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A GENERALIZATION OF THE HANDLE ADDITION THEOREM

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ABSTRACT. We will generalize Jaco's Handle Addition Theorem to the n-compressibility of surfaces on the boundary of 3-manifolds. Several corollaries are given that show how the theorem can be applied to different situations.

The Handle Addition Theorem was first proved by Przytycki [6] in the case when M is a handlebody. In [4] Jaco proved the general version below.

Handle Addition Theorem [4]. *Suppose M is a 3-manifold with compressible boundary, and J is a simple closed curve on ∂M such that $\partial M - J$ is incompressible. Then the manifold obtained by adding a 2-handle to M along J has incompressible boundary.*

Note that in the theorem M can be noncompact. So the theorem is still true when ∂M is replaced by a surface S on ∂M. Several alternative proofs have been published [1, 5, 7]. And it has been applied very successfully in dealing with incompressible surfaces, surgeries, and other related topics. (See for example [2, 3, 4, 5, 8].) In this paper we will discuss the n-compressibility of surfaces with respect to a specified 1-manifold, and prove a generalized handle addition theorem for this situation. While 0-compressibility is the usual notion of compressibility of surfaces, 2-compressibility includes ∂-compressibility. Our original motivation is to prove Corollary 3, which says that under certain conditions an essential surface in a 3-manifold will remain essential after handle addition. Some other corollaries are given, which illustrate how the theorem is applied in different situations.

1. The main theorem

We work in smooth category. All manifolds and surfaces are assumed orientable, and submanifolds are assumed intersecting each other transversely. Let F be an arbitrary surface on the boundary of a 3-manifold M, and let γ be a 1-manifold in F. In applications γ is usually a union of disjoint essential circles on F. A compressing disc of F is a disc D properly embedded in M so that ∂D is an essential curve in F. We call D an *n-compressing disc* (with respect to γ) if ∂D intersects γ in n points. We also call a compressing disc D of $F - \gamma$ to be a *0-compressing disc* of F. Note that D may fail to be a

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compressing disc of F. But this does not happen when γ consists of essential circles. F is called *n-compressible* if an n-compressing disc exists. Otherwise it is n-incompressible. By definition F is 0-compressible if and only if $F - \gamma$ is compressible.

Given a simple closed curve J on F, let $M' = \tau(M; J)$ be the manifold obtained by attaching a 2-handle $D^2 \times I$ to M so that $\partial D^2 \times I$ is identified with a regular neighborhood of J in F. Let $F' = \sigma(F; J)$ be the surface $(F - \partial D^2 \times I) \cup (D^2 \times \partial I)$ on the boundary of M'.

Theorem 1. *Let γ be a 1-manifold in F, and let J be a circle in F disjoint from γ. Suppose that $F - \gamma$ is compressible.*

(a) *If F' is n-compressible, then $F - J$ is k-compressible for some $k \leq n$.*

(b) *If F' has an n-compressing disc D with ∂D a nonseparating curve on F', then either $F - J$ is 0-compressible, or it has a k-compressing disc B such that $k \leq n$ and ∂B is nonseparating on $F - J$.*

When $n = 0$ and $\gamma = \varnothing$, the theorem reduces to the Handle Addition Theorem. We will try to follow Jaco's argument to prove this generalized form. Some care has to be taken in the second step: The process in [4] does not work in our setting because a compressing disc of $F - \gamma$ may be changed to a disc that has nontrivial intersection with γ, and hence will no longer be a compressing disc of $F - \gamma$. Notice that Assertion (2) in our proof is weaker than that of [4]. The gap is amended in the proof of Assertion (3), in which we do not assume that the arc α is essential in P. We will pay attention to those steps that must be modified in our setting, referring the reader to [4] for details of other steps. Another remark is that the first possibility in the conclusion of part (b) of the theorem cannot be dropped as it is easy to find counterexamples otherwise.

