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h_0-TORSION BOUNDS IN THE COHOMOLOGY OF THE STEENROD ALGEBRA

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ABSTRACT. In this paper we use a technique of M. Hopkins to prove that the cohomology of the finite Hopf subalgebra of the mod 2 Steenrod algebra generated by $\mathrm{Sq}(2^i)$ with $i \leq n$, has h_0-torsion bound $2^{n+1} - n - 2$ for $n \geq 1$.

I. INTRODUCTION

Let R be any commutative algebra and $x \in R$. An element $y \in R$ is said to be *x-torsion* if $x^k y = 0$ for some k. R has *x-torsion bound m* if $x^m y = 0$ for any x-torsion $y \in R$. Let $\mathscr{A}_n$ be the Hopf subalgebra of the mod 2 Steenrod algebra generated by $\mathrm{Sq}(2^i)$ with $i \leq n$. For any graded $\mathbb{Z}_2$-algebra Γ, define the *cohomology of* Γ to be $H^{*,*}(\Gamma) \equiv \mathrm{Ext}_\Gamma^{*,*}(\mathbb{Z}_2, \mathbb{Z}_2)$. Let h_0 be the unique nonzero element in $H^{1,1}(\mathscr{A}_n)$. In this paper we prove:

Theorem. *$H^{*,*}(\mathscr{A}_n)$ has h_0-torsion bound $2^{n+1} - n - 2$ for $n \geq 1$.*

Davis [1] proved that $h_0^{2^i} h_{i+1} = 0$ and $h_0^{2^i - 1} h_{i+1} \neq 0$ in $H^{*,*}(\mathscr{A}_{i+1})$. So $H^{*,*}(\mathscr{A}_n)$ must have an h_0-torsion bound $\geq 2^{n-1}$. Thus the best possible h_0-torsion bound must lie somewhere between 2^{n-1} and $2^{n+1} - n - 2$. While the result of this paper does not provide the best possible bound, it dramatically improves on those previously obtainable from the global torsion bounds computed in [6] (see Table 1). The theorem correctly predicts the best possible h_0-torsion bound for $H^{*,*}(\mathscr{A}_1)$. For $H^{*,*}(\mathscr{A}_2)$ the theorem gives a bound of 4, while the best possible value is 3 (which can be determined from the description of $H^{*,*}(\mathscr{A}_2)$ given in [2]). The best possible h_0-torsion bounds for $H^{*,*}(\mathscr{A}_n)$ are not known for $n \geq 3$. These bounds can be interpreted as upper bounds on the height of finite h_0-towers in the usual diagram of $H^{*,*}(\mathscr{A}_n)$ [2]. It is hoped that this result will be valuable to others in computing the stable homotopy groups of spheres via the Adams spectral sequence, whose E_2-term is isomorphic to

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TABLE 1. h_0-torsion bounds for $H^{*,*}(\mathscr{A}_n)$.

n	Bound from [6]	Bound from Theorem I
0	1	–
1	3	1
2	9	4
3	96	11
4	3,294	26
5	267,282	57
6	60,896,016	120

the $H^{*,*}(\mathscr{A}_n)$ through a range of dimensions [4].

II. BACKGROUND

Let E_n be the normal exterior Hopf subalgebra of $\mathscr{A}_n$ generated by $\{Q_i: i \leq n\}$ where $Q_i = \mathrm{Sq}(0, \dots, 0, 1)$ (the 1 in the $(i+1)$st place, e.g. $Q_0 = \mathrm{Sq}(1)$) in the Milnor basis for $\mathscr{A}$ [5]. Then $H^{*,*}(E_n)$ is a polynomial algebra on generators v_i, $i \leq n$ (corresponding to the Q_i). Since E_n is normal in $\mathscr{A}_n$, we can form the quotient algebra, $\mathscr{A}_n//E_n \equiv \mathscr{A}_n/(\overline{\mathscr{A}E_n})$. This is isomorphic as an algebra to $\mathscr{A}_{n-1}$ via the map (of algebras, not graded algebras)

$$D: \mathscr{A}_{n-1} \to \mathscr{A}_n//E_n$$
$$\mathrm{Sq}(r_1, r_2, r_3, \dots) \mapsto [\mathrm{Sq}(2r_1, 2r_2, 2r_3, \dots)],$$

where $[x]$ means the equivalence class of the element x. The map is extended linearly to an arbitrary element of $\mathscr{A}_{n-1}$.

There is a change of rings map

$$\phi: \mathrm{Ext}^{*,*}_{E_n}(\mathbb{Z}_2, \mathbb{Z}_2) \to \mathrm{Ext}^{*,*}_{\mathscr{A}_n}(\mathscr{A}_n//E_n, \mathbb{Z}_2),$$

which is an isomorphism of graded $\mathbb{Z}_2$-modules. Thus $\mathrm{Ext}^{*,*}_{\mathscr{A}_n}(\mathscr{A}_n//E_n, \mathbb{Z}_2)$ is also isomorphic as a graded $\mathbb{Z}_2$-vector space to the $\mathbb{Z}_2$-polynomial algebra on generators v_i, $i \leq n$ (corresponding to the Q_i in E_n). Additionally, it is an $\mathrm{Ext}^{*,*}_{\mathscr{A}_n}(\mathbb{Z}_2, \mathbb{Z}_2)$-module with free h_0 action. All of this material can be found in [3].

