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COMPOSITION OPERATORS ON POTENTIAL SPACES

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ABSTRACT. By a result of B. Dahlberg, the composition operators $T_H f = H \circ f$ need not be bounded on some of the Sobolev spaces (or spaces of Bessel potentials) even for very smooth functions $H = H(t)$, $H(0) = 0$, unless of course, $H(t) = ct$. In this note a natural domain is found for T_H that is, in a sense, maximal and on which the $\{T_H\}$ form an algebra of bounded operators. Here the functions $H(t)$ need not be bounded though they are required to have a sufficient number of bounded derivatives.

1. INTRODUCTION

For $H\colon \mathbb{R} \to \mathbb{R}$, let T_H be the associated composition operator that takes $f\colon \mathbb{R}^n \to \mathbb{R}$ to $H(f)$. We consider the action of T_H on the Bessel potential spaces $L^p_\alpha(\mathbb{R}^n)$ for $\alpha > 0$ and $1 < p < \infty$. We say that H is α-admissible if

$$H(0) = 0$$

and

$$M \equiv \max_k \sup_{t\in\mathbb{R}} |H^{(k)}(t)| < \infty,$$

where the max is taken over $k \in \{1, \dots, m\}$ if $\alpha = m \in \mathbb{Z}^+$, and over $k \in \{1, \dots, m+1\}$ if $m < \alpha < m+1$, $m \in \mathbb{Z}^+$. If in addition, H is bounded (an assumption not required for our results), then T_H is often referred to as a *smooth truncation operator*; cf. [2].

Let $\dot{L}^p_\alpha$, $\alpha > 0$, $1 < p < \infty$, be the Riesz potential space, i.e. the homogeneous counterpart to the inhomogeneous space L^p_α. Complete definitions are given below; for now we note that if $\alpha = m \in \mathbb{Z}^+$, then

$$\|f\|_{\dot{L}^p_m} \approx \sum_{|\gamma|=m} \|D^\gamma f\|_{L^p},$$

while

$$\|f\|_{L^p_m} \approx \sum_{|\gamma|\le m} \|D^\gamma f\|_{L^p} \approx \|f\|_{L^p} + \|f\|_{\dot{L}^p_m};$$

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see (2.5) and (2.6) below. Our main result is

Theorem A. *Suppose* $\alpha > 1$, $1 < p < \infty$, *and* H *is* α-*admissible. If* $f \in L^p_\alpha \cap \dot{L}^{\alpha p}_1$, *then* $H(f) \in L^p_\alpha \cap \dot{L}^{\alpha p}_1$ *and*

$$\|H(f)\|_{L^p_\alpha} \leq cM(\|f\|_{L^p_\alpha} + \|f\|^\alpha_{\dot{L}^{\alpha p}_1}), \tag{1.1}$$

and

$$\|H(f)\|_{\dot{L}^{\alpha p}_1} \leq M\|f\|_{\dot{L}^{\alpha p}_1} \tag{1.2}$$

The nonlinearity of estimate (1.1) is a reflection of the nonlinearity of the operator T_H; cf. (1) of [3] and (1.6) of [2].

For $0 < \alpha \leq 1$ and $1 < p < \infty$, $H(f) \in L^p_\alpha$ for all $f \in L^p_\alpha$ and any α-admissible H. In fact, we see below in §3 (see (3.1), (3.2), and (3.3)) that in this case, we have

$$\|H(f)\|_{L^p_\alpha} \leq cM\|f\|_{L^p_\alpha}.$$

And, in particular, it is easy to see that (1.2) holds. We also have $H(f) \in L^p_\alpha$ for all $f \in L^p_\alpha$ if $\alpha \geq n/p$, as noted in [1], or as a consequence of Theorem A, since the Sobolev (potential) imbedding theorem implies that L^p_α is continuously imbedded into $\dot{L}^{\alpha p}_1$ whenever $\alpha p \geq n$, $\alpha > 1$ and $1 < p < \infty$. However, for $\alpha \in \mathbb{Z}^+$, Dahlberg [5] has shown that if $1 < \alpha < n/p$, $1 < p < \infty$, and $H(f) \in L^p_\alpha$ for every $f \in L^p_\alpha$, H α-admissible, then $H(t) = ct$ for some $c \in \mathbb{R}$.

In view of Dahlberg's negative result, a natural question is to determine what extra conditions of $f \in L^p_\alpha$ guarantee that $H(f) \in L^p_\alpha$ for α-admissible H. The first result of this type is obtained from the Gagliardo–Nirenberg lemma (see [7] and Lemma 2.2 below), which implies that $H(f) \in L^p_\alpha$ for every $f \in L^p_\alpha \cap L^\infty$; see [2]. This, along with an extension of the Fefferman–Stein decomposition of BMO, was used in [2] to show that $H(f) \in L^p_\alpha$ for every $f \in L^p_\alpha \cap \mathrm{BMO}$. This result also follows from Theorem A because of the imbedding

$$\dot{L}^p_\alpha \cap \mathrm{BMO} \to \dot{L}^{\alpha p}_1$$

if $\alpha > 1$ and $1 < p < \infty$; see Lemma 2.2 below. It should also be noted that the methods used here to prove Theorem A are considerably simpler than those used in [2]. As for the sharpness of our main result, we also prove

Theorem B. *Let* $\alpha = m \in \mathbb{Z}^+$, $m \geq 1$ *and suppose that* $f: \mathbb{R}^n \to \mathbb{R}$ *and* $H(f) \in L^p_m$ *for all* m-*admissible* H. *Then* $f \in L^p_m \cap \dot{L}^{mp}_1$.

