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A RIEMANN TYPE DEFINITION OF A VARIATIONAL INTEGRAL

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ABSTRACT. We present a Riemann type definition of a coordinate free integral for which a general divergence theorem holds. The definition is particularly simple in dimension one.

In [P_2] we defined a coordinate free variational integral on bounded sets of finite perimeter, and used it to establish a very general Gauss-Green theorem. In this note we show (Corollary 3.4) that the variational integral of [P_2] has a simple Riemann type definition, appreciably more transparent than that of [P_2, §7]. In dimension one, it enlightens the relationship between the variational integral of [P_2] and the generalized Riemann integral of Henstock and Kurzweil (see [P_2, Definition 2.1]).

1. Preliminaries

All functions we consider are real-valued. If f is a function on a set A and $B \subset A$, we denote by $f \restriction B$ the restriction of f to B; when no confusion can arise we write f instead of $f \restriction B$.

Throughout this note, $m \geq 1$ is a fixed integer. The set of all real numbers is denoted by $\mathbf{R}$, and the m-fold Cartesian product of $\mathbf{R}$ is denoted by $\mathbf{R}^m$. For $x = (\xi_1, \ldots, \xi_m)$ and $\varepsilon > 0$, we let $|x| = \max\{|\xi_1|, \ldots, |\xi_m|\}$ and $U(x, \varepsilon) = \{y \in \mathbf{R}^m : |x - y| < \varepsilon\}$. If $E \subset \mathbf{R}^m$, then $\operatorname{cl} E$, $\operatorname{int} E$, $\operatorname{bd} E$, and $|E|$ denote, respectively, the closure, interior, boundary, and outer Lebesgue measure of E; furthermore $d(E) = \sup\{|x - y| : x, y \in E\}$.

Let $E \subset \mathbf{R}^m$. We say that an $x \in \mathbf{R}^m$ is, respectively, a *density* or *dispersion* point of E whenever

$$\liminf_{\varepsilon \to 0+} \frac{|E \cap U(x, \varepsilon)|}{(2\varepsilon)^m} = 1 \quad \text{or} \quad \limsup_{\varepsilon \to 0+} \frac{|E \cap U(x, \varepsilon)|}{(2\varepsilon)^m} = 0.$$

The set of all density points of E is called the *essential interior* of E, denoted by $\operatorname{int}_e E$, and the set of all nondispersion points of E is called the *essential closure* of E, denoted by $\operatorname{cl}_e E$. The *essential boundary* of E is the set $\operatorname{bd}_e E = \operatorname{cl}_e E - \operatorname{int}_e E$. Clearly $\operatorname{int} E \subset \operatorname{int}_e E \subset \operatorname{cl}_e E \subset \operatorname{cl} E$, and so $\operatorname{bd}_e E \subset \operatorname{bd} E$. If

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$\mathrm{cl}_e E$ equals E or $\mathrm{cl}\, E$, the set E is called *essentially closed* or *nondispersed*, respectively.

The $(m-1)$-dimensional outer Hausdorff measure $\mathscr{H}$ in $\mathbf{R}^m$ is defined so that it is the counting measure if $m = 1$, and agrees with the Lebesgue measure in $\mathbf{R}^{m-1}$ if $m > 1$. A bounded set $A \subset \mathbf{R}^m$ is called a BV set (BV for *bounded variation*) whenever the number $\|A\| = \mathscr{H}(\mathrm{bd}_e A)$, called the *perimeter* of A, is finite. By [Fe, §2.10.6 and Theorem 4.5.11], the family BV of all BV sets coincides with the collection of all bounded measurable subsets of $\mathbf{R}^m$ whose De Giorgi's perimeter in $\mathbf{R}^m$, defined in [M-M, §2.1.2], is finite. If $E \subset \mathbf{R}^m$ we denote by BV_E the family of all BV subsets of E.