III. PROOF OF THE THEOREM

Define t_n to be the top-dimensional class in $\mathscr{A}_n$, i.e.

$$t_n = \mathrm{Sq}(2^{n+1} - 1, 2^n - 1, \dots, 7, 3, 1).$$

Then by the doubling isomorphism the highest nonzero grading in $\mathscr{A}_n//E_n$ is a $\mathbb{Z}_2$ generated by

$$D(t_{n-1}) = [\mathrm{Sq}(2(2^n - 1), 2(2^{n-1} - 1), \dots, 14, 6, 2)].$$

Let $H(n) = |D(t_{n-1})| = 2 \cdot \sum_{i+j=n+1}(2^i - 1)(2^j - 1)$. Then we have nontrivial maps

$$\Sigma^{H(n)}\mathscr{A}_n//E_n \xrightarrow{\alpha} \Sigma^{H(n)}\mathbb{Z}_2 \xrightarrow{\beta} \mathscr{A}_n//E_n,$$

where Σ denotes suspension. Define γ to be the composition $\beta \circ \alpha$. These maps induce

$$\mathrm{Ext}^{*,*}_{\mathscr{A}_n}(\mathscr{A}_n//E_n, \mathbb{Z}_2) \xrightarrow{\beta^*} \mathrm{Ext}^{*,*}_{\mathscr{A}_n}(\Sigma^{H(n)}\mathbb{Z}_2, \mathbb{Z}_2) \xrightarrow{\alpha^*} \mathrm{Ext}^{*,*}_{\mathscr{A}_n}(\Sigma^{H(n)}\mathscr{A}_n//E_n, \mathbb{Z}_2)$$

and the composite $\gamma^* = \alpha^* \circ \beta^*$.

Suppose there is some $x \in \mathrm{Ext}^{*,*}_{\mathscr{A}_n}(\mathscr{A}_n//E_n, \mathbb{Z}_2)$ such that $\gamma^*(x) = h_0^m$ (and thus $\beta^*(x) = h_0^m$) and that $y \in H^{*,*}(\mathscr{A}_n)$ is h_0-torsion, that is, there is some k such that $h_0^k y = 0$. Then

$$h_0^k \cdot (y \cdot x) = h_0^k y \cdot x = 0 \cdot x = 0.$$

But $\mathrm{Ext}^{*,*}_{\mathscr{A}_n}(\mathscr{A}_n//E_n, \mathbb{Z}_2)$ has free h_0 action, so $h_0^k \cdot (y \cdot x) = 0 \Rightarrow y \cdot x = 0$. Therefore,

$$h_0^m y = y h_0^m = y \cdot h_0^m = y \cdot \beta^*(x) = \beta^*(y \cdot x) = \beta^*(0) = 0,$$

where '$\cdot$' indicates the action of an element on the module structure. So m is an h_0-torsion bound for $H^{*,*}(\mathscr{A}_n)$. Thus, to prove the theorem, it suffices to show that for a given n, h_0^m is in the image of γ^*, where $m = 2^{n+1} - n - 2$.

Consider the minimal $\mathscr{A}_n$-resolution of $\mathscr{A}_n//E_n$: $\mathscr{A}_n//E_n \xleftarrow{\mathscr{Q}} C_0 \xleftarrow{\delta_0} C_1 \xleftarrow{\delta_1} \cdots$ where C_s is a free $\mathscr{A}_n$-module on generators denoted by all symbols of the form $g(m_0, \ldots, m_n)$ satisfying $\sum m_j = s$ and $m_i \geq 0$, the map $\mathscr{Q}$ is just the quotient map (since $C_0 = \mathscr{A}_n$), and the maps δ_s are given by $\delta_s(g(m_0, \ldots, m_n)) = \sum_{i=0}^n Q_i g(m_0, \ldots, m_i - 1, \ldots, m_n)$ (and extended $\mathscr{A}_n$-linearly). We can also suspend this resolution to obtain a resolution for $\Sigma^{H(n)}\mathscr{A}_n//E_n$ and will refer to the suspended generators by the same names as the original generators, as the meaning will be clear from context. Notice that in these resolutions the generator $g(s)$ yields the element h_0^s in Ext. Thus it suffices to show that there is a map of resolutions

$$\begin{array}{ccccccc}
\Sigma^{H(n)}\mathscr{A}_n//E_n & \xleftarrow{\mathscr{Q}} & \Sigma^{H(n)}C_0 & \xleftarrow{\delta_0} & \Sigma^{H(n)}C_1 & \xleftarrow{\delta_1} & \cdots \\
\downarrow \gamma & & \downarrow \gamma_0 & & \downarrow \gamma_1 & & \\
\mathscr{A}_n//E_n & \xleftarrow{\mathscr{Q}} & C_0 & \xleftarrow{\delta_0} & C_1 & \xleftarrow{\delta_1} & \cdots
\end{array}$$

such that for some $m_0, \ldots, m_n$, $\gamma_m(g(m)) = g(m_0, \ldots, m_n) +$ other terms, where $m = 2^{n+1} - n - 2$.