In §2 we give the necessary preliminaries, definitions, and statements for the needed lemmas. The proofs of Theorems A and B are given in §3. We discuss the proofs of the lemmas in §4.

2. Preliminaries

For $\alpha > 0$, let G_α be the Bessel potential kernel of order α, i.e. the positive $L^1(\mathbb{R}^n)$ function that satisfies $\widehat{G}_\alpha(\xi) = (1 + |\xi|^2)^{-\alpha/2}$ for $\xi \in \mathbb{R}^n$, $\hat{}$ denoting the Fourier transform on L^1. For $1 < p < \infty$, let $L^p_\alpha = L^p_\alpha(\mathbb{R}^n)$ be the space of all $f = G_\alpha * g$, for some $g \in L^p(\mathbb{R}^n)$; in this case $\|f\|_{L^p_\alpha} = \|g\|_{L^p}$. The homogeneous space $\dot{L}^p_\alpha = \dot{L}^p_\alpha(\mathbb{R}^n)$ is the space of Riesz potentials $f = I_\alpha * g$, $g \in L^p$. Here $\widehat{I}_\alpha(\xi) = |\xi|^{-\alpha}$. However, for simplicity below, we use the equivalent Littlewood–Paley characterization of $\dot{L}^p_\alpha$ for its definition; see [8, 11].

Let $\phi: \mathbb{R}^n \to \mathbb{C}$ satisfy

$$\phi \in \mathscr{S}, \tag{2.1}$$

$$\operatorname{supp} \hat{\phi} \subseteq \{\xi : 1/2 \le |\xi| \le 2\}, \tag{2.2}$$

$$|\hat{\phi}(\xi)| \ge c > 0 \quad \text{if } 3/5 \le |\xi| \le 5/3, \tag{2.3}$$

and

$$\sum_{\nu \in \mathbb{Z}} \hat{\phi}(2^\nu \xi) = 1 \quad \text{for } \xi \ne 0. \tag{2.4}$$

For $\nu \in \mathbb{Z}$, set $\phi_\nu(x) = 2^{\nu n}\phi(2^\nu x)$. For $1 < p < \infty$ and $\alpha \ge 0$, let

$$\|f\|_{\dot{L}^p_\alpha} = \left\| \left(\sum_{\nu \in \mathbb{Z}} |2^{\nu\alpha}\phi_\nu * f|^2 \right)^{1/2} \right\|_{L^p}.$$

When $\alpha = 0$, $\dot{L}^p_0 = L^p$, $1 < p < \infty$ by Littlewood-Paley theory [8, 11]. With this definition, the following lemma is a simple consequence of Hölder's inequality.

Lemma 2.1. *Suppose* $0 \le \alpha_1 < \alpha_2 < \infty$, $0 < p_1, p_2 < \infty$, $0 < \theta < 1$, $\alpha = (1-\theta)\alpha_1 + \theta\alpha_2$, *and* $1/p = (1-\theta)/p_1 + \theta/p_2$. *Then*

$$\|f\|_{\dot{L}^p_\alpha} \le \|f\|^{1-\theta}_{\dot{L}^{p_1}_{\alpha_1}} \|f\|^{\theta}_{\dot{L}^{p_2}_{\alpha_2}}.$$

In general, $\dot{L}^p_\alpha$ is a space of tempered distributions modulo polynomials. However, in this paper we consider $\|f\|_{\dot{L}^p_\alpha}$ only for functions f belonging to L^q for some $q \in (1, \infty)$. This eliminates any ambiguity regarding polynomials.

The following two facts about $\dot{L}^p_\alpha$ are well known (see e.g. [8] or [11]). For $1 < p < \infty$ and $\alpha > 0$,

$$\|f\|_{L^p_\alpha} \approx \|f\|_{L^p} + \|f\|_{\dot{L}^p_\alpha}, \tag{2.5}$$

while for $1 \le m \le \alpha < m+1$, $m \in \mathbb{Z}$,

$$\|f\|_{\dot{L}^p_\alpha} \approx \sum_{|\gamma|=m} \|D^\gamma f\|_{\dot{L}^p_{\alpha-m}}. \tag{2.6}$$