We shall need a lemma proved by G. Congedo and I. Tamanini (cf. [C-T] and [T-G], or [P_2, Proposition 3.2]).

Lemma 1.1. *For each BV set A there are nondispersed sets $A_n \in BV_A$ such that $\|A_n\| \leq \|A\|$ and $|A - A_n| \leq \|A\|/n$, $n = 1, 2, \ldots$.*

Let $A \in BV$. An $x \in \mathbf{R}^m$ is called a *perimeter dispersion point* of A whenever

$$\lim_{\varepsilon \to 0+} \frac{\mathscr{H}[\mathrm{bd}_e A \cap U(x, \varepsilon)]}{(2\varepsilon)^{m-1}} = 0.$$

The set of all $x \in \mathrm{int}_e A$ which are perimeter dispersion points of A is called the *critical interior* of A, denoted by $\mathrm{int}_c A$. It has been proved in [V, §4], that $\mathscr{H}(\mathrm{int}_e A - \mathrm{int}_c A) = 0$. The *regularity* of A is the number

$$r(A) = \begin{cases} \dfrac{|A|}{d(A)\|A\|} & \text{if } d(A)\|A\| > 0, \\[2ex] 0 & \text{otherwise.} \end{cases}$$

The usual concept of regularity defined in [S, Chapter 4, §2] is related to $r(A)$ by the isoperimetric inequality (see [M-M, §2.2]).

2. Partitions

A *dyadic cube* is the product $\prod_{i=1}^m [k_i 2^{-n}, (k_i + 1)2^{-n})$ where $k_1, \ldots, k_m$ and n are integers with $n \geq 0$. Given a function δ on a set E, we let $N_\delta = \{x \in E : \delta(x) = 0\}$ and call it the *null set* of δ. A *caliber* is any sequence $\eta = \{\eta_j\}$ of positive real numbers. A set $N \subset \mathbf{R}^m$ is called *thin* if its $\mathscr{H}$ measure is σ-finite.

Let $A \in BV$. A nonnegative function δ on $\mathrm{cl}_e A$ is called a *gage* in A whenever its null set N_δ is thin. A *partition* in A is a collection (possibly empty) $P = \{(A_1, x_1), \ldots, (A_p, x_p)\}$ where $A_1, \ldots, A_p$ are disjoint BV subsets of A and $x_i \in \mathrm{cl}_e A_i$, $i = 1, \ldots, p$; the set $\bigcup_{i=1}^p A_i$ is called the *body* of P, denoted by $\bigcup P$.

Definition 2.1. Let $\varepsilon > 0$, let η be a caliber, and let δ be a gage in a BV set A. We say that a partition $P = \{(A_1, x_1), \ldots, (A_p, x_p)\}$ in A is:

(1) *dyadic* if $A_1, \ldots, A_p$ are dyadic cubes;
(2) *ε-regular* if $r(A_i) > \varepsilon$, $i = 1, \ldots, p$;
(3) *δ-fine* if $d(A_i) < \delta(x_i)$, $i = 1, \ldots, p$;
(4) *(ε, η)-approximating* if $A - \bigcup P$ is the union of disjoint BV sets $B_1, \ldots, B_k$ such that $\|B_j\| < 1/\varepsilon$ and $|B_j| < \eta_j$, $j = 1, \ldots, k$.

The family of all ε-regular δ-fine (ε, η)-approximating partitions in A is denoted by $\Pi(A, \varepsilon; \delta, \eta)$.

Remark 2.2. Let δ be a gage in a BV set A, and let $\delta_+ = \delta \restriction (\mathrm{cl}_e A - N_\delta)$. If $\varepsilon > 0$ and P is an ε-regular δ-fine partition in A, then in the terminology of [P$_2$, Definition 7.1], P is a tight δ_+-fine ε-partition in A mod N_δ.

