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ε-SELECTIONS

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ABSTRACT. Some complete and some partial characterizations of continua are obtained in terms of the existence of an ε-selection on one or another of their hyperspaces for each $\varepsilon > 0$.

1. INTRODUCTION

Throughout this paper, X denotes a continuum (nonempty, compact, connected, metric space), d denotes a metric for X,

$$d(A, B) = \inf\{d(a, b): a \in A \text{ and } b \in B\}$$

whenever A and B are nonempty subsets of X (if $A = \{p\}$, we write $d(p, B)$), 2^X is the hyperspace of all nonempty compact subsets of X with the Hausdorff metric H_d [9] and

$C(X) = \{A \in 2^X: A \text{ is a continuum}\}$;

$F_n(X) = \{A \in 2^X: A \text{ has at most } n \text{ points}\}$, $n = 1, 2, \ldots$.

Let $\mathfrak{C} \subset 2^X$. If $\varepsilon > 0$, then an *ε-selection on* $\mathfrak{C}$ is a continuous function $\sigma: \mathfrak{C} \to X$ such that $d(A, \sigma(A)) < \varepsilon$ for all $A \in \mathfrak{C}$. Clearly, σ is an ε-selection for every $\varepsilon > 0$ if and only if σ is what is called a (continuous) *selection*.

We note the following known results about selections. There is a selection on $F_2(X)$ or on 2^X if and only if X is an arc or a point ([7, 8]). If there is a selection on $C(X)$, then X is a dendroid [10]. If X is a Peano continuum, then there is a selection on $C(X)$ if and only if X is a dendrite [10].

In this paper, we obtain some characterizations of those continua X for which there are ε-selections on $\mathfrak{C} \subset 2^X$ for each $\varepsilon > 0$. When $\mathfrak{C} = F_2(X)$, we show that such continua must be a-triodic (2.2) and hereditarily unicoherent (2.3). Thus, if X is hereditarily decomposable, X is arc-like (2.4) and, if X is arcwise connected, X is an arc or a point (2.5). When $\mathfrak{C} = 2^X$, we obtain a complete characterization in terms, of interior approximation (2.6). Using this characterization, we give an example of a nondegenerate indecomposable continuum X for which there is an ε-selection on 2^X (hence, on $F_2(X)$) for each

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$\varepsilon > 0$ (2.7). When $\mathfrak{C} = C(X)$, we show that such continua have trivial shape (3.2) and are hereditarily unicoherent (3.5). We then derive some consequences (3.6–3.8) and discuss a possible characterization (3.9).

2. ε-SELECTIONS ON $F_2(X)$ AND 2^X

The following technical lemma is general enough to yield easy proofs of 2.2 and 2.3.

2.1. **Lemma.** *Let X be a continuum such that there is an ε-selection on $F_2(X)$ for each $\varepsilon > 0$. Then, there do not exist three points p_1, p_2, p_3 in X and three subcontinua A_{12}, A_{13}, A_{23} of X such that $p_k \in A_{ij}$ if and only if $k = i$ or $k = j$.*

Proof. Suppose that there do exist such points p_1, p_2, p_3 and subcontinua A_{12}, A_{13}, A_{23}. Let

$$\varepsilon = 2^{-1} \cdot \min\{d(p_k, A_{ij}): k \neq i,\ k \neq j, \text{ and } i, j, k \in \{1, 2, 3\}\}.$$

For each $x \in X$, let $B(x) = \{y \in X: d(x, y) < \varepsilon\}$. Now, let σ be an ε-selection on $F_2(X)$. Then, $\sigma(\{p_1, p_2\}) \in B(p_1) \cup B(p_2)$. Hence, we may assume without loss of generality that

(1) $\sigma(\{p_1, p_2\{) \in B(p_1)$.

Let $E = \{x \in A_{23}: \sigma(p_1, x\}) \in B(p_1)\}$. Note that for all $x \in E$, $B(x) \cap B(p_1) = \varnothing$ and $\sigma(\{p_1, x\}) \in B(p_1) \cup B(x)$.

Hence, using the continuity of σ, it follows easily that E is both open and closed in A_{23}. Thus, since $E \neq \varnothing$ (by (1)) and A_{23} is connected, $E = A_{23}$. Therefore,

(2) $\sigma(\{p_1, p_3\}) \in B(p_1)$.

