, then  $\pi = \kappa \circ \omega$ ,  $X \xrightarrow{\omega} Z \xrightarrow{\kappa} Y$ , where  $Z \xrightarrow{\kappa} Y$  is the largest almost periodic extension (actually a group extension) of  $Y$  under  $\pi$ , and  $x \xrightarrow{\omega} Z$  is RIC and weakly mixing.

2. DEFINITIONS, A REVIEW OF SOME THEOREMS, AND SOME BASIC LEMMAS

We refer to [E, V, A, G2] for more details on the algebraic theory of minimal flows; we now describe some of its aspects. Assume  $T$  to be a discrete group, and as usual, let  $\beta T$  denote the Stone-Ćech compactification of  $T$ . Let  $M$  be a fixed minimal ideal in  $\beta T$  and  $J \subset M$  the set of idempotents in  $M$ . We fix an element  $u \in J$ , and let  $G$  be the group  $uM$ .

$\beta T$  acts on every flow  $(X, T)$  and the orbit closure of a point  $x \in X$  is given by  $\bar{o}(x) = \{px : p \in \beta T\}$ . A point  $x \in X$  is almost periodic (a.p.) if  $\bar{o}(x)$  is minimal, iff  $\bar{o}(x) = \{px : p \in M\}$ .

A flow  $(X, T)$  satisfies the Bronstein condition (B.c.) if the a.p. points are dense in  $X$ . Given  $X \xrightarrow{\pi} Y$ , a homomorphism of minimal flows, we let

$$R_{\pi} = \{(x, x') \in X \times X : \pi(x) = \pi(x')\}.$$

The homomorphism  $\pi$  satisfies Bc if  $R_{\pi}$  is a Bc flow. Let  $\Gamma_{\pi} = \{\psi \in \text{Aut}(X, T) : \pi \circ \psi = \pi\}$ .

Here  $\text{Aut}(X, T)$  is the group of all self-homeomorphisms of  $X$  such that  $\psi \circ t = t \circ \psi \forall t \in T$ .

We provide  $\text{Aut}(X, T)$  with the topology of uniform convergence of homomorphisms and their inverses. When  $X$  is metric,  $\text{Aut}(X, T)$  is a polish topological group.

A pointed flow is a flow  $(X, T)$  with a distinguished base point  $x_0 \in X$ . We adopt the convention of always choosing a base point  $x_0$  satisfying  $ux_0 = x_0$ .

When  $(X, x_0)$  is a pointed flow its Ellis group is the subgroup  $\mathcal{G}(X) = \mathcal{G}(X, x_0) = \{\alpha \in G : \alpha x_0 = x_0\}$  of  $G$ .

A homomorphism  $(X, x_0) \xrightarrow{\pi} (Y, y_0)$  (this means  $\pi x_0 = y_0$ ) is proximal (i.e.  $\pi x = \pi x' \Rightarrow x$  and  $x'$  are proximal) iff  $\mathcal{G}(X, x_0) = \mathcal{G}(Y, y_0)$ .

For the definition of the  $\tau$ -topology on  $G$ , we refer to e.g. [G2]. If  $F$  is a  $\tau$ -closed subgroup of  $G$ , then

$$F' = \bigcap \{\tau - \text{clr}(O) : O \text{ is a } \tau - \text{ncd of } u \text{ in } F\}.$$

$F'$  is a  $\tau$ -closed subgroup of  $F$ , which is invariant under all  $\tau$ -continuous automorphisms of  $F$ ; in particular  $F' \triangleleft F$ ,  $F'' \triangleleft F$  etc.  $F/F'$  with the quo-

tient  $\tau$ -topology is a compact Hausdorff topological group. A homomorphism  $(X, x_0) \xrightarrow{\pi} (Y, y_0)$  of minimal flows is almost periodic iff it is *distal* (i.e.  $\pi x = \pi x', x \neq x' \Rightarrow x, x'$  are not proximal) and  $F' \subset A$ , where  $A = \mathcal{G}(X, x_0)$ ,  $F = \mathcal{G}(Y, y_0)$ . An almost periodic homomorphism is a *group homomorphism* if in addition  $A \triangleleft F$ , and in that case, the topological groups  $\Gamma_\pi$  and  $F/A$  are isomorphic and  $(Y, T) \cong (X/\Gamma_\pi, T)$ .