Lemma 2.3. *Let $K = \prod_{i=1}^m [k_i, k_i + 2^h)$ where $k_1, \dots, k_m$ and h are integers, let δ be a gage in K, and let η be a caliber. There is a $\tau > 0$, depending only on the dimension m, such that a δ-fine (τ, η)-approximating dyadic partition in K exists.*

Proof. We employ the idea of E. J. Howard (cf. [H]). According to [Fa, Theorem 1.6(a)], there is a sequence $\{N_j\}$ of sets such that $N_\delta = \bigcup_j N_j$ and $\mathscr{H}(N_j) < 2$, $j = 1, 2, \dots$. Fix an integer $j \geq 1$. It follows from [Fa, Theorem 5.1] that there is an $\alpha > 0$, depending only on the dimension m, and a countable family $\mathscr{C}_j$ of dyadic cubes of diameters less than η_j/α such that $N_j \subset \mathrm{int}(\bigcup \mathscr{C}_j)$ and $\sum_{C \in \mathscr{C}_j} [d(C)]^{m-1} < \alpha$. Thus if $\mathscr{C} \subset \mathscr{C}_j$ and $E = \bigcup \mathscr{C}$, then

$$\|E\| \leq \sum_{C \in \mathscr{C}} \|C\| \leq 2m \sum_{C \in \mathscr{C}_j} [d(C_j)]^{m-1} < 2m\alpha,$$

$$|E| \leq \sum_{C \in \mathscr{C}} |C| \leq \frac{\eta_j}{\alpha} \sum_{C \in \mathscr{C}_j} [d(C)]^{m-1} < \eta_j .$$

Now let $\mathscr{C}$ be a disjoint subfamily of $\bigcup_j \mathscr{C}_j$ such that $\bigcup \mathscr{C} = \bigcup_j (\cup \mathscr{C}_j)$. We define a positive function δ_+ on $\mathrm{cl}\, K$ by setting

$$\delta_+(x) = \begin{cases} \delta(x) & \text{if } x \in \mathrm{cl}\, K - N_\delta, \\ \min\{d(C) : C \in \mathscr{C},\ x \in \mathrm{cl}\, C\} & \text{if } x \in N_\delta. \end{cases}$$

By classical Cousin's lemma (see [MS, Chapter 4, Theorem 3–1]), there is a δ_+-fine dyadic partition $P = \{(K_1, x_1), \dots, (K_p, x_p)\}$ in K with $\bigcup P = K$. Let $\mathscr{D}$ consist of all $C \in \mathscr{C}$ such that $C \subset K$ and $K_i \subset C$ for some $i = 1, \dots, p$. As $\mathscr{C}$ is disjoint $\mathscr{D}$ is finite, and as $N_\delta \subset \mathrm{int}(\bigcup \mathscr{C})$, our definition of δ_+ on N_δ implies that $K_i \subset \bigcup \mathscr{D}$ whenever $x_i \in N_\delta$. If K_i meets a $D \in \mathscr{D}$ and $K_i \not\subset D$, then $D \subset K_i$ because D and K_i are dyadic cubes. This is, however, impossible, for by the definition of $\mathscr{D}$, the set D contains a K_j disjoint from K_i. We conclude that for each $i = 1, \dots, p$ either $K_i \subset \bigcup \mathscr{D}$ or $K_i \cap (\bigcup \mathscr{D}) = \varnothing$. Thus after a suitable reordering, $\bigcup \mathscr{D} = K - \bigcup_{i=1}^q K_i$ for a nonnegative integer $q \leq p$, and $Q = \{(K_1, x_1), \dots, (K_q, x_q)\}$ is a δ-fine dyadic partition in K. We complete the proof by showing that Q is (τ, η)-approximating for $\tau = 1/(2m\alpha)$. To this end, let $\mathscr{D}_1 = \mathscr{D} \cap \mathscr{C}_1$ and for $j = 1, 2, \dots$, let $\mathscr{D}_j = \mathscr{D} \cap \mathscr{C}_j - \bigcup_{i=1}^{j-1} \mathscr{D}_i$. The family $\mathscr{D}$, being finite, is the union of disjoint families $\mathscr{D}_1, \dots, \mathscr{D}_k$. If $D_j = \bigcup \mathscr{D}_j$ then $\bigcup \mathscr{D}$ is the union of disjoint sets $D_1, \dots, D_k$. As $\|D_j\| < 1/\tau$ and $|D_j| < \eta_j$, $j = 1, \dots, k$ our assertion is established.