Here "$\approx$" means that the ratio of the two sides is bounded above and below by finite positive constants independent of f. We are using the standard multi-index notation: $\gamma = (\gamma_1, \dots, \gamma_n)$, $\gamma_i \ge 0$, $\gamma_i \in \mathbb{Z}$, $i = 1, \dots, n$, $|\gamma| = \gamma_1 + \cdots + \gamma_n$, and $D^\gamma = (\partial/\partial x_1)^{\gamma_1} \cdots (\partial/\partial x_n)^{\gamma_n}$. Obviously, (2.5) and (2.6) imply the classical result of Calderón [4]: if $1 < p < \infty$ and $1 \le m \le \alpha < m+1$, $m \in \mathbb{Z}$, then

$$\|f\|_{L^p_\alpha} \approx \|f\|_{L^p} + \sum_{|\gamma|=m} \|D^\gamma f\|_{\dot{L}^p_{\alpha-m}}. \tag{2.7}$$

For $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, let $\|f\|_{\mathrm{BMO}} = \sup_Q (1/(|Q|) \int_Q |f - f_Q|\, dx$, where $f_Q = (1/|Q|) \int_Q f$ and the supremum is over all cubes $Q \subset \mathbb{R}^n$ with sides parallel to the axes.

Lemma 2.2. *Suppose* $0 < \theta < 1$, $\alpha > 0$, *and* $1 < p < \infty$. *Then*

$$\|f\|_{\dot{L}^{p/\theta}_{\alpha\theta}} \le c\|f\|_{\mathrm{BMO}}^{1-\theta} \cdot \|f\|_{\dot{L}^p_\alpha}^{\theta}. \tag{2.8}$$

Clearly $\|f\|_{\mathrm{BMO}} \le 2\|f\|_{L^\infty}$; using this inequality in (2.8) results in the Gagliardo–Nirenberg lemma [7].

In his study of the fractional order potential spaces, Strichartz [10] introduced the operator S_α, defined for $0 < \alpha < 1$ and $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ by

$$S_\alpha f(x) = \left(\int_0^\infty \left[\int_{|y|<1} |f(x+ry) - f(x)|\, dy\right]^2 r^{-1-2\alpha}\, dr\right)^{1/2}.$$

Polking [9] introduced a class of variants of S_α; here we consider only the special case D_t^α, $1 \le t < \infty$, $0 < \alpha < 1$, defined for $f \in L^t_{\mathrm{loc}}(\mathbb{R}^n)$ by

$$D_t^\alpha f(x) = \left(\int_0^\infty \left[\int_{|y|<1} |f(x+ry) - f(x)|^t\, dy\right]^2 r^{-1-2t\alpha}\, dr\right)^{1/2t}.$$

Obviously, $D_1^\alpha f = S_\alpha f$.

Lemma 2.3. *Suppose* $0 < \alpha < 1$ *and* $1 < p < \infty$. *If* $f \in L^q$ *for some* $q \in (1, \infty)$, *then*

$$\|f\|_{\dot{L}^p_\alpha} \approx \|S_\alpha f\|_{L^p}, \tag{2.9}$$

and, if $1 \le t < p$,

$$\|D_t^\alpha f\|_{L^p} \le c(\alpha, p, t)\|f\|_{\dot{L}^p_\alpha}. \tag{2.10}$$

In (2.9) the estimate $\|f\|_{\dot{L}^p_\alpha} \le c(\alpha, p)\|S_\alpha f\|_{L^p}$ holds for any $f \in L^1_{\mathrm{loc}}$; for the converse of (2.9) and (2.10), an assumption like $f \in L^q$ is needed since polynomials have norm zero in $\dot{L}^p_\alpha$ whereas S_α and D_t^α only annihilate constants.

For $h \in \mathbb{R}^n$, set $\Delta_h f(x) = f(x+h) - f(x)$. Following Polking [9], we easily have

$$\Delta_h(fg)(x) = f(x)\Delta_h g(x) + g(x)\Delta_h f(x) + \Delta_h f(x) \cdot \Delta_h g(x). \tag{2.11}$$

As a result, we can estimate the L^p_α norm of a product of functions in terms of their individual norms in other potential spaces. This method gives

Lemma 2.4. *Suppose* $0 < \theta < 1$, $1 < p < \infty$, $k \in \mathbb{Z}$, $k \ge 2$; *let* $p_i \ge 1$, $\gamma_i \ge 1$, $i = 1, \dots, k$, *and* $1/p_j + \sum_{i\ne j} 1/\gamma_i = 1$, *for* $j = 1, \dots, k$. *Then*

$$\left\|\prod_{i=1}^k f_i\right\|_{\dot{L}^p_\theta} \le c\sum_{j=1}^k \|f_j\|_{\dot{L}^{pp_j}_\theta} \cdot \prod_{\substack{i=1\\ i\ne j}}^k \|f_i\|_{L^{\gamma_i p}}.$$

3. Proofs of the theorems

Proof of Theorem A. Since $H(0) = 0$, we have $|H(f)(x)| = |H(f(x)) - H(0)| \le \|H'\|_{L^\infty} \cdot |f(x)|$. Hence

$$\|H(f)\|_{L^p} \le \|H'\|_{L^\infty}\|f\|_{L^p}, \qquad 0 < p \le \infty. \tag{3.1}$$