A *quasifactor* of a minimal flow  $(X, T)$  is a minimal subflow of the flow  $(2^X, T)$  induced on the space  $2^X$  of closed subsets of  $X$  by  $T$ . For  $p \in \beta T$ ,  $K \in 2^X$  the action of  $p$  on  $K$  is written as  $p \circ K$ . This is to distinguish it from the subset  $pK = \{px : x \in K\}$  of  $X$ , which is usually not even closed. We have, however,  $pK \subset p \circ K$ . Given a homomorphism  $(X, x_0) \xrightarrow{\pi} (Y, y_0)$  of minimal sets with  $\mathcal{G}(X, x_0) = A \subset F = \mathcal{G}(Y, y_0)$ , we let  $\tilde{Y} = \{p \circ Fx_0 : p \in M\}$ . This is a quasifactor of  $X$ , and we say that  $\pi$  is a *RIC-homomorphism* (or extension) if the sets  $p \circ Fx_0$  are exactly the fibers of the map  $\pi$ ; i.e.  $\pi^{-1}(py_0) = p \circ Fx_0 \forall p \in M$ . In any case  $\theta : p \circ Fx_0 \mapsto py_0$  ( $p \in M$ ) is a homomorphism of  $\tilde{Y}$  onto  $Y$ , and it is an isomorphism iff  $\pi$  is RIC. A RIC homomorphism is open and satisfies *Bc*. Let  $\tilde{y}_0 = u \circ Fx_0$  be the base point of  $\tilde{Y}$ , and let  $\tilde{X} = X \vee \tilde{Y} = \bar{o}(x_0, \tilde{y}_0) \subset X \times \tilde{Y}$ . Then  $\tilde{X} = \{(x, \tilde{y}) : x \in \tilde{y}\} = \{(px_0, p \circ Fx_0) : p \in M\}$ . If we let  $\hat{\theta}(x, \tilde{y}) = \tilde{y}$  and  $\tilde{\pi}(x, \tilde{y}) = x$ , then in the *shadow diagram*

$$\begin{array}{ccc} X & \xleftarrow{\hat{\theta}} & \tilde{X} \\ \pi \downarrow & & \downarrow \tilde{\pi} \\ Y & \xleftarrow{\theta} & Y \end{array}$$

the maps  $\theta, \hat{\theta}$  are proximal and  $\tilde{\pi}$  is RIC. When  $X \xrightarrow{\pi} Y$  is RIC, a commutative diagram

$$\begin{array}{ccc} X & & \\ & \searrow \sigma & \\ \pi \downarrow & & Z \\ & \swarrow \rho & \\ Y & & \end{array}$$

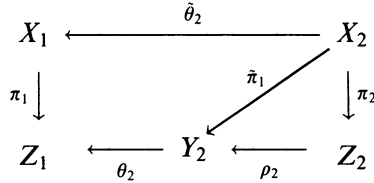
can be constructed such that  $\rho$  is almost periodic and  $\mathcal{G}(Z, z_0) = F'A$ .

The latter condition means that  $Z$  is the largest almost periodic extension of  $Y$  that is a factor of  $X$ . In particular, when  $X$  is metric,  $\pi$  is *weakly mixing* (i.e.  $R_\pi$  is topologically ergodic) iff  $\rho$  is trivial.