Letting $F = \{x \in A_{12}: \sigma(\{x, p_3\}) \in B(x)\}$ and using (2), an argument similar to the one used to prove (2) shows that $F = A_{12}$ and, hence,

(3) $\sigma(\{p_2, p_3\}) \in B(p_2)$.

Letting $G = \{x \in A_{13}: \sigma(\{p_2, x\}) \in B(p_2)\}$ and using (3), an argument similar to the one used to prove (2) shows that $G = A_{13}$ and, hence,

(4) $\sigma(\{p_2, p_1\}) \in B(p_2)$.

Since $B(p_1) \cap B(p_2) = \varnothing$, (4) contradicts (1). Therefore, we have proved 2.1.

The notions of triod, a-triodic, and hereditarily unicoherent are as in, e.g., [1]. After proving the following two results, we derive some specific consequences of them.

2.2. **Theorem.** *If X is a continuum such that there is an ε-selection on $F_2(X)$ for each $\varepsilon > 0$, then X is a-triodic.*

Proof. Suppose that X contains a triod T. Let N denote a subcontinuum of T such that $T - N = S_1 \cup S_2 \cup S_3$ where S_i and S_j are nonempty and mutually separated in T for $i \neq j$. For each i, let $p_i \in S_i$ and let C_i denote the component of S_i containing p_i. It follows using Theorem 2 of [6, p. 172] that $C_i \cup N$ is a continuum for each i. For each i, $j \in \{1, 2, 3\}$ such that $i \neq j$, let

$$A_{ij} = C_i \cup N \cup C_j.$$

Then, we see that we have a contradiction to 2.1.

2.3. **Theorem.** *If X is a continuum such that there is an ε-selection on $F_2(X)$ for each $\varepsilon > 0$, then X is hereditarily unicoherent.*

Proof. Suppose that there is a nonunicoherent subcontinuum Y of X. Then, $Y = K \cup L$ where K and L are continua and $K \cap L$ is not connected. Let B and C be two components of $K \cap L$. By using Theorem 2 of [6, p. 172], we see that there are subcontinua D of K and E of L such that

$$D \supset B \neq D, \qquad E \supset B \neq E, \qquad D \cap C = \varnothing, \qquad E \cap C = \varnothing.$$

Hence, there exist $p_1 \in D - L$ and $p_2 \in E - K$. Let $p_3 \in C$. Now, letting

$$A_{12} = D \cup E, \qquad A_{13} = K, \quad \text{and} \quad A_{23} = L,$$

we have a contradiction to 2.1.

With respect to the following corollary, we call to the reader's attention the example in 2.7 and the remark in 2.8.

2.4. **Corollary.** *If X is a hereditarily decomposable continuum such that there is an ε-selection on $F_2(X)$ for each $\varepsilon > 0$, then X is arc-like.*

Proof. Use 2.2 and 2.3, and apply Theorem 11 of [1].

2.5. **Corollary.** *Let X be a nondegenerate arcwise connected continuum. Then, there is an ε-selection on $F_2(X)$ for each $\varepsilon > 0$ if and only if X is an arc.*

Proof. For the "only if part," use 2.2 and 2.3 in order to apply Theorem 3.2 of [11, p. 456]. The "if part" is trivial.

By using our results for $F_2(X)$, we obtain the following characterization concerning the existence of ε-selections on 2^X.

2.6. **Theorem.** *Let X be a nondegenerate continuum. Then: There is an ε-selection on 2^X for each $\varepsilon > 0$ if and only if there is a sequence $\{\varphi_i\}_{i=1}^{\infty}$ of continuous functions $\varphi_i \colon X \to X$ such that $\{\varphi_i\}_{i=1}^{\infty}$ converges uniformly to the identity map on X and $\varphi_i(X)$ is an arc for each i.*

Proof. Assume that there is an ε-selection on 2^X for each $\varepsilon > 0$. Then, since $F_2(X) \subset 2^X$, there is an ε-selection on $F_2(X)$ for each $\varepsilon > 0$. Hence, by 2.2 and 2.3, we have:

(1) X is a-triodic and hereditarily unicoherent.