For $i = 1, \ldots, n$, let $D_i = \partial/\partial x_i$. Then $D_i H(f) = H'(f) \cdot D_i f$, and hence by (2.6),

$$\|H(f)\|_{\dot{L}_1^p} \leq c\|H'\|_{L^\infty} \|f\|_{\dot{L}_1^p}, \qquad 1 < p < \infty. \tag{3.2}$$

If $0 < \alpha < 1$, applying the mean value theorem to $H(f(x+ry)) - H(f(x))$ gives $S_\alpha H(f) \leq \|H'\|_{L^\infty} S_\alpha(f)$. Therefore by (2.9),

$$\|H(f)\|_{\dot{L}_\alpha^p} \leq c\|H'\|_{L^\infty} \|f\|_{\dot{L}_\alpha^p}. \tag{3.3}$$

(Estimates (3.1)–(3.3) give the analogue of Theorem A for $0 < \alpha \leq 1$, $1 < p < \infty$, as noted in the introduction.)

Now, suppose that $\alpha = m \in \mathbb{Z}$, $m \geq 2$. For a multi-index γ with $|\gamma| = m$ and $k \in \{1, \ldots, m\}$, let $A(k, \gamma)$ be the collection of all ordered k-tuples of multi-indices $\overline{\beta} = \{\beta_1, \ldots, \beta_k\}$ with $\beta_i \in (\mathbb{Z}^+)^n$, $|\beta_i| \geq 1$, and $\sum_1^k \beta_i = \gamma$. By the Leibniz rule, there exist scalars $c(\overline{\beta}, k, \gamma)$ such that

$$D^\gamma H(f) = \sum_{k=1}^m H^{(k)}(f) \sum_{\overline{\beta} \in A(k,\gamma)} c(\overline{\beta}, k, \gamma) D^{\beta_1} f \cdots D^{\beta_k} f. \tag{3.4}$$

For $\gamma_i = m/|\beta_i|$, $|\beta_1| + \cdots + |\beta_k| = m$, Hölder's inequality together with (2.6) gives

$$\left\| \prod_{i=1}^k D^{\beta_i} f \right\|_{L^p} \leq \prod_{i=1}^k \|D^{\beta_i} f\|_{L^{\gamma_i p}} \leq \prod_{i=1}^k \|f\|_{\dot{L}^{\gamma_i p}_{|\beta_i|}}.$$

By Lemma 2.1, with $\theta_i = (|\beta_i| - 1)/(\alpha - 1)$, $i = 1, \ldots, k$,

$$\|f\|_{\dot{L}^{\gamma_i p}_{|\beta_i|}} \leq \|f\|^{1-\theta_i}_{\dot{L}_1^{mp}} \|f\|^{\theta_i}_{\dot{L}_m^p}.$$

Summing $1 - \theta_i$ and θ_i, respectively, over $i = 1, \ldots, k$ gives

$$\left\| \prod_{i=1}^k D^{\beta_i} f \right\|_{L^p} \leq c\|f\|^{m(k-1)/(m-1)}_{\dot{L}_1^{mp}} \|f\|^{(m-k)/(m-1)}_{\dot{L}_m^p}. \tag{3.5}$$

The terms on the right side of (3.5) for $k = 1, \ldots, m$ are dominated by the term for either $k = 1$ or $k = m$ depending on whether $\|f\|^{m/(m-1)}_{\dot{L}_1^{mp}}$ exceeds $\|f\|^{1/(m-1)}_{\dot{L}_m^p}$ or vice-versa. Thus by (2.6), (3.4), and (3.5),

$$\|H(f)\|_{\dot{L}_m^p} \leq cM(\|f\|_{\dot{L}_m^p} + \|f\|^m_{\dot{L}_1^{mp}}). \tag{3.6}$$

This with (2.5) and (3.1) establishes the desired result for $\alpha \in \mathbb{Z}^+$.

Now suppose that $1 \leq m < \alpha < m+1$, $m \in \mathbb{Z}$. In view of (2.7), (3.1), and (3.4), fix γ with $|\gamma| = m$, $k \in \{1, \ldots, m\}$, and $\overline{\beta} \in A(k, \gamma)$. By Lemma 2.3, or more precisely, by the remark immediately following its statement,

$$\|H^{(k)}(f) D^{\beta_1} f \cdots D^{\beta_k} f\|_{\dot{L}^p_{\alpha-m}} \leq c\|S_{\alpha-m}(H^{(k)}(f) \cdot D^{\beta_1} f \cdots D^{\beta_k} f)\|_{L^p}.$$

As noted by Polking [9], the identity

$$\Delta_h(FG)(x) = F(x+h)\Delta_h G(x) + G(x)\Delta_h F(x)$$

yields

$$S_\theta(FG) \leq \|F\|_{L^\infty} S_\theta G + |G| S_\theta F$$

for $0 < \theta < 1$. Hence by (3.4)

$$\|D^\gamma H(f)\|_{\dot{L}^p_{\alpha-m}} \le c \sum_{k=1}^{m} \sum_{\overline{\beta}\in A(k,\gamma)} (\|I\|_{L^p} + \|II\|_{L^p}), \tag{3.7}$$

where

$$I = D^{\beta_1} f \cdots D^{\beta_k} f \cdot S_{\alpha-m}(H^{(k)}(f))$$

and

$$II = \|H^{(k)}\|_{L^\infty} S_{\alpha-m}(D^{\beta_1} f \cdots D^{\beta_k} f).$$