The construction of the shadow diagram producing RIC-homomorphisms and the construction of the maximal almost periodic extension within a RIC-homomorphism are the basic blocks of the *canonical PI-tower* associated with a homomorphism  $X \xrightarrow{\pi} Y$ . Thus the first stage of this tower is the diagram

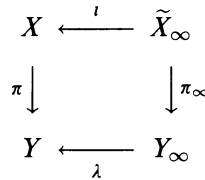
$$\begin{array}{ccccc} X & \xleftarrow{\hat{\theta}_1} & & & X_1 \\ \pi \downarrow & & & \swarrow \tilde{\pi} & \downarrow \pi_1 \\ Y & \xleftarrow{\theta_1} & Y_1 & \xleftarrow{\rho_1} & Z_1 \end{array}$$

where  $Y_1 = \tilde{Y}$ ,  $X_1 = \tilde{X}$ , and  $Z_1 \xrightarrow{\rho_1} Y_1$  is the largest almost periodic extension of  $Y_1$  under the RIC extension  $\tilde{\pi}$ . The next stage is the diagram



where  $\tilde{\pi}_1$  is the RIC shadow of  $\pi_1$  and  $\rho_2$  the largest almost periodic extension of  $Y_2$  under  $\tilde{\pi}_1$  etc.

If necessary, one utilizes inverse limits to construct inductively flows  $\{Y_\nu, Z_\nu, X_\nu : \nu \leq \eta\}$ , where  $\eta$  is the first nonlimit ordinal for which  $Z_\eta = Y_\eta$ . The top of the tower consists of the flows  $X_\infty = X_\eta, Y_\infty = Y_\eta = Z_\eta$ , where the diagram



is achieved. Here  $\pi_\infty$  is RIC and weakly mixing,  $Y_\infty \xrightarrow{\lambda} Y$  is a *strictly* PI homomorphism (i.e. can be decomposed into possibly transfinite sequence of alternating proximal and a.p. homomorphism), and  $\iota$  is a proximal extension. When  $X$  is metrizable, each flow in the PI-tower is metrizable and the ordinal  $\eta$  is countable.

We say that  $\pi$  is a *PI homomorphism* if  $\pi_\infty$  is an isomorphism. Though we are not going to use it, we mention the following characterization of PI homomorphism due to I. U. Bronstein (see [A]). When  $X$  is metric,  $X \xrightarrow{\pi} Y$  is PI iff every closed invariant subset of  $R_\pi$  that is both topologically ergodic and satisfies *Bc* is minimal.

A homomorphism  $X \xrightarrow{\pi} Y$  of minimal flows is called *regular* if whenever  $(x, x') \in R_\pi$  is an a.p. point, there exists an automorphism  $\psi \in \Gamma_\pi$  with  $x' = \psi(x)$ .  $X$  is called *regular* if this condition holds for every a.p.  $(x, x') \in X \times X$ .

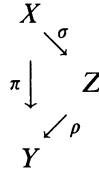
**Lemma 1.** *Let  $(X, x_0) \xrightarrow{\pi} (Y, y_0)$  be a regular homomorphism of minimal flows. Let  $A = \mathcal{G}(X, x_0)$ ,  $F = \mathcal{G}(Y, y_0)$ , then  $A \triangleleft F$  and there exists a natural algebraic homomorphism  $\gamma \rightarrow \psi_\gamma$  of  $F$  onto  $\Gamma_\pi$  with kernel  $A$ . Thus  $\Gamma_\pi$  is algebraically isomorphic to  $F/A$ .*

*Proof.* Let  $\gamma \in F$ ; then  $(x_0, \gamma x_0)$  is an almost periodic point in  $R_\pi$ . By the regularity of  $\pi$ ,  $\exists \psi_\gamma \in \Gamma_\pi$  for which  $\psi_\gamma^{-1}(x_0) = \gamma x_0$ . For  $p \in M$  we have

$$\psi_\gamma^{-1}(px_0) = p\psi_\gamma^{-1}(x_0) = p\gamma x_0.$$

If  $\delta \in F$ , then  $\psi_{\gamma\delta}^{-1}(px_0) = p\gamma\delta x_0 = \psi_\delta^{-1}(p\gamma x_0) = \psi_\delta^{-1} \circ \psi_\gamma^{-1}(px_0) = (\psi_\gamma \circ \psi_\delta)^{-1}(px_0)$  and  $\gamma \mapsto \psi_\gamma$  is a homomorphism. Clearly the kernel of this homomorphism is  $A$ . Finally if  $\psi \in \Gamma_\pi$  then  $u\psi(x_0) = \psi(ux_0) = \psi(x_0)$ , and therefore,  $\psi(x_0) = \gamma^{-1}x_0$  for some  $\gamma \in G$ . Since  $\gamma^{-1}y_0 = \gamma^{-1}\pi(x_0) = \pi(\gamma^{-1}x_0) = \pi \circ \psi(x_0) = \pi(x_0) = y_0$ , we have  $\gamma \in F$  and  $\psi = \psi_\gamma$ .  $\square$