Lemma 2.4. *Let A be a BV set, $x \in \mathrm{int}_c A$, and let $\{C_n\}$ be a sequence of dyadic cubes such that $x \in \mathrm{cl}\, C_n$, $n = 1, 2, \dots$, and $\lim d(C_n) = 0$. Then $x \in \mathrm{cl}_e(A \cap C_n)$, $n = 1, 2, \dots$, and*

$$\liminf r(A \cap C_n) \geq \frac{1}{2m} .$$

This lemma, which has been proved in [P_2, Lemma 3.6], allows us to apply Lemma 2.3 to BV sets.

Proposition 2.5. *Let δ be a gage in a BV set A and let η be a caliber. There is a $\kappa > 0$, depending only on the dimension m, such that $\Pi(A, \kappa; \delta, \eta) \neq \varnothing$.*

Proof. Assume first that $\|A\| \leq 2$. By Lemma 1.1, there is a nondispersed BV set $B \subset A$ with $\|B\| \leq 2$ and $|A-B| < \eta_1$. Choose integers $k_1, \ldots, k_m$ and h so that $K = \prod_{i=1}^{m}[k_i, k_i + 2^h)$ contains B. If $x \in \operatorname{cl} K - \operatorname{cl} B$ there is a $\gamma_x > 0$ with $B \cap U(x, \gamma_x) = \varnothing$. If $x \in \operatorname{int}_c B$ and $\beta = 1/(4m)$, Lemma 2.4 implies the existence of $\gamma_x > 0$ such that $r(B \cap C) > \beta$ for each dyadic cube C with $x \in \operatorname{cl} C$ and $d(C) < \gamma_x$. Since $\operatorname{cl} B - \operatorname{int}_c B$ equals to a thin set $\operatorname{cl}_e B - \operatorname{int}_c B$, setting

$$\delta^\circ(x) = \begin{cases} \gamma_x & \text{if } x \in \operatorname{cl} K - \operatorname{cl} B, \\ \min\{\gamma_x, \delta(x)\} & \text{if } x \in \operatorname{int}_c B, \\ 0 & \text{if } x \in \operatorname{cl} B - \operatorname{int}_c B, \end{cases}$$

defines a gage δ° in K. Let $\eta^\circ = \{\eta_2, \eta_3, \ldots\}$ and use Lemma 2.3 to find a δ°-fine (τ, η°)-approximating dyadic partition Q in K. Then $P = \{(B \cap L, x) : (L, x) \in Q,\ x \in \operatorname{int}_c B\}$ is a δ-fine β-regular partition in B and $B - \bigcup P = B \cap (K - \bigcup Q)$. As $K - \bigcup Q$ is the union of disjoint BV sets $D_2, \ldots, D_k$ with $\|D_j\| < 1/\tau$ and $|D_j| < \eta_j$, $j = 2, \ldots, k$, an easy calculation shows that $P \in \Pi(A, \kappa; \delta, \eta)$ for $\kappa = \min\{\beta, \tau/(1+2\tau)\}$.

An arbitrary BV set A is the union of disjoint BV sets $A_1, \ldots, A_n$ whose perimeters are less than or equal to 2. For $i = 0, \ldots, n-1$, let $\eta^i = \{\eta_{nj-i}\}_j$ and use the first part of the proof to find a $P_i \in \Pi(A_i, \kappa; \delta, \eta^i)$. It is clear that $P = \bigcup_{i=1}^{n} P_i$ belongs to $\Pi(A, \kappa; \delta, \eta)$.