Now, fix $\varepsilon > 0$. Let σ_ε be an ε-selection on 2^X, and let $A_\varepsilon = \sigma_\varepsilon(2^X)$. Define a map φ_ε on X by

$$\varphi_\varepsilon(x) = \sigma_\varepsilon(\{x\}) \quad \text{for each } x \in X.$$

Clearly, φ_ε is continuous, $d(x, \varphi_\varepsilon(x)) < \varepsilon$ for all $x \in X$, and $\varphi_\varepsilon(X) \subset A_\varepsilon$. Since 2^X is arcwise connected [9, 1.13], A_ε is arcwise connected. Hence, by (1) and [11], A_ε is an arc or a point and, thus, so is $\varphi_\varepsilon(X)$. It now follows from what we have shown about φ_ε that the desired sequence $\{\varphi_i\}_{i=1}^{\infty}$ exists. Conversely, assume that there is a sequence $\{\varphi_i\}_{i=1}^{\infty}$ as in the statement of the theorem. Let $\varepsilon > 0$. Fix i such that

$$d(\varphi_i(x), x) < \varepsilon \quad \text{for all } x \in X,$$

and let $Y = \varphi_i(X)$. Since Y is an arc (by assumption), there is a selection σ on 2^Y. Define $\varphi_i^* \colon 2^X \to 2^Y$ by

$$\varphi_i^*(K) = \{\varphi_i(x) \colon x \in K\} \quad \text{for each } K \in 2^X.$$

Observe that φ_i^* is continuous and that φ_i^* is within ε of the identity map on 2^X. Therefore, it is easy to see that $\sigma \circ \varphi_i^*$ is an ε-selection on 2^X. This completes the proof of 2.6.

2.7. **Example.** We give an example of a nondegenerate indecomposable continuum X for which there is an ε-selection on 2^X for each $\varepsilon > 0$. Let

$$X = \varprojlim \{X_n, f_n\}_{n=1}^{\infty} \quad \text{(inverse limit)}$$

where each $X_n = [0, 1]$ and

$$f_n(t) = \begin{cases} 2t, & 0 \le t \le 1/2, \\ 2 - 2t, & 1/2 \le t \le 1. \end{cases}$$

It is well known that X is an indecomposable continuum (see, e.g., [9, 1.209.3]). In fact, X is the continuum pictured in Figure 4 of [6, p. 205] from which it can be seen that there is a sequence $\{\varphi_i\}_{i=1}^{\infty}$ as in 2.6. More precisely, such a sequence $\{\varphi_i\}_{i=1}^{\infty}$ can also be obtained by using the inverse limit representation for X as follows: For each $i = 1, 2, \ldots$ and each $(x_n)_{n=1}^{\infty} \in X$, let

$$\varphi_i((x_n)_{n=1}^{\infty}) = (y_n)_{n=1}^{\infty} \quad \text{where} \begin{cases} y_n = x_n, & n \le i, \\ y_{i+j} = 2^{-j} \cdot x_i, & j = 1, 2, \ldots. \end{cases}$$

Clearly, each φ_i maps X continuously into X, each $\varphi_i(X)$ is an arc since it is an inverse limit of arcs with homeomorphisms as the bonding maps, and, assuming the usual product metric (so that the projection of X onto X_n is a 2^{-n}-map for each n), each φ_i is within 2^{-i} of the identity map on X (we thank W. J. Charatonik for the formula for φ_i). Therefore by 2.6, there is an ε-selection on 2^X for each $\varepsilon > 0$. We remark that a similar procedure shows that for any inverse limit X of arcs with open (onto) bonding maps, there is an ε-selection on 2^X for each $\varepsilon > 0$.

2.8. *Remark.* It is immediate from 2.6 that the continua X for which there is an ε-selection on 2^X for each $\varepsilon > 0$ must be arc-like. We do not know if the same is true when considering ε-selections on $F_2(X)$-comp., 2.4.

3. ε-SELECTIONS ON $C(X)$

By an ANR we mean a compact, metrizable, absolute neighborhood retract [2]. We say that X *is* crANR provided that every continuous function from X into any ANR is inessential (= homotopic to a constant map). Note the following general lemma which is probably known. It will be applied to the situation in 3.2.

3.1. **Lemma.** *Let $Y \subset Z$ be continua such that Z is crANR. If, for each $\varepsilon > 0$, there is a continuous function $\varphi_\varepsilon \colon Z \to Y$ such that $\varphi_\varepsilon | Y \colon Y \to Y$ is within ε of the identity map on Y, then Y is crANR.*

Proof. Let A be a (compact) ANR, and let $f \colon Y \to A$ be continuous. There is an $\eta > 0$ such that if $g \colon Y \to A$ is continuous and $\rho(f, g) < \eta$ ($\rho =$ supremum metric), then f and g are homotopic (as follows from 1.1 of [2, p. 101]). Since f is uniformly continuous, there is a $\delta > 0$ such that if $y_1, y_2 \in Y$ are less than δ apart, then $f(y_i)$ and $f(y_2)$ are less than η apart. Now, let

$\varphi_\delta: Z \to Y$ be as above. Since Z is crANR, $f \circ \varphi_\delta: Z \to A$ is inessential. Hence,

$$g = f \circ \varphi_\delta | Y: Y \to A \quad \text{is inessential.}$$

Since $\varphi_\delta | Y: Y \to Y$ is within δ of the identity map on Y, clearly $\rho(f, g) < \eta$. Therefore, f is inessential.