Proof of Theorem 1. Suppose J is a curve in a surface S. A curve $K \subset S$ is called *coplanar* with J in S if either K bounds a disc, K is parallel to J, or K bounds a once-punctured torus containing J as a nonseparating curve. In our setting a *pre-disc* with respect to (γ, J) is defined to be a properly embedded planar surface P in M satisfying

(1) $\partial P \subset F - J$;

(2) one component, $s = s(P)$, of ∂P is not coplanar with J in F; and

(3) each component of $\partial P - s$ is coplanar with J in $F - \gamma$.

Suppose B is an n-compressing disc in F'. Then $Q = B \cap M$ is a pre-disc with respect to (γ, J). (In case (b) Q satisfies an extra condition: $s(Q)$ is nonseparating on $F - J$.) Denote by $|X|$ the number of components of X. Among all pre-discs with $|\partial P \cap \gamma| \leq n$, choose one, say P, such that the complexity $(|s(P) \cap \gamma|, |\partial P|)$ of P is minimal in the lexicographic order. (In case (b), P is chosen among those with $s(P)$ nonseparating on $F - J$, and the minimum is taken within this class.) We will eventually prove that either $F - J$ is 0-compressible, or $|\partial P| = 1$, and hence P will be a compressing disc of $F - J$ intersecting γ at most n times.

Suppose that $|\partial P| > 1$. As is pointed out in [4], we may assume that some component of ∂P is parallel to J. Choose a compressing disc D of $F - \gamma$ in M so that $|D \cap P|$ is minimized. Since $D \cap J \neq \varnothing$, we have $|D \cap P| \neq 0$.

(1) Assertion. *No component of $D \cap P$ is a simple closed curve.*

This is a standard cut and paste argument (cf. [4]). Note that ∂P is unchanged, therefore, in case (b) the nonseparability of s is preserved.

We need a definition for the next step: An arc α in P is *γ-inessential* if there is an arc $\beta \subset \partial P$ such that $\alpha \cup \beta$ bounds a disc Δ in P, and β is disjoint from γ. Note that this is automatic if α is an inessential arc with $\partial\alpha \subset \partial P - s$.

(2) Assertion. *No component of $D \cap P$ is a γ-inessential arc in P.*

If α is such an arc, let β and Δ be as in the definition. A boundary compression of D along Δ results in two discs D_1 and D_2 with $|D_i \cap P| < |D \cap P|$. Since $\beta \subset F - \gamma$, we have $\partial D_i \subset F - \gamma$. Hence one of the D_i is a compressing disc of $F - \gamma$, contradicting the minimality of $|D \cap P|$.

Now let α be an outermost component of $D \cap P$ in D. Let β be an arc on ∂D such that $\alpha \cup \beta$ bounds a disc Δ with interior disjoint from P. Since J is parallel to some component of ∂P, it can be isotoped off $\beta \cup \partial P$. So we can assume β is disjoint from J.

(3) Assertion. *The arc α cannot have both end points in s.*

Compressing P along Δ, we obtain two surfaces P_1 and P_2. Since by Assertion (2) α is F-essential, both P_i have less complexity than P. One of the P_i is a pre-disc. Moreover, in case (b), since $s(P)$ is nonseparating in $F - J$, we can choose P_i so that $s(P_i)$ is nonseparating in $F - J$. Note that this implies that $s(P_i)$ is not coplanar to J, and hence P_i is a pre-disc. In both cases, P_i will contradict the minimality of the complexity of P.

(4–6) Assertion. *The arc α cannot be an essential arc with at least one end on $\partial P - s$.*

If α has endpoints on different components of ∂P, a compression of P along Δ produces a new essential pre-disc P' with less complexity, and $s(P')$ is nonseparating if $s(P)$ is. If α has both endpoints in a same component of $\partial P - s$, then a compression of P along Δ yields two new surfaces P_1 and P_2 with $s \subset P_2$, say. Let s' be the component of P_1 that is not a component of ∂P. If s' is coplanar with J, then P_2 is a pre-disc with less complexity, contradicting the choice of P. If s' is not coplanar with J, P_1 would be a pre-disc with less complexity. In case (a) this violates the choice of P, and completes the proof. In case (b) notice that P_1 induces a 0-compressing disc for F'. Therefore by (a) we conclude that $F - J$ is 0-compressible. □

2. Some applications

Recall that a set of disjoint simple closed curves $\{C_1, \dots, C_n\}$ on the boundary of a handlebody is primitive if there exist disjoint discs $\{D_1, \dots, D_n\}$ in the handlebody such that $|C_i \cap D_j| = \delta_{ij}$. The following result is proved by Gordon in [3].