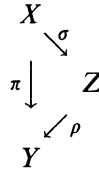
**Lemma 2.** *In the diagram*



let  $\rho$  be proximal. Then  $\Gamma_\pi = \Gamma_\sigma$ .

*Proof.* Suppose  $\psi \in \Gamma_\sigma$  i.e.  $\sigma \circ \psi = \sigma$ . Then  $\pi \circ \psi = \rho \circ \sigma \circ \psi = \rho \circ \sigma = \pi$  and  $\psi \in \Gamma_\pi$ . Thus  $\Gamma_\sigma \subset \Gamma_\pi$ . On the other hand, if  $\psi \in \Gamma_\pi$ , then for  $x \in X$ ,  $\rho(\sigma(\psi(x))) = \pi \circ \psi(x) = \pi x = \rho(\sigma x)$ , and we conclude that  $\sigma(\psi(x))$  and  $\sigma(x)$  are proximal. However,  $(x, \psi(x))$  is an a.p. point and therefore, so is  $(\sigma x, \sigma(\psi(x)))$ . This can happen only when  $\sigma x = \sigma(\psi x)$ , and since  $x$  is an arbitrary point of  $X$ , we have  $\sigma = \sigma \circ \psi$ , so that  $\psi \in \Gamma_\sigma$  and  $\Gamma_\pi \subset \Gamma_\sigma$ .  $\square$

**Lemma 3.** *In the diagram*



suppose  $\pi$  and  $\rho$  are regular,  $\sigma$  is proximal, and  $X$  is metrizable. Then  $\Gamma_\pi$  and  $\Gamma_\rho$  are topologically isomorphic.

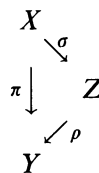
*Proof.* Since  $\sigma$  is proximal  $\mathcal{G}(X, x_0) = \mathcal{G}(Z, z_0) = A$ . Thus by Lemma 1,  $\Gamma_\pi$ ,  $\Gamma_\rho$ , and  $F/A$  where  $F = \mathcal{G}(Y, y_0)$ , are all algebraically isomorphic. Moreover, it is clear that the canonical isomorphism  $J: \Gamma_\pi \rightarrow \Gamma_\rho$  is continuous. Since  $\Gamma_\pi$  and  $\Gamma_\rho$  are polish groups and  $J$  is 1-1, by a theorem of Souslin [K, Vol. I, p. 487],  $J$  is a Borel isomorphism. In particular,  $J^{-1}: \Gamma_\rho \rightarrow \Gamma_\pi$  is a Borel measurable homomorphism of polish groups. By a theorem of Banach [B], such a homomorphism is continuous, and we conclude that  $J$  is a topological isomorphism as claimed.  $\square$

### 3. THE PI-TOWER FOR A REGULAR HOMOMORPHISM

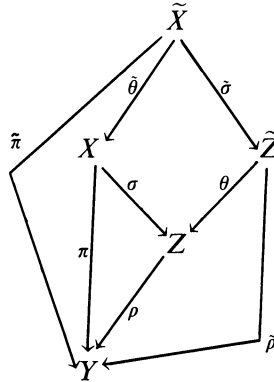
**Theorem 2.** *Let  $X \xrightarrow{\pi} Y$  be a regular homomorphism of minimal flows; then each of the homomorphisms onto  $Y$  of the flows  $X_\nu, Y_\nu, Z_\nu$  ( $\nu \leq \eta$ ) constructed in the canonical PI-tower for  $\pi$  is regular.*

The proof is a consequence of the following three lemmas.