3.2. **Theorem.** *If X is a continuum such that there is an ε-selection on $C(X)$ for each $\varepsilon > 0$, then X is crANR (equivalently, X has trivial shape).*

Proof. By 1.181 of [9], $C(X)$ is crANR. Let $Y = F_1(X)$, $Z = C(X)$, and, for any given ε-selection σ_ε on Z, define $\varphi_\varepsilon: Z \to Y$ by

$$\varphi_\varepsilon(A) = \{\sigma_\varepsilon(A)\} \quad \text{for all } A \in C(X) = Z.$$

Then by 3.1, Y is crANR. Therefore, since Y is homeomorphic to X, X is crANR.

Now, note the following general result. A special case (3.4) is used in the proof of our next main result (3.5).

3.3. **Proposition.** *Let X be a continuum such that there is an ε-selection on $C(X)$ for each $\varepsilon > 0$. Assume that Y is a subcontinuum of X such that for each $\varepsilon > 0$, there exist an open subset U_ε of X, with $Y \subset U_\varepsilon$, and a continuous function $f_\varepsilon: U_\varepsilon \to Y$ such that*

$$d(f_\varepsilon(x), x) < \varepsilon \quad \textit{for all } x \in U_\varepsilon.$$

Then, there is an ε-selection on $C(Y)$ for each $\varepsilon > 0$.

Proof. Let $\varepsilon > 0$. Let U_ε and f_ε be as above. Then, by using the compactness of Y, there exists δ, $0 < \delta < \varepsilon$, such that $Y(\delta) \subset U_\varepsilon$ where

$$Y(\delta) = \{x \in X: d(x, Y) < \delta\}.$$

Now, let σ be a δ-selection on $C(X)$. Since $Y(\delta) \subset U_\varepsilon$, clearly $f_\varepsilon \circ (\sigma | C(Y))$ is defined on all of $C(Y)$. Let $A \in C(Y)$. Since σ is a δ-selection,

$$d(a, \sigma(A)) < \delta \quad \text{for some } a \in A$$

and, from our assumption about f_ε,

$$d(\sigma(A), f_\varepsilon(\sigma(A))) < \varepsilon.$$

Thus, since $\delta < \varepsilon$, $d(a, f_\varepsilon(\sigma(A))) < 2\varepsilon$. Therefore, we have proved that $f_\varepsilon \circ (\sigma | C(Y))$ is a 2ε-selection on $C(Y)$. This proves 3.3.

The following corollary is immediate from 3.3.

3.4. **Corollary.** *Let X be a continuum such that there is an ε-selection on $C(X)$ for each $\varepsilon > 0$. If Y is a subcontinuum of X such that Y is a retract of a neighborhood of Y in X, then there is an ε-selection on $C(Y)$ for each $\varepsilon > 0$.*

We remark that the proof of 3.3 shows that 3.3 and 3.4 remain true with $C(X)$ and $C(Y)$ replaced by $F_2(X)$ and $F_2(Y)$ respectively.

Three corollaries to the following theorem are in 3.6–3.8.

3.5. **Theorem.** *If X is a continuum such that there is an ε-selection on $C(X)$ for each $\varepsilon > 0$, then X is hereditarily unicoherent.*