To estimate I, let $r = \alpha/(\alpha - m)$ and $\gamma_i = \alpha/|\beta_i|$, $i = 1, \dots, k$. Hölder's inequality gives

$$\|I\|_{L^p} \le \|D^{\beta_1} f\|_{L^{\gamma_1 p}} \cdots \|D^{\beta_k} f\|_{L^{\gamma_k p}} \|S_{\alpha-m}(H^{(k)}(f))\|_{L^{rp}},$$

since $|\beta_1| + \cdots + |\beta_k| = m$. Let $\theta_i = (|\beta_i| - 1)/(\alpha - 1)$ for $i = 1, \dots, k$. Then by (2.6) and Lemma 2.1,

$$\|D^{\beta_i} f\|_{L^{\gamma_i p}} \le c\|f\|_{\dot{L}^{\gamma_i p}_{|\beta_i|}} \le c\|f\|^{1-\theta_i}_{\dot{L}^{\alpha p}_1} \|f\|^{\theta_i}_{\dot{L}^p_\alpha}. \tag{3.8}$$

Summing on i yields

$$\prod_{i=1}^{k} \|D^{\beta_i} f\|_{L^{\gamma_i p}} \le c\|f\|^{(k\alpha-m)/(\alpha-1)}_{\dot{L}^{\alpha p}_1} \|f\|^{(m-k)/(\alpha-1)}_{\dot{L}^p_\alpha}.$$

By (2.9), Lemma 2.2, and the trivial inequality $\|F\|_{\mathrm{BMO}} \le 2\|F\|_{L^\infty}$, we have

$$\begin{aligned} \|S_{\alpha-m}(H^{(k)}(f))\|_{L^{rp}} &\approx \|H^{(k)}(f)\|_{\dot{L}^{rp}_{\alpha-m}} \\ &\le c\|H^{(k)}(f)\|^{m+1-\alpha}_{L^\infty} \cdot \|H^{(k)}(f)\|^{\alpha-m}_{\dot{L}^{\alpha p}_1}. \end{aligned}$$

Note that the equivalence above follows since constants are annhilated by both the $S_{\alpha-m}$ operator and the $\dot{L}^{rp}_{\alpha-m}$ norm, hence we can apply (2.9) to $(H^{(k)}(f) - H^{(k)}(0))$, which is in L^p, since $|H^{(k)}(f) - H^{(k)}(0)| \le \|H^{(k+1)}\|_{L^\infty} \cdot |f|$. So applying (3.2) to $H^{(k)}(f)$ gives

$$\|S_{\alpha-m}(H^{(k)}(f))\|_{L^{rp}} \le cM\|f\|^{\alpha-m}_{\dot{L}^{\alpha p}_1},$$

and then we have

$$\|I\|_{L^p} \le cM \cdot \|f\|^{\alpha(1-\lambda)}_{\dot{L}^{\alpha p}_1} \|f\|^{\lambda}_{\dot{L}^p_\alpha}, \tag{3.9}$$

where $\lambda = (m - k)/(\alpha - 1)$.

To estimate II, we claim that we can apply (2.9) and conclude that

$$\|II\|_{L^p} \le cM \cdot \|D^{\beta_1} f \cdots D^{\beta_k} f\|_{\dot{L}^p_{\alpha-m}}. \tag{3.10}$$

To justify this, note first that $f \in L^p_\alpha \subset L^p_m \subset \dot{L}^p_m$, since $\alpha > m$. Also note that $D_i f \in L^{\alpha p} \cap L^p$, $i = 1, \dots, n$, since we are assuming $f \in \dot{L}^{\alpha p}_1 \cap L^p_\alpha$ and $\alpha > 1$. Thus it follows that $D_i f \in L^{mp}$ since $1 < m < \alpha$, or $f \in \dot{L}^{mp}_1$. Now applying (3.5), we have $\prod_{i=1}^k D^{\beta_i} f \in L^p$.

Now if $k = 1$ in (3.10), then $|\beta_1| = m$ and (2.6) implies $\|D^{\beta_1} f\|_{\dot{L}^p_{\alpha-m}} \le c\|f\|_{\dot{L}^p_\alpha}$. If $2 \le k \le m$, we apply Lemma 2.4 with $\gamma_i = \alpha/|\beta_i|$ and $p_i = \alpha/(\alpha - m + |\beta_i|)$, $i = 1, \ldots, k$. Notice that

$$1/p_j + \sum_{i \neq j} 1/\gamma_i = \left(\alpha - m + \sum_{i=1}^{k} |\beta_i|\right) / \alpha = 1.$$