Let χ be the canonical antiautomorphism of $\mathscr{A}$ [5]. Define $\mathrm{Sq}(r_1, \ldots, r_n) = 0$ if $r_i < 0$ for some i. Then it suffices to prove:

Lemma 1. *There is a map of the above resolutions such that*

$$\gamma_s(g(s)) = \sum_{\Sigma m_j = s} \chi\,\mathrm{Sq}(2r_1 - 2m_1, \ldots, 2r_n - 2m_n) g(m_0, \ldots, m_n),$$

where $r_i = 2^{n+1-i} - 1$.

Proof. We will proceed by induction on s. Let g be the generator of $\mathscr{A}_n//E_n$. Then $\gamma(g) = [2t_{n-1}] = [\chi 2t_{n-1}]$ and so we can define $\gamma_0(g(0)) = \chi 2t_{n-1} g(0) =$

$\chi\,\mathrm{Sq}(2r_1, \dots, 2r_n)g(0)$ where $r_i = 2^{n+1-i} - 1$. So the lemma is true for $s = 0$. Now assume that the lemma is true for γ_{s-1}.

It is easy to see from the multiplication formula that

$$\mathrm{Sq}(2r_1, \dots, 2r_n)Q_0 = \sum_{i=0}^{n} Q_i\,\mathrm{Sq}(2r_1, \dots, 2r_i - 2, \dots, 2r_n),$$

(where we take $\mathrm{Sq}(s_1, \dots, s_k) = 0$ if $s_j < 0$ for some j). Applying χ to this relation (and using the facts that $\chi(ab) = \chi(b)\chi(a)$ and $\chi(Q_j) = Q_j$ for any j) we obtain

$$Q_0\chi\,\mathrm{Sq}(2r_1, \dots, 2r_n) = \sum_{i=0}^{n} \chi\,\mathrm{Sq}(2r_1, \dots, 2r_i - 2, \dots, 2r_n)Q_i.$$

So for example

$$\begin{aligned} &Q_0\chi\,\mathrm{Sq}(4, 6, 2) \\ &\quad = \chi\,\mathrm{Sq}(4, 6, 2)Q_0 + \chi\,\mathrm{Sq}(2, 6, 2)Q_1 + \chi\,\mathrm{Sq}(4, 4, 2)Q_2 + \chi\,\mathrm{Sq}(4, 6)Q_3. \end{aligned}$$

Then

$$\begin{aligned} &\gamma_{s-1} \circ \delta_{s-1}(g(s)) = \gamma_{s-1}(Q_0 g(s)) = Q_0(\gamma_{s-1}(g(s))) \\ &\quad = Q_0 \sum_{\Sigma m_j = s-1} \chi\,\mathrm{Sq}(2r_1 - 2m_1, \dots, 2r_n - 2m_n)g(m_0, \dots, m_n) \\ &\quad = \sum_{\Sigma m_j = s-1} Q_0\chi\,\mathrm{Sq}(2r_1 - 2m_1, \dots, 2r_n - 2m_n)g(m_0, \dots, m_n) \\ &\quad = \sum_{\Sigma m_j = s-1} \sum_{i=0}^{n} \chi\,\mathrm{Sq}(2r_1 - 2m_1, \dots, 2r_i - 2m_i - 2, \dots, 2r_n - 2m_n) \\ &\qquad\qquad \times Q_i g(m_0, \dots, m_n) \\ &\quad = \sum_{\Sigma n_j = s} \chi\,\mathrm{Sq}(2r_1 - 2n_1, \dots, 2r_n - 2n_n) \sum_{i=0}^{n} Q_i g(n_0, \dots, n_i - 1, \dots, n_n) \\ &\quad = \sum_{\Sigma n_j = s} \chi\,\mathrm{Sq}(2r_1 - 2n_1, \dots, 2r_n - 2n_n)\delta_{s-1}(g(n_0, \dots, n_n)) \\ &\quad = \delta_{s-1}\left(\sum_{\Sigma n_j = s} \chi\,\mathrm{Sq}(2r_1 - 2n_1, \dots, 2r_n - 2n_n)(g(n_0, \dots, n_n))\right) \\ &\quad = \delta_{s-1} \circ \gamma_s(g(s)). \end{aligned}$$

Thus $\gamma_m(g(m)) = g(0, r_1, \dots, r_n) +$ other terms for

$$m = \sum_{i=1}^{n} r_i = \sum_{i=1}^{n}(2^{n+1-i} - 1) = 2^{n+1} - n - 2. \quad \square$$

Notice that there are no smaller powers of h_0 in the image of γ^*, and so our bounds are the best possible in that sense.

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