Gordon's Theorem. *Let $\mathscr{C}$ be a set of disjoint simple loops on the boundary of a handlebody X, such that $\tau(X; \mathscr{C}')$ is a handlebody for all $\mathscr{C}' \subset \mathscr{C}$. Then $\mathscr{C}$ is primitive.*

The theorem was proved inductively by applying the Handle Addition Theorem. One gets stuck when the genus of the handlebody is reduced to 2. Fortunately this special case has been proved in [2, §2.3]. Our first application is an alternative proof of this special case. It follows that the inductive proof in [3] can be carried over the genus 2 case if we use Theorem 1 instead.

Corollary 1 [2, §2.3]. *Let C_1, C_2 be disjoint simple loops in the boundary of a handlebody X of genus 2. If there are discs D_i such that $|C_i \cap \partial D_i| = 1$, and $\partial X - C_1 \cup C_2$ is incompressible, then $\tau(X; C_1 \cup C_2)$ is a punctured less space.*

Proof. Since C_1 is primitive, $X' = \tau(X; C_1)$ is a solid torus. Since C_2 is primitive, $\partial X - C_2$ is compressible. If $\tau(X; C_1 \cup C_2) = \tau(X'; C_2)$ was not a punctured lens space, then there would be a compressing disc of $\partial X'$ intersecting $C_2 = \gamma$ at most once. By Theorem 1, there is a compressing disc B of $\partial X - C_1$ intersecting C_2 at most once. If $|\partial B \cap C_2| = 1$, then the boundary of a regular neighborhood of $\partial B \cup C_2$ would bound a compressing disc of $\partial X - C_1 \cup C_2$. If $|\partial B \cap C_2| = 0$, B itself is a compressing disc of $\partial X - C_1 \cup C_2$. Both cases contradict the incompressibility of $\partial X - C_1 \cup C_2$. □

The following result for Heegaard diagrams of lens spaces has a similar nature to Gordon's theorem above. It says that under certain conditions there exists a "nice" cutting system.

Corollary 2. *Let $\mathscr{C} = \{C_1, \dots, C_m\}$ be a set of disjoint simple loops on the boundary of a handlebody X of genus m. If $\tau(X; \mathscr{C}')$ is a handlebody for all proper subsets $\mathscr{C}' \subset \mathscr{C}$, and $\tau(X; \mathscr{C})$ is a punctured lens space $L(p, q)$, then there is a set of disjoint disc $\mathscr{D} = \{D_1, \dots, D_m\}$ cutting X into a 3-cell, such that C_1 intersects D_1 p times in the same direction, and $|C_i \cap D_j| = \delta_{ij}$ when $i > 1$.*

Proof. We first show that $\partial X - \mathscr{C}$ is incompressible. Suppose D is a compressing disc of $\partial X - \mathscr{C}$. Since D remains a proper disk in the lens space $\tau(X, \mathscr{C})$, it is separating, so it cuts X into two handlebodies Y_1, Y_2. Let $\mathscr{C}_i = \mathscr{C} \cap Y_i$. If $\mathscr{C} = \mathscr{C}_1$, say, then $\tau(X, \mathscr{C})$ has Y_2 as a boundary connected summand, which is absurd. Thus both $\mathscr{C}_1$, $\mathscr{C}_2$ are proper subsets of $\mathscr{C}$. By assumption $\tau(Y_i, \mathscr{C}_i)$ are handlebodies, so $\tau(X, \mathscr{C}) = \tau(Y_1, \mathscr{C}_1) \cup \tau(Y_2, \mathscr{C}_2)$ is also a handlebody. But this is impossible because $\tau(X, \mathscr{C})$ is a lens space.