Hence

$$\left\| \prod_{i=1}^{k} D^{\beta_i} f \right\|_{\dot{L}^p_{\alpha-m}} \le c_k \sum_{j=1}^{k} \|D^{\beta_j} f\|_{\dot{L}^{pp_j}_{\alpha-m}} \cdot \prod_{\substack{i=1 \\ i \neq j}}^{k} \|D^{\beta_i} f\|_{L^{\gamma_i p}}.$$

By (2.6) and Lemma 2.1,

$$\|D^{\beta_j} f\|_{\dot{L}^{pp_j}_{\alpha-m}} \le c\|f\|_{\dot{L}^{pp_j}_{\alpha-m+|\beta_j|}} \le c\|f\|^{1-\lambda_j}_{\dot{L}^{\alpha p}_1} \|f\|^{\lambda_j}_{\dot{L}^p_\alpha},$$

where $\lambda_j = 1 - (m - |\beta_j|)/(\alpha - 1)$. We finally estimate $\|D^{\beta_i} f\|_{L^{\gamma_i p}}$ by (3.8) with the same values of θ_i. Altogether, this gives

$$\|II\|_{L^p} \le cM\|f\|^{\alpha(1-\mu)}_{\dot{L}^{\alpha p}_1} \cdot \|f\|^{\mu}_{\dot{L}^p_\alpha} \tag{3.11}$$

with $\mu = (\alpha - k)/(\alpha - 1)$, which as noted above, is even correct when $k = 1$.

Putting estimates (3.9) and (3.11) into (3.7), we see that the term $cM\|f\|^{\alpha}_{\dot{L}^{\alpha p}_1}$ arises in (3.9) when $k = m$, while $cM\|f\|_{\dot{L}^p_\alpha}$ arises in (3.11) when $k = 1$. The other terms for $k \in \{1, \ldots, m\}$ in (3.9) and (3.11) are dominated by one or the other of these extreme terms, so that

$$\|D^\gamma H(f)\|_{\dot{L}^p_{\alpha-m}} \le cM(\|f\|_{\dot{L}^p_\alpha} + \|f\|^{\alpha}_{\dot{L}^{\alpha p}_1}),$$

for $|\gamma| = m$. With (2.5) and (3.1), the proof of Theorem A is complete. □

Proof of Theorem B. First notice that $f \in L^p_m$ since $H(t) = t$ is m-admissible for any m. Let $\chi \in C^\infty(\mathbb{R}^n)$ satisfy $\operatorname{supp} \chi \subseteq [-1, 1]$ and $\chi(t) = 1$ for $-2/3 \le t \le 2/3$. For $j = 1, \ldots, m$, let

$$H_j(t) = \sum_{l \in \mathbb{Z}} (t - 2l)^j \chi(t - 2l)$$

and $A = \bigcup_{l \in \mathbb{Z}} (-2/3 + 2l, 2/3 + 2l)$. Also let $\widetilde{H}_j(t) = H_j(t - 1)$ and $\widetilde{A} = \bigcup_{l \in \mathbb{Z}} (1/3 + 2l, 5/3 + 2l)$. Clearly $A \cup \widetilde{A} = \mathbb{R}$, and

$$H^{(j)}_j(t) = j! \quad \text{for } t \in A, \qquad \text{and} \qquad \widetilde{H}^{(j)}_j(t) = j! \quad \text{for } t \in \widetilde{A}. \tag{3.12}$$

Obviously, by periodicity both H_j and $\widetilde{H}_j$ are m-admissible for all j and m.

Now fix an $i \in \{1, \ldots, n\}$. By the chain rule

$$D^{(m)}_i H(f) = \sum_{k=1}^{m} H^{(k)}(f) F_k, \tag{3.13}$$

where $F_k = \sum c_{m_1, \ldots, m_k} D^{(m_1)}_i f \cdots D^{(m_k)}_i f$, with the sum extending over all $m_1, \ldots, m_k$ such that $\sum_{j=1}^{k} m_j = m$, $m_j \ge 1$ for each j. (Recall $D_i = \partial/\partial x_i$). Note that F_k depends on f and i, but not on H. By assumption,

$D_i^{(m)}H(f) \in L^p$ for $H = H_j$ or $H = \widetilde{H}_j$, $j = 1, \dots, m$. We show by induction that this implies that $F_j \in L^p$, $j = 1, \dots, m$. First, since we have noted that $f \in L^p_m$, we have $F_1 = D_i^{(m)} f \in L^p$. Now suppose $F_1, \dots, F_{j-1} \in L^p$. Let H be H_j in (3.13). By (3.12), which obviously implies that $H_j^{(k)}(t) = 0$ for $t \in A$ and $k \geq j+1$, we have

$$D_i^{(m)} H_j(f)(x) = j! F_j(x) + \sum_{k=1}^{j-1} H_j^{(k)}(f(x)) \cdot F_k(x)$$

for $x \in f^{-1}(A)$. Since $H_j^{(k)} \in L^\infty$ and $F_1, \dots, F_{j-1} \in L^p$, it follows that $F_j \in L^p(f^{-1}(A))$. By the same argument with $\widetilde{H}_j$ in place of H_j, $F_j \in L^p(f^{-1}(\widetilde{A}))$; hence $F_j \in L^p(\mathbb{R}^n)$. This completes the induction; after finitely many steps, we obtain that $F_m = (D_i f)^m \in L^p$, or $D_i f \in L^{mp}$. Since $i \in \{i, \dots, n\}$ is arbitrary, we have $f \in \dot{L}_1^{mp}$. □

The analogue of Theorem B for $0 < \alpha \notin \mathbb{Z}^+$ remains open.