3. The integral

If A is a BV set and f is a function on $\operatorname{cl}_e A$, we set

$$\sigma(f, P) = \sum_{i=1}^{p} f(x_i)|A_i|$$

for each partition $P = \{(A_1, x_1), \ldots, (A_p, x_p)\}$ in A.

Definition 3.1. Let A be a BV set and let f be a function on $\operatorname{cl}_e A$. We say that f is *integrable* in A if there is a real number I having the following property: given $\varepsilon > 0$, we can find a gage δ in A and a caliber η so that $|\sigma(f, P) - I| < \varepsilon$ for each $P \in \Pi(A, \varepsilon; \delta, \eta)$.

The family of all integrable functions in a BV set A is denoted by $\mathscr{I}(A)$. It follows from Proposition 2.5 that the number I of Definition 3.1 is determined uniquely by $f \in \mathscr{I}(A)$. We call it the *integral* of f over A, denoted by $\int_A f$.

Let A be a BV set. A *division* of A is a finite disjoint family of BV sets whose union is A. A function F on BV_A is called:

(i) *additive* if $F(A) = \sum_{D \in \mathscr{D}} F(D)$ for each division $\mathscr{D}$ of A;
(ii) *continuous* if given $\varepsilon > 0$, there is a $\nu > 0$ such that $|F(B)| < \varepsilon$ for each $B \in BV_A$ with $\|B\| < 1/\varepsilon$ and $|B| < \nu$.

Proposition 3.2. *Let $A \in BV$ and $f \in \mathscr{I}(A)$. Then $f_B = f \restriction \mathrm{cl}_e B$ belongs to $\mathscr{I}(B)$ for each $B \in BV_A$, and the map $F : B \mapsto \int_B f_B$ is an additive continuous function on BV_A.*

Proof. Choose a positive $\varepsilon \leq \kappa$ where κ is the constant from Proposition 2.5. There is a gage δ in A and a caliber η such that $|\sigma(f, P) - \int_A f| < \varepsilon/2$ for each $P \in \Pi(A, \varepsilon; \delta, \eta)$.

Let $\eta^e = \{\eta_{2j}\}$ and $\eta^o = \{\eta_{2j-1}\}$. Given a $B \in BV_A$, select partitions Q_1 and Q_2 in $\Pi(B, \varepsilon; \delta, \eta^e)$. Using Proposition 2.5, find a partition $Q \in \Pi(A - B, \varepsilon; \delta, \eta^o)$. Now $P_i = Q_i \cup Q$, $i = 1, 2$, belongs to $\Pi(A, \varepsilon; \delta, \eta)$ and

$$|\sigma(f, Q_1) - \sigma(f, Q_2)| = |\sigma(f, P_1) - \sigma(f, P_2)|$$
$$\leq \left|\sigma(f, P_1) - \int_A f\right| + \left|\int_A f - \sigma(f, P_2)\right| < \varepsilon.$$

Thus f_B belongs to $\mathscr{I}(B)$ since it satisfies Cauchy's test for integrability (cf. [P$_2$, Lemma 7.4]).

If $\{A_1, \dots, A_n\}$ is a division of A, let $\eta^k = \{\eta_{nj-k}\}_j$, $k = 0, \dots, n-1$. By the first part of the proof and Proposition 2.5, there is a $P_k \in \Pi(A_k, \varepsilon; \delta, \eta^k)$ such that $|\sigma(f, P_k) - F(A_k)| < \varepsilon/(2n)$. Because $P = \bigcup_{k=1}^n P_k$ belongs to $\Pi(A, \varepsilon; \delta, \eta)$, we have

$$\left|F(A) - \sum_{k=1}^{n} F(A_k)\right| \leq |F(A) - \sigma(f, P)| + \sum_{k=1}^{n} |\sigma(f, P_k) - F(A_k)| < \varepsilon,$$

and the additivity of F follows from the arbitrariness of ε.