The corollary is obvious when $m = 1$. So assume $m > 1$. Let $F = \partial X - (C_2 \cup \cdots \cup C_{m-1})$. Let $\gamma = C_1$, and let $J = C_m$. By Gordon's Theorem, any proper subsets of $\mathscr{C}$ are primitive. Therefore $F - \gamma$ is compressible. After attaching a 2-handle along C_m, the new manifold X' is a handlebody of genus $m - 1$. By induction, there is a nonseparating compressing disc D' in X', disjoint from C_j for $j = 2, \dots, m-1$, intersecting γ in p points. Thus D' is a nonseparating compressing disc for $F' = \sigma(F, J) = \partial X' - C_2 \cup \cdots \cup C_{m-1}$. We have shown that $F - (\gamma \cup J) = \partial X - \bigcup C_i$ is incompressible. Therefore by Theorem 1(b), there is a *nonseparating* compressing disc D_1 for $F - J = \partial X - C_2 \cup \cdots \cup C_m$ which meets C_1 at most p times.

Cutting X along D_1, we get a handlebody Y of genus $m - 1$. For any subset $\mathscr{C}'$ of $\{C_2, \dots, C_m\}$, $\tau(Y; \mathscr{C}')$ is $\tau(X, \mathscr{C}')$ cut along D_1, and hence is a handlebody. By Gordon's Theorem $\{C_2, \dots, C_m\}$ is primitive. So there are discs $\{D_2, \dots, D_m\}$ cutting Y into a 3-cell, with $D_i \cap C_j = \delta_{ij}$ $(i, j \geq 2)$. We have shown that $|D_1 \cap C_1| \leq p$. For homological reasons, the algebraic intersection number of C_1 with ∂D_1 is p. So C_1 must intersect ∂D_1 exactly in p points, and all the intersections have the same sign. □

Now suppose M is a 3-manifold with compressible boundary, and J is a curve on ∂M such that $\partial M - J$ is incompressible. Remember that a properly embedded surface P is called essential if P is incompressible and ∂-incompressible. The following result shows that an essential surface P in M remains essential after 2-handle addition along some essential curve J.

Corollary 3. *Let M be a 3-manifold with compressible boundary. Let J be a simple closed curve on ∂M such that $\partial M - J$ is incompressible. If P is an essential surface in M with boundary disjoint from J, then P is essential in $\tau(M; J)$.*

Proof. Let X be the manifold obtained by cutting M along P. Let γ be the two copies of ∂P on $F = \partial X$. As P is essential, and ∂M is compressible, $F - \gamma$ must be compressible in X. Since $\partial M - J$ and P are incompressible, $\partial X - J$ is 0-incompressible. Since γ is separating on ∂X, $\partial X - J$ is 1-incompressible. Since P is ∂-incompressible, $\partial X - J$ is 2-incompressible. Now we can apply Theorem 1 and conclude that no compressing disc for the boundary of $\tau(X; J)$ can intersect γ in at most two points. This implies that P is essential in $\tau(M; J)$. □

A set of simple loops $\mathscr{C} = \{C_1, \dots, C_{n+1}\}$ in the boundary of a handlebody X of genus n is called *standard* if $\bigcup C_i$ bounds a planar surface P in X such that $(X, P) \cong (P \times I, P \times \{1/2\})$. Our next application is a simple proof of a theorem of Gordon's [3].

Corollary 4. *Let $\mathscr{C} = \{C_1, \dots, C_{n+1}\}$ be a set of disjoint simple loops in the boundary of a handlebody X of genus n. If $\tau(X; \mathscr{C}')$ is a handlebody for all proper subsets $\mathscr{C}'$ of $\mathscr{C}$, then $\mathscr{C}$ is standard.*

Proof. The result is true when $n = 1$ because in this case $\tau(X; C_i)$ is a handlebody, which implies that C_i is a longitude, $i = 1, 2$. So we assume $n > 1$ and proceed by induction. Let $F = \partial X - C_3 \cup \cdots \cup C_n$, let $J = C_{n+1}$, and let $\gamma = C_1 \cup C_2$. The proof is based on the following observation:

Claim. If $F - J$ has a nonseparating compressing disc D with $|\partial D \cap \gamma| \leq 2$, then C is standard.