4. Proof of the Lemmas

Proof of Lemma 2.1. Applying Hölder's inequality with exponents $1/(1-\theta)$ and $1/\theta$ gives

$$\begin{aligned} \|f\|_{\dot{L}^p_\alpha} &= \left\| \left[\sum_{\nu \in \mathbb{Z}} (2^{\nu(1-\theta)\alpha_1} |\phi_\nu * f|^{1-\theta} 2^{\nu\theta\alpha_2} |\phi_\nu * f|^\theta)^2 \right]^{1/2} \right\|_{L^p} \\ &\leq \left\| \left[\sum_{\nu \in \mathbb{Z}} (2^{\nu\alpha_1} |\phi_\nu * f)^2 \right]^{(1-\theta)/2} \left[\sum_{\nu \in \mathbb{Z}} (2^{\nu\alpha_2} |\phi_\nu * f|)^2 \right]^{\theta/2} \right\|_{L^p}. \end{aligned}$$

Hölder's inequality again, this time with conjugate exponents $p_1/(1-\theta)p$ and $p_2/\theta p$, yields the result. □

Proof of Lemma 2.2. This can be derived from the complex interpolation result $[\mathrm{BMO}, \dot{L}^p_\alpha]_\theta \approx \dot{L}^{p/\theta}_{\alpha\theta}$. Also we could give a proof based on the theory of the "ϕ-transform" S_ϕ in [6] and the Calderón product (see [6, Theorems 2.2, 9.2, 8.2]). However, we omit the proof since all that is used in the proof of Theorem A is the well-known Gagliardo–Nirenberg inequality [7], i.e. (2.8) with L^∞ in place of BMO. □

Proof of Lemma 2.3. The inhomogeneous analogues of (2.9) and (2.10), i.e.

$$\|f\|_{L^p_\alpha} \approx \|f\|_{L^p} + \|S_\alpha\|_{L^p} \quad \text{and} \quad \|D_t^\alpha f\|_{L^p} \leq c\|f\|_{L^p_\alpha},$$

are proved in Strichartz [10], and Polking [9], respectively. Undoubtedly, (2.9) and (2.10) are implicit in their arguments. However, we want to note another proof based on the Littlewood–Paley definition of $\dot{L}^p_\alpha$ given in §2. The analogue of (2.10) in the inhomogeneous case for $t = 1$ is done in Triebel [11, Theorem 2.5.11]. With a few modifications, the same argument gives (2.10). The assumption $f \in L^q$ is used to prove the pointwise a.e. equality of f and $\sum_v \phi_\nu * f$ under assumptions (2.1)–(2.4). We omit the details.

We do, however, give a proof of the lower bound of $S_\alpha f$ in (2.9). This proof appears to be more direct than the corresponding proofs contained in the literature. For $r > 0$, let

$$A_{r,\alpha}f(x) = r^{-\alpha}\int_{|y|<1}|f(x+ry)-f(x)|\,dy = r^{-\alpha-n}\int_{|y|<r}|f(x+y)-f(x)|\,dy.$$

It is easy to see that if $2^\mu \le r \le 2^{\mu+1}$, then

$$2^{-\alpha-n}A_{2^\mu,\alpha}f(x) \le A_{r,\alpha}f(x) \le 2^{\alpha+n}A_{2^{\mu+1},\alpha}f(x).$$

Thus a simple discretization of the integral from 0 to ∞ shows that

$$S_\alpha f(x) \approx \left(\sum_{\mu\in\mathbb{Z}}|A_{2^\mu,\alpha}f(x)|^2\right)^{1/2}, \tag{4.1}$$

with constants depending only on n and α.

By (2.2), $\int\phi = 0$, so for $\nu \in \mathbb{Z}$,

$$\phi_\nu * f(x) = \int[f(x-y)-f(x)]\phi_\nu(y)\,dy = \int[f(x-2^{-\nu}y)-f(x)]\phi(y)\,dy.$$

For $\mu \in \mathbb{Z}$, let $R_\mu = \{x \in \mathbb{R}^n : 2^{\mu-1} \le |x| \le 2^\mu\}$ and $B_\mu = \{x \in \mathbb{R}^n : |x| \le 2^\mu\}$. Selecting $M > n+\alpha$, we have

$$\begin{aligned}
2^{\nu\alpha}|\phi_\nu * f(x)| &= 2^{\nu\alpha}\sum_{\mu\in\mathbb{Z}}\int_{R_\mu}|\phi(y)|\cdot|f(x-2^{-\nu}y)-f(x)|\,dy\\
&\le 2^{\nu\alpha}\sum_{\mu\in\mathbb{Z}}\sup_{R_\mu}|\phi|\cdot\int_{B_\mu}|f(x-2^{-\nu}y)-f(x)|\,dy\\
&\le c_M 2^{\nu\alpha}\sum_{\mu\in\mathbb{Z}}(1+2^\mu)^{-M}2^{\nu n}\int_{B_{\mu-\nu}}|f(x+y)-f(x)|\,dy\\
&= c_M\sum_{\mu\in\mathbb{Z}}2^{\mu(n+\alpha)}(1+2^\mu)^{-M}\cdot A_{2^{\mu-\nu},\alpha}f(x).
\end{aligned}$$