Choose a $B \in BV_A$ with $\|B\| < 1/\varepsilon$ and $|B| < \eta_1$. Let $C = A - B$, $\eta^\circ = \{\eta_2, \eta_3, \dots\}$, and let $Q \in \Pi(C, \varepsilon; \delta, \eta^\circ)$ be such that $|\sigma(f, Q) - F(Q)| < \varepsilon/2$. Since Q also belongs to $\Pi(A, \varepsilon; \delta, \eta)$, we obtain

$$|F(B)| = |F(A) - F(C)| \leq |F(A) - \sigma(f, Q)| + |\sigma(f, Q) - F(C)| < \varepsilon$$

which establishes the continuity of F.

Theorem 3.3. *Let $A \in BV$ and let f be a function on $\mathrm{cl}_e A$. Then $f \in \mathscr{I}(A)$ if and only if there is an additive continuous function F on BV_A which satisfies the following condition: given $\varepsilon > 0$, we can find a gage δ in A so that*

$$\sum_{i=1}^{p} \big| f(x_i)|A_i| - F(A_i)\big| < \varepsilon$$

for each ε-regular δ-fine partition $\{(A_1, x_1), \dots, (A_p, x_p)\}$ in A. In particular, we have $\int_A f = F(A)$.

Proof. Assume first that $f \in \mathscr{I}(A)$ and let $F(B) = \int_B f$ for each $B \in BV_A$. By Proposition 3.2, F is an additive continuous function on BV_A. Choose a positive $\varepsilon \leq \kappa$ where κ is the constant from Proposition 2.5. There is a gage δ in A and a caliber η such that $|\sigma(f, P) - F(A)| < \varepsilon/3$ for each $P \in \Pi(A, \varepsilon; \delta, \eta)$. Let $\eta^e = \{\eta_{2j}\}$ and $\eta^o = \{\eta_{2j-1}\}$. It follows from Proposition 2.5 that each ε-regular δ-fine partition in A is a subset of a partition from $\Pi(A, \varepsilon; \delta, \eta^o)$. Thus it suffices to prove the required inequality for a partition $\{(A_1, x_1), \dots, (A_p, x_p)\}$ in $\Pi(A, \varepsilon; \delta, \eta^o)$.

To this purpose let $\eta^i = \{\eta_{2pj-2i}\}_j$, $i = 0, \dots, p-1$, and use Propositions 2.5 and 3.2 to find a $P_i \in \Pi(A_i, \varepsilon; \delta, \eta^i)$ so that $|\sigma(f, P_i) - F(A_i)| < \varepsilon/(3p)$. After a suitable reordering there is a nonnegative integer $k \le p$ such that $f(x_i)|A_i| \ge F(A_i)$ for $i = 1, \dots, k$ and $f(x_i)|A_i| < F(A_i)$ for $i = k+1, \dots, p$. By our choice of the η^i's, it is easy to see that

$$P_+ = \{(A_1, x_1), \dots, (A_k, x_k)\} \cup \bigcup_{i=k+1}^{p} P_i,$$

$$P_- = \{(A_{k+1}, x_{k+1}), \dots, (A_p, x_p)\} \cup \bigcup_{i=1}^{k} P_i$$

belong to $\Pi(A, \varepsilon; \delta, \eta)$. Hence

$$\frac{\varepsilon}{3} > \sigma(f, P_+) - F(A) = \sum_{i=1}^{k} \bigl|f(x_i)|A_i| - F(A_i)\bigr| + \sum_{i=k+1}^{p} [\sigma(f, P_i) - F(A_i)]$$

$$\ge \sum_{i=1}^{k} \bigl|f(x_i)|A_i| - F(A_i)\bigr| - \frac{\varepsilon(p-k)}{3p}$$

and similarly

$$\frac{\varepsilon}{3} > \sum_{i=k+1}^{p} \bigl|f(x_i)|A_i| - F(A_i)\bigr| - \frac{\varepsilon k}{3p}.$$

The desired inequality is obtained by adding the last two inequalities.