**Lemma 4.** *Let*



be a diagram where  $\pi$  and  $\rho$  are regular, and let



be the same diagram with the attached RIC shadow diagram for  $\pi$ . Then the maps  $\tilde{\pi}$  and  $\tilde{\rho}$  are regular.

*Proof.* Denote  $A = \mathcal{G}(X, x_0)$ ,  $F = \mathcal{G}(Y, y_0)$ , and  $B = \mathcal{G}(Z, z_0)$ , then Lemma 1 implies  $A \triangleleft F$  and  $B \triangleleft F$ . We recall the definitions  $\tilde{Z} = \{p \circ Bx_0 : p \in M\}$ ,  $\tilde{z}_0 = u \circ Bx_0$ , and  $\tilde{X} = \{p(x_0, u \circ Bx_0) : p \in M\} = \{(x, \tilde{z}) : x \in \tilde{z} \in \tilde{Z}\}$ . Given  $\psi \in \Gamma_\pi$ , we define a map  $\Psi: \tilde{Z} \rightarrow 2^X$  by  $\Psi(p \circ Bx_0) := \psi(p \circ Bx_0) = p \circ B\psi x_0$ . If  $\psi(x_0) = \gamma x_0$  for  $\gamma \in G$ , then

$$\gamma y_0 = \gamma \pi(x_0) = \pi(\gamma x_0) = \pi \circ \psi(x_0) = \pi(x_0) = y_0$$

so that  $\gamma \in F$ . Now  $\Psi(p \circ Bx_0) = p \circ B\psi(x_0) = p \circ \gamma Bx_0 = p\gamma \circ Bx_0 \in \tilde{Z}$ , and  $\Psi$  maps  $\tilde{Z}$  into itself. Moreover, if also  $\psi(x_0) = \delta x_0$  for  $\delta \in F$ , then  $\gamma x_0 = \delta x_0$  implies  $\gamma^{-1}\delta \in A \subset B$  and  $\delta B = \gamma B$ . Thus for every  $p \in M$ ,

$$p\gamma \circ Bx_0 = p \circ \gamma Bx_0 = p \circ \delta Bx_0 = p\delta \circ Bx_0,$$

and  $\Psi$  is well defined. It is also clear now that  $\Psi$  is a continuous automorphism of  $(\tilde{Z}, T)$  where the inverse  $\Psi^{-1}$  is induced by  $\psi^{-1}$ . We now define a map  $\tilde{\psi}: X \times \tilde{Z} \rightarrow X \times \tilde{Z}$  by  $\tilde{\psi}(x, \tilde{z}) = (\psi(x), \Psi(\tilde{z}))$ . Since for  $p \in M$

$$\tilde{\psi}(p(x_0, \tilde{z}_0)) = p(\psi(x_0), \Psi(u \circ Bx_0)) = p(\gamma x_0, \gamma \circ Bx_0) = p\gamma(x_0, \tilde{z}_0) \in \tilde{X},$$

we see that  $\tilde{\psi}$  is an automorphism of  $(\tilde{X}, T)$ .

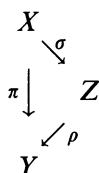
Now let  $((x, \tilde{z}), (x', \tilde{z}'))$  be an a.p. point of  $R_{\tilde{\pi}}$ . Then there exist an idempotent  $v \in J$  and  $\delta, \zeta \in G$  such that

$$(x, \tilde{z}) = v\zeta(x_0, \tilde{z}_0), \quad (x', \tilde{z}') = v\delta(x_0, \tilde{z}_0).$$