This last sum is a discrete convolution and $\{2^{\mu(\alpha+n)}(1+2^\mu)^{-M}\} \in l^1(\mathbb{Z})$, so Young's inequality gives

$$\left(\sum_{\nu\in\mathbb{Z}}|2^{\nu\alpha}\phi_\nu * f(x)|^2\right)^{1/2} \le c_{\mu,\alpha}\left(\sum_{\mu\in\mathbb{Z}}|A_{2^\mu,\alpha}f(x)|^2\right)^{1/2}.$$

Taking L^p-norms and using (4.1) gives $\|f\|_{L^p_\alpha} \le c\|S_\alpha f\|_{L^p}$. □

Proof of Lemma 2.4. We first prove the case $k = 2$. Using (2.11) and applying the Cauchy–Schwarz inequality twice to the final term, gives

$$S_\theta(f_1f_2)(x) \le |f_1(x)|S_\theta f_2(x) + |f_2(x)|S_\theta f_1(x) + D_2^{\theta/2}f_1(x)D_2^{\theta/2}f_2(x).$$

(This is a special case of Polking's formula in [9].) Hence by (2.9) and the remark following Lemma 2.3,

$$\|f_1f_2\|_{L^p_\theta} \le c\|f_1S_\theta f_2\|_{L^p} + c\|f_2S_\theta f_1\|_{L^p} + c\|D_2^{\theta/2}f_1D_2^{\theta/2}f_2\|_{L^p}.$$

Since $1/p_1 + 1/\gamma_2 = 1$, Hölder's inequality and (2.9) imply

$$\|f_2 S_\theta f_1\|_{L^p} \le \|S_\theta f_1\|_{L^{p_1 p}} \|f_2\|_{L^{\gamma_2 p}} \le c\|f_1\|_{\dot{L}_\theta^{p_1 p}} \|f_2\|_{L^{\gamma_2 p}}.$$

(Note that (2.9) and (2.10) apply to f_1 and f_2 since $f_j \in L^{\gamma_j p}$, $j = 1, 2$, else there is nothing to prove.) Similarly,

$$\|f_1 S_\theta f_2\|_{L^p} \le c\|f_2\|_{\dot{L}_\theta^{p_2 p}} \|f_1\|_{L^{\gamma_1 p}}.$$

Define conjugate indices s and s' so that $1/s = (1/\gamma_1 + 1/p_1)/2$ and $1/s' = (1/\gamma_2 + 1/p_2)/2$. By Hölder's inequality and (2.10),

$$\|D_2^{\theta/2} f_1 D_2^{\theta/2} f_2\|_{L^p} \|D_2^{\theta/2} f_1\|_{L^{sp}} \|D_2^{\theta/2} f_2\|_{L^{s'p}} \le c\|f_1\|_{\dot{L}_{\theta/2}^{sp}} \|f_2\|_{\dot{L}_{\theta/2}^{s'p}}.$$

By Lemma 2.1, we have

$$\|f_1\|_{\dot{L}_{\theta/2}^{sp}} \le \|f_1\|_{L^{\gamma_1 p}}^{1/2} \|f_1\|_{\dot{L}_\theta^{p_1 p}}^{1/2}$$

and

$$\|f_2\|_{\dot{L}_{\theta/2}^{s'p}} \le \|f_2\|_{L^{\gamma_2 p}}^{1/2} \|f_2\|_{\dot{L}_\theta^{p_2 p}}^{1/2}.$$

Thus since $|abcd|^{1/2} \le |ab| + |cd|$,

$$\|f_1 f_2\|_{\dot{L}_\theta^p} \le c(\|f_1\|_{\dot{L}_\theta^{p_1 p}} \|f_2\|_{L^{\gamma_2 p}} + \|f_2\|_{\dot{L}_\theta^{p_2 p}} \|f_1\|_{L^{\gamma_1 p}}),$$

as desired in the case $k = 2$. The general case follows by induction on k. We only sketch the details. Apply the case $k = 2$ with p_1, γ_2, p_2, γ_1, f_1, f_2 replaced by $\gamma_k' = \gamma_k/(\gamma_k - 1)$, γ_k, p_k, $p_k' = p_k/(p_k - 1)$, $\prod_{i=1}^{k-1} f_i$, and f_k, respectively. This yields an estimate for $\| \prod_{i=1}^k f_i\|_{\dot{L}_\theta^p}$ involving two products of the form $\prod_{i=1}^{k-1}$. Using the $k - 1$ case for one, and Hölder's inequality for the other, yields the result. □

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