Conversely, suppose that an additive continuous function F on BV_A satisfying the conditions of the theorem exists. Choose an $\varepsilon > 0$ and find a gage δ in A so that

$$\sum_{i=1}^{p} \bigl|f(x_i)|A_i| - F(A_i)\bigr| < \frac{\varepsilon}{2}$$

for each ε-regular δ-fine partition $\{(A_1, x_1), \dots, (A_p, x_p)\}$ in A. Since F is continuous, there is a caliber η such that $|F(B)| < \varepsilon 2^{-j-1}$ for each $B \in BV_A$ with $\|B\| < 1/\varepsilon$ and $|B| < \eta_j$, $j = 1, 2, \dots$. Now given a partition $P = \{(A_1, x_1), \dots, (A_p, x_p)\}$ in $\Pi(A, \varepsilon; \delta, \eta)$, the set $A - \bigcup P$ is the union of disjoint BV sets $B_1, \dots, B_k$ such that $\|B_j\| < 1/\varepsilon$ and $|B_j| < \eta_j$ for $j = 1, \dots, k$. Thus

$$|\sigma(f, A) - F(A)| \le \sum_{i=1}^{p} \bigl|f(x_i)|A_i| - F(A_i)\bigr| + \sum_{j=1}^{k} |F(B_j)| < \frac{\varepsilon}{2} + \sum_{j=1}^{k} \varepsilon 2^{-j-1} < \varepsilon,$$

and we see that $f \in \mathscr{I}(A)$ and $\int_A f = F(A)$.

From the previous theorem and [P_2, Propositions 7.7 and 7.8] we obtain immediately the following corollary.

Corollary 3.4. *The integral of Definition* 3.1 *coincides with the tight variational integral defined in* [P_2, Remark 5.2, 4(a)].

Each equivalence class of BV sets modulo, the sets of Lebesgue measure zero contains a unique essentially closed set. In dimension one such a set is a finite union of nondegenerate compact intervals (see [V, §6]). This observation enables us to present a particularly simple definition of the integral when $m = 1$—an assumption we shall make for the remainder of the paper.

By an *interval* we mean a compact interval $[a, b] \subset \mathbf{R}$ with $a < b$. A *figure* is a finite union of intervals, and the family of all figures is denoted by $\mathcal{F}$. We say that figures A and B *overlap* if $|A \cap B| > 0$. An *$\mathcal{F}$-partition* of a figure A is a collection $\{(A_1, x_1), \dots, (A_p, x_p)\}$ such that $A_1, \dots, A_p$ are nonoverlapping figures, $\bigcup_{i=1}^p A_i = A$, and $x_i \in A_i$, $i = 1, \dots, p$. The ε-regular and δ-fine $\mathcal{F}$-partitions of a figure A are defined in the obvious way. For $\mathcal{F}$-partitions, the obvious meaning is also given to the symbol $\sigma(f, P)$.

Proposition 3.5. *Let f be a function defined on a figure A. Then $f \in \mathcal{I}(A)$ with $\int_A f = I$ if and only if for each $\varepsilon > 0$ there is a* positive *gage δ in A such that $|\sigma(f, P) - I| < \varepsilon$ for every ε- regular δ-fine $\mathcal{F}$-partition P of A.*