Proof. Gordon's Theorem (at the beginning of this section) implies that all proper subsets of $\mathscr{C}$ are primitive. Thus $\{C_1, C_3, \dots, C_{n+1}\}$ generates the first homology of X. Since D is nonseparating and $D \cap C_i = \varnothing$ for $i \geq 3$, for homological reasons we must have $D \cap C_1 \neq \varnothing$. Similarly $D \cap C_2 \neq \varnothing$. Therefore D must intersect each of C_1 and C_2 in a single point.

Cutting X along D, we get a handlebody Y. Let D_1 and D_2 be the two copies of D on ∂Y, and let α_1 and α_2 be the arcs on ∂Y corresponding to C_1 and C_2, respectively. Choose an arc β_i in D_i connecting an end of α_1 to an end of α_2. Then $C_1' = \alpha_1 \cup \beta_1 \cup \alpha_2 \cup \beta_2$ is a simple loop on ∂Y. It is easy to see that $\tau(Y, C_1') = \tau(X; \{C_1, C_2\})$. Therefore, $\tau(Y; \mathscr{C}')$ is a handlebody for all proper subsets $\mathscr{C}'$ of $\{C_1', C_3, \dots, C_{n+1}\}$. By induction, $\{C_1', C_3, \dots, C_{n+1}\}$ bounds a planar surface Q in Y, and $(Y, Q) \cong (Q \times I, Q \times \{1/2\})$. The homeomorphism can be chosen so that $(D_i, \beta_i) \cong (\beta_i \times I, \beta_i \times \{1/2\})$. So we can glue D_1 to D_2 to induce a homeomorphism $(X, P) \cong (P \times I, P \times \{1/2\})$, where P is the surface obtained from Q by identifying β_1 with β_2 so that $\partial P = \bigcup C_i$. This proves the claim.

We need to show that $\partial X - \bigcup C_i$ is incompressible. Suppose D is a compressing disc of $\partial X - \bigcup C_i$. Let $N(D)$ be a regular neighborhood of D. Whether D is separating or not, there is a component Y of $X - \operatorname{Int} N(D)$

such that $|\mathscr{C} \cap Y| > g(Y)$, the genus of Y. Let $\mathscr{C}'$ be a set of $g(Y)+1$ curves in $\mathscr{C} \cap Y$. Then $\mathscr{C}'$ is a proper subset of $\mathscr{C}$, and $\tau(Y, \mathscr{C}')$ is not a handlebody. It follows that $\tau(X; \mathscr{C}')$ is not a handlebody, contradicting the hypothesis.

Now attach a 2-handle along $J = C_{n+1}$, getting a handlebody X' of genus $n-1$. We have $F' = \sigma(F; J) = \partial X' - C_3 \cup \cdots \cup C_n$. For any proper subset $\mathscr{C}'$ of $\{C_1, \ldots, C_n\}$, $\tau(X', \mathscr{C}') = \tau(X; \mathscr{C}' \cup \{C_{n+1}\})$ is a handlebody. By induction, $\{C_1, \ldots, C_n\}$ is standard. Therefore there is a nonseparating compressing disc of F' that meets each of C_1 and C_2 exactly once. Since $\{C_1, \ldots, C_n\}$ are primitive on X, $F - \gamma$ is compressible. So we can apply Theorem 1 (b), and conclude that either $F - (\gamma \cup J) = \partial X - \bigcup C_i$ is compressible, or $F - J$ has a nonseparating compressing disc which meets γ at most twice. We have just ruled out the first possibility. So the second possibility holds. The corollary now follows from the claim above. □

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