Clearly  $\zeta y_0 = \delta y_0$  and therefore,  $\zeta^{-1}\delta \in F$ . Now  $(x, x')$  is an a.p. point of  $R_\pi$  and by regularity of  $\pi$ , a  $\psi \in \Gamma_\pi$  exists for which  $x' = \psi(x)$ . Let  $\xi \in F$  be such that  $\psi(x_0) = \xi x_0$ ; then  $\Psi(v\zeta \circ Bx_0) = v\zeta\xi \circ Bx_0$  and  $v\delta x_0 = x' = \psi(x) = \psi(v\zeta x_0) = v\zeta\psi(x_0) = v\zeta\xi x_0$ , implies  $\delta^{-1}\zeta\xi \in A \subset B$ . Thus  $\delta^{-1}\zeta\xi \circ Bx_0 = u \circ Bx_0$ , hence  $\zeta\xi \circ Bx_0 = \delta \circ Bx_0$  and  $\Psi(\tilde{z}) = \psi(v\zeta \circ Bx_0) = v\zeta\xi \circ Bx_0 = v\delta \circ Bx_0 = \tilde{z}'$ . It follows that  $\tilde{\psi}(x, \tilde{z}) = (x', \tilde{z}')$  and  $\tilde{\pi}$  is regular.

Finally if  $(\tilde{z}, \tilde{z}') = (v\zeta\tilde{z}_0, v\delta\tilde{z}_0)$  is an a.p. point in  $R_{\tilde{\rho}}$ , then  $((x, \tilde{z}), (x', \tilde{z}')) = ((v\zeta x_0, v\zeta\tilde{z}_0), (v\delta x_0, v\delta\tilde{z}_0))$  is an a.p. point in  $R_\pi$  and  $\tilde{z}' = \Psi(\tilde{z})$  where  $\Psi \in \Gamma_{\tilde{\rho}}$  is constructed as above. This proves that also  $\tilde{\rho}$  is regular.  $\square$

**Lemma 5.** *In the diagram*



suppose  $\rho$  is regular and  $\sigma$  is distal. If  $\mathcal{G}(X, x_0) = A \triangleleft F = \mathcal{G}(Y, y_0)$ , then  $\pi$  is regular

*Proof.* Given  $\gamma \in F$ , by Lemma 1 there exists an automorphism  $\psi_\gamma \in \Gamma_\rho$  defined by  $\psi_\gamma(pz_0) = p\gamma z_0$  ( $p \in M$ ). Define  $\psi_\gamma \in \Gamma_\pi$  by  $\psi_\gamma(px_0) = p\gamma z_0$ . We first show that  $\psi_\gamma$  is well defined on  $X$ . Suppose  $px_0 = qx_0$ ,  $p, q \in M$ . Then a fortiori  $pz_0 = qz_0$  and  $\psi_\gamma(pz_0) = p\gamma z_0$ ,  $\psi_\gamma(qz_0) = q\gamma z_0$  implies  $p\gamma z_0 = q\gamma z_0$ . Hence

$$\sigma(q\gamma x_0) = q\gamma\sigma(x_0) = q\gamma z_0 = p\gamma z_0 = p\gamma\sigma(x_0) = \sigma(p\gamma x_0).$$

If  $v \in J$  satisfies  $vp = p$ , then

$$v\sigma(p\gamma x_0) = vp\gamma z_0 = p\gamma z_0 = \sigma(p\gamma x_0),$$

and the distality of  $\sigma$  implies  $vq\gamma x_0 = q\gamma x_0$  as well.

Denote  $\alpha = (up)^{-1}(uq)$ , then  $\alpha \in A$ . Since  $A \triangleleft F$ ,  $\gamma A = A\gamma$ , and there exists  $\alpha' \in A$  with  $\gamma\alpha' = \alpha\gamma$ . Thus  $p\gamma x_0 = vp\gamma\alpha'x_0 = v(up)\alpha\gamma x_0 = v(uq)\gamma x_0 = q\gamma x_0$  and  $\psi_\gamma$  is well defined on  $X$ . It is clearly an element of  $\Gamma_\pi$ , and if  $(v\zeta x_0, v\xi x_0)$  is an a.p. point of  $R_\pi$ , then  $\gamma = \zeta^{-1}\xi \in F$  and  $\psi_\gamma(v\zeta x_0) = v\zeta\gamma x_0 = v\xi x_0$  so that  $\pi$  is regular.  $\square$

**Lemma 6.** *If  $(X_\nu, x_\nu) \xrightarrow{\pi_\nu} (Y, y_0)$  is an inverse system of regular homomorphisms of pointed minimal flows, then  $\varprojlim X_\nu = X \xrightarrow{\pi} Y$  is also regular.*