Proof. Let $f \in \mathcal{I}(A)$ and $\varepsilon > 0$. There is a gage δ in A and a caliber η such that $|\sigma(f, Q) - \int_A f| < \varepsilon/2$ for each $Q \in \Pi(A, \varepsilon/2; \delta, \eta)$. As $m = 1$, the gage δ is positive outside a countable set $N = \{z_2, z_3, \dots\}$. With no loss of generality we may assume that $\eta_j|f(z_j)| \le \varepsilon 2^{-j}$ for $j = 2, 3, \dots$. We define a positive gage δ° in A by setting

$$\delta^\circ = \begin{cases} \delta(x) & \text{if } x \in A - N, \\ \eta_j & \text{if } x = z_j,\ j = 2, 3, \dots, \end{cases}$$

and choose an ε-regular δ°-fine $\mathcal{F}$-partition $P = \{(A_1, x_1), \dots, (A_p, x_p)\}$ of A. Clearly, $Q = \{(\operatorname{int} A_i, x_i) : x_i \in A - N\}$ is an ε-regular δ-fine partition in A. To show that Q is also $(\varepsilon/2, \eta)$-approximating, let $B_1 = \bigcup_{i=1}^p (\operatorname{bd} A_i)$ and $B_j = \bigcup\{\operatorname{int} A_i : x_i = z_j\}$, $j = 2, 3, \dots$. The sets B_j are disjoint, $|B_j| < \eta_j$ for $j = 1, 2, \dots$, and $A - \bigcup Q = \bigcup_{j=1}^k B_j$ for an integer $k \ge 1$. Moreover, $\|B_j\| < 2/\varepsilon$ since $r(A_i) > \varepsilon$ and the map $i \mapsto x_i$ is at most two-to-one. Thus

$$\left|\sigma(f, P) - \int_A f\right| \le \sum_{j=2}^k |f(z_j)| \cdot |B_j| + \left|\sigma(f, Q) - \int_A f\right| < \sum_{j=2}^k \varepsilon 2^{-j} + \frac{\varepsilon}{2} < \varepsilon.$$

To prove the converse, for every figure $B \subset A$, denote by $F(B)$ a real number which satisfies the following condition: given an $\varepsilon > 0$, there is a positive gage δ in B such that $|\sigma(f, Q) - F(B)| < \varepsilon$ for each ε-regular δ-fine $\mathcal{F}$-partition Q of B. If the condition of the proposition is satisfied, it is completely routine to show that a unique number $F(B)$ exists for each figure $B \subset A$, and that the map $B \mapsto F(B)$ extends uniquely to an additive function F on BV_A. The following Henstock lemma (cf. [P_1, Lemma 2.5]) is also obtained by a standard argument.

Given $\varepsilon > 0$, there is a positive gage δ in A such that

$$\sum_{i=1}^p \big|f(x_i)|A_i| - F(A_i)\big| < \varepsilon$$

for each ε-regular δ-fine partition $\{(A_1, x_1), \dots, (A_p, x_p)\}$ in A.

Therefore, in view of Theorem 3.3, it suffices to show that F is a continuous function. Choose an interval $[a, b]$ containing A, and for each $x \in A$ let $\varphi(x) = F(A \cap [a, x])$. If $B \in BV_A$ and $[a_i, b_i]$, $i = 1, \dots, n$, are the connected components of $\mathrm{cl}_e B$, then $F(B) = \sum_{i=1}^{n}[\varphi(b_i) - \varphi(a_i)]$. Now given $\varepsilon > 0$ and $x \in A$, it follows from the Henstock lemma quoted above that there is a positive $\nu < \varepsilon/[|f(x)| + 1]$ such that $\left|f(x)(y-x) - [\varphi(y) - \varphi(x)]\right| < \varepsilon$ for each $y \in A$ with $|y - x| < \nu$. Hence

$$|\varphi(y) - \varphi(x)| < |f(x)| \cdot |y - x| + \varepsilon < 2\varepsilon$$

whenever $y \in A$ and $|y - x| < \nu$, and we see that φ is continuous at x. It follows that φ is uniformly continuous in A, and this implies the continuity of F.

Note. In dimension one, the integral lies properly in between the Lebesgue and Denjoy-Perron integrals (see [P_2, Proposition 6.8]). A detailed analysis of this situation is given in the forthcoming paper [B-G-P].

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