Proof. Suppose that X contains a nonunicoherent subcontinuum Y. Then, $Y = Y_1 \cup Y_2$ where Y_1 and Y_2 are continua and $Y_1 \cap Y_2 = M \cup N$ with M and N being nonempty, disjoint, compact sets. It is easy to see that there is an $\varepsilon > 0$ such that if Z is a subcontinuum of X with $d(Z, M) < \varepsilon$ and $d(Z, N) < \varepsilon$, then there exists $z \in Z$ such that $d(z, Y_i) \geq \varepsilon$ for some $i = 1$ or 2 (the proof can be done by contradiction using the compactness of $C(X)$ [9, p. 7]). Let σ be an ε-selection on $C(X)$. Let $p \in M$ and $q \in N$. For each $i = 1$ and 2, $C(Y_i)$ is arcwise connected [9, 1.13] and, hence, there is an arc $\mathfrak{C}_i$ in $C(Y_i)$ from $\{p\}$ and $\{q\}$. Thus, for each $i = 1$ and 2, there is an arc A_i in $\sigma(\mathfrak{C}_i)$ from $\sigma(\{p\})$ to $\sigma(\{q\})$. Now, let $Z = A_1 \cap A_2$. We show that Z is not connected. First, note that $Z \subset \sigma(\mathfrak{C}_i)$ for each i. Thus, since $\mathfrak{C}_i \subset C(Y_i)$ for each i and σ is an ε-selection, we see that

(1) $d(z, Y_i) < \varepsilon$ for each $z \in Z$ and $i = 1, 2$.

Note that $\sigma(\{p\}), \sigma(\{q\}) \in Z$, and recall that $p \in M$ and $q \in N$. Thus, since σ is an ε-selection, we see that

(2) $d(Z, M) < \varepsilon$ and $d(Z, N) < \varepsilon$.

By (1), (2), and our choice of ε, Z is not connected. Therefore, since A_1 and A_2 are arcs and $Z = A_1 \cap A_2$, it follows easily that $A_1 \cup A_2$ contains a simple closed curve S. Since S is an ANR, we see from 3.4 that there is an ε-selection on $C(S)$ for each $\varepsilon > 0$. However, this contradicts 3.2. Therefore, X is hereditarily unicoherent.

Recall that a *dendriod* is an arcwise connected, hereditarily unicoherent continuum. Thus by 3.5, we have

3.6. **Corollary.** *Let X be an arcwise connected continuum. If there is an ε-selection on $C(X)$ for each $\varepsilon > 0$, then X is a dendroid.*

We also have the following characterization for Peano continua.

3.7. **Corollary.** *If X is a Peano continuum, then there is an ε-selection on $C(X)$ for each $\varepsilon > 0$ if and only if X is a dendrite.*

Proof. The "only if part" follows from 3.5 (or 3.6), and the "if part" is due to the fact that there is a selection on $C(X)$ whenever X is a dendrite ([10, p. 371] or [3]).

The following corollary is discuseed in 3.9.

3.8. **Corollary.** *Let X be a continuum. If there is an ε-selection on $C(X)$ for each $\varepsilon > 0$, then there is a sequence $\{\varphi_i\}_{i=1}^{\infty}$ of continuous functions $\varphi_i\colon X \to X$ such that $\{\varphi_i\}_{i=1}^{\infty}$ converges uniformly to the identity map on X and $\varphi_i(X)$ is a dendroid for each i.*

Proof. Fix $\varepsilon > 0$. Let σ_ε be an ε-selection on $C(X)$, and let $A_\varepsilon = \sigma_\varepsilon(C(X))$. Define φ_ε as in the proof of 2.6. Observe that since A_ε is arcwise connected [9, 1.13] and hereditarily unicoherent (3.5), A_ε is a dendroid and, thus, since $\varphi_\varepsilon(X) \subset A_\varepsilon$, $\varphi_\varepsilon(X)$ is a dendroid. Clearly, as in the proof of 2.6, φ_ε is a continuous function within ε of the identity map on X. The corollary now follows.

3.9 *Remark.* We do not know if the converses of 3.6 and 3.8 are true. If the converse of 3.6 is true, then so is the converse of 3.8 (by only minor changes in the second half of the proof of 2.6). In [5, p. 261], it was asked if every dendroid can be ε-retracted onto a tree for each $\varepsilon > 0$. If this question is answered affirmatively, then the converse of 3.6 follows easily and 3.8 becomes a characterization. We remark that the answer to the question is affirmative for any fan [4, p. 120] and for all smooth dendroids [5, p. 261].

3.10. *Remark.* The notion of an ε-selection can be generalized to topological spaces using open covers. For example: If $\mathscr{U}$ is an open cover of X and $\mathfrak{C} \subset 2^X$ with the Vietoris topology [9], then a continuous function $\sigma: \mathfrak{C} \to X$ is called a *$\mathscr{U}$-selection* provided that for each $A \in \mathfrak{C}$, $\sigma(A) \in U$ for some $U \in \mathscr{U}$ such that $U \cap A \neq \varnothing$. Using this idea, many of the results in this paper have straightforward analogues in the setting of Hausdorff continua.

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