*Proof.* If  $\gamma \in F = \mathcal{G}(Y, y_0)$ , then by Lemma 1  $\psi_\gamma(p x_\nu) = p\gamma x_\nu$  ( $p \in M$ ) defines an automorphism on each  $X_\nu$ . This and the definition of the inverse limit automatically define  $\psi_\gamma$  on  $X$  as well. If  $(x, x') \in R_\pi$  is an almost periodic point, then for some  $v \in J$ ,  $\zeta, \xi \in F$ ,  $x = v\zeta x_0$ ,  $x' = v\xi x_0$  where  $x_0 \in X$  is the point corresponding to the points  $x_\nu \in X_\nu$ . If we let  $\gamma = \zeta^{-1}\xi \in F$ , then clearly  $\psi_\gamma(x) = x'$ .  $\square$

The proof of Theorem 2 follows by transfinite induction on the canonical PI-tower for  $X \xrightarrow{\pi} Y$ , using Lemmas 4, 5, and 6. For the use of Lemma 5, we recall that  $F^\nu \triangleleft F$  for all  $\nu$ , where  $F^{\nu+1} = (F^\nu)'$ , and that therefore, also  $\mathcal{G}(Z_\nu, z_\nu) = F^\nu A \triangleleft F$ .  $\square$

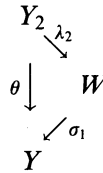
#### 4. A PROOF OF THEOREM 1

We are given a metric minimal flow  $X$  and a regular PI homomorphism  $X \xrightarrow{\pi} Y$ . Thus in the canonical PI-tower for  $\pi$ ,  $X_\infty = Y_\infty$ . In the notations of §2, consider the chain

$$Z_2 \xrightarrow{\rho_2} Y_2 \xrightarrow{\theta_2} Z_1 \xrightarrow{\rho_1} Y_1 \xrightarrow{\theta_1} Y.$$

We have  $\mathcal{G}(Y) = \mathcal{G}(Y_1) = F$ ,  $\mathcal{G}(Z_1) = \mathcal{G}(Y_2) = F'A$ , and  $\mathcal{G}(Z_2) = F''A$ , where by Lemma 1,  $\mathcal{G}(X) = A \triangleleft F$ .

Let  $\theta = \theta_1 \circ \rho_1 \circ \theta_2$  and  $\rho = \theta_1 \circ \rho_1$ . By Theorem 2,  $\rho$  and  $\theta$  are regular, and by Lemma 3,  $\Gamma_\rho$  and  $\Gamma_\theta$  are topologically isomorphic. By Lemma 2,  $\Gamma_\rho = \Gamma_{\rho_1}$ . Now since  $A \triangleleft F$ ,  $F'A \triangleleft F$ , the homomorphism  $\rho_1$  is a group homomorphism. Thus the topological group  $\Gamma_{\rho_1}$  is a compact group (topologically isomorphic to the compact Hausdorff topological group  $F/F'A$  with its  $\tau$ -topology). We also conclude that  $\Gamma_\theta$  is a compact group. Let  $W$  be the quotient flow  $Y_2/\Gamma_\theta$  and let

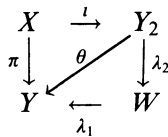


be the associated diagram.  $\mathcal{S}(W) \subset F$  and since every  $\gamma \in F$  defines an element  $\psi_\gamma \in \Gamma_\theta$ , we also have  $F \subset \mathcal{S}(W)$ . Thus  $\mathcal{S}(W) = F = \mathcal{S}(Y)$  and  $\lambda_1$  is proximal. We now have the alternative chain

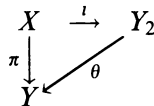
$$Z_2 \xrightarrow{\rho_2} Y_2 \xrightarrow{\lambda_2} W \xrightarrow{\lambda_1} Y,$$

where  $\rho_2$  and  $\lambda_2$  are group extensions and  $\lambda_1$  is proximal. Put  $\kappa_1 = \lambda_1 \circ \lambda_2 \circ \rho_2$  and  $\kappa_2 = \lambda_2 \circ \rho_2$ . Then  $\kappa_2$  is a distal homomorphism.

If  $(x, x')$  is an a.p. point in  $R_{\kappa_2} \subset R_{\kappa_1}$ , then by regularity of  $\kappa_1$  (Theorem 2)  $\exists \psi \in \Gamma_{\kappa_1}$  with  $x' = \psi(x)$ . However, by Lemma 2,  $\Gamma_{\kappa_2} = \Gamma_{\kappa_1}$  so that  $\psi \in \Gamma_{\kappa_2}$  and we conclude that  $\kappa_2$  is also regular. It follows from [G1, Proposition 3.1] that  $Z_2 \xrightarrow{\kappa_2} W$  is almost periodic, and since  $\mathcal{S}(Z_2) = F''A$ ,  $\mathcal{S}(W) = F$ , this implies  $F' \subset F''A$ . Since  $\mathcal{S}(Y_2) = F'A$ , we now have  $\mathcal{S}(Z_2) = F''A = F'A = \mathcal{S}(Y_2)$ , whence  $Y_2 = Z_2 = Y_\infty = X_\infty$ . Thus our PI-tower is of height two and we have the following diagram



where  $\iota$  is proximal. Apply Lemma 3 again to the diagram



to get the topological isomorphism of the compact  $\Gamma_\theta$  with  $\Gamma_\pi$ . Thus  $\Gamma_\pi$  is also compact and denoting the quotient flow  $X/\Gamma_\pi$  by  $L$  and the quotient map  $X \rightarrow L$  by  $\nu$ . We find that the map  $L \xrightarrow{\mu} Y$  is proximal.

Thus  $\pi = \mu \circ \nu$  is the required decomposition. It is easy to check that when  $\pi$  is RIC  $\mu$  must be trivial and  $\pi = \nu$  is a group homomorphism. This completes the proof of Theorem 1.  $\square$

**Theorem 3.** *Let  $(X, T)$  be a metrizable minimal flow and  $X \xrightarrow{\pi} Y$  a regular RIC homomorphism. Let  $X \xrightarrow{\omega} Z \xrightarrow{\kappa} Y$ ,  $\pi = \kappa \circ \omega$ , where  $Z$  is the largest equicontinuous extension of  $Y$  under  $\pi$ . Then  $\omega$  is RIC and weakly mixing, and  $\kappa$  is a group homomorphism. In particular, when  $X$  is regular and  $T$*



abelian, the homomorphism  $X \xrightarrow{\omega} Z$  of  $X$  onto its largest equicontinuous factor is RIC and weakly mixing and  $Z$  is a compact group rotation.

*Proof.* Repeating the proof of Theorem 1 without the assumption  $X_\infty = Y_\infty$ , we arrive at the diagram

$$\begin{array}{ccccc}
 X & & \xleftarrow{\iota} & & X_\infty = X \vee Y_\infty \\
 \pi \downarrow & & & & \downarrow \tilde{\pi} \\
 Y & \xleftarrow{\lambda_1} & W & \xleftarrow{\lambda_2} & Z_2 = Y_2 = Y_\infty
 \end{array}$$

where  $\lambda_1$  is proximal,  $\lambda_2$  a group homomorphism, and  $\tilde{\pi}$  RIC and weakly mixing. From the simple structure of  $Y_\infty$  over  $Y$  we immediately deduce (in the notation of the canonical tower of §2) that  $W = Y_1 = \tilde{Y}$  and  $Z_2 = Z_1$ . However, the assumption that  $\pi$  is RIC means that  $\tilde{Y} = Y = W$ , and therefore,  $Z = Z_2$  is the largest almost periodic extension of  $Y$  that is a factor of  $X$ . Hence  $X_\infty = X \vee Y_\infty = X \vee Z = X$ . Put  $\omega = \tilde{\pi}$  and  $\kappa = \lambda_2$  to get the decomposition  $\pi = \kappa \circ \omega$ .  $\square$

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