

COMPOSITION OPERATORS ON POTENTIAL SPACES

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ABSTRACT. By a result of B. Dahlberg, the composition operators $T_H f = H \circ f$ need not be bounded on some of the Sobolev spaces (or spaces of Bessel potentials) even for very smooth functions $H = H(t)$, $H(0) = 0$, unless of course, $H(t) = ct$. In this note a natural domain is found for T_H that is, in a sense, maximal and on which the $\{T_H\}$ form an algebra of bounded operators. Here the functions $H(t)$ need not be bounded though they are required to have a sufficient number of bounded derivatives.

1. INTRODUCTION

For $H: \mathbb{R} \rightarrow \mathbb{R}$, let T_H be the associated composition operator that takes $f: \mathbb{R}^n \rightarrow \mathbb{R}$ to $H(f)$. We consider the action of T_H on the Bessel potential spaces $L_\alpha^p(\mathbb{R}^n)$ for $\alpha > 0$ and $1 < p < \infty$. We say that H is α -admissible if

$$H(0) = 0$$

and

$$M \equiv \max_k \sup_{t \in \mathbb{R}} |H^{(k)}(t)| < \infty,$$

where the max is taken over $k \in \{1, \dots, m\}$ if $\alpha = m \in \mathbb{Z}^+$, and over $k \in \{1, \dots, m+1\}$ if $m < \alpha < m+1$, $m \in \mathbb{Z}^+$. If in addition, H is bounded (an assumption not required for our results), then T_H is often referred to as a *smooth truncation operator*; cf. [2].

Let \dot{L}_α^p , $\alpha > 0$, $1 < p < \infty$, be the Riesz potential space, i.e. the homogeneous counterpart to the inhomogeneous space L_α^p . Complete definitions are given below; for now we note that if $\alpha = m \in \mathbb{Z}^+$, then

$$\|f\|_{\dot{L}_m^p} \approx \sum_{|\gamma|=m} \|D^\gamma f\|_{L^p},$$

while

$$\|f\|_{L_m^p} \approx \sum_{|\gamma| \leq m} \|D^\gamma f\|_{L^p} \approx \|f\|_{L^p} + \|f\|_{\dot{L}_m^p};$$

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see (2.5) and (2.6) below. Our main result is

Theorem A. *Suppose $\alpha > 1$, $1 < p < \infty$, and H is α -admissible. If $f \in L^p_\alpha \cap \dot{L}^{\alpha p}_1$, then $H(f) \in L^p_\alpha \cap \dot{L}^{\alpha p}_1$ and*

$$(1.1) \quad \|H(f)\|_{L^p_\alpha} \leq cM(\|f\|_{L^p_\alpha} + \|f\|_{\dot{L}^{\alpha p}_1}^\alpha),$$

and

$$(1.2) \quad \|H(f)\|_{\dot{L}^{\alpha p}_1} \leq M\|f\|_{\dot{L}^{\alpha p}_1}$$

The nonlinearity of estimate (1.1) is a reflection of the nonlinearity of the operator T_H ; cf. (1) of [3] and (1.6) of [2].

For $0 < \alpha \leq 1$ and $1 < p < \infty$, $H(f) \in L^p_\alpha$ for all $f \in L^p_\alpha$ and any α -admissible H . In fact, we see below in §3 (see (3.1), (3.2), and (3.3)) that in this case, we have

$$\|H(f)\|_{L^p_\alpha} \leq cM\|f\|_{L^p_\alpha}.$$

And, in particular, it is easy to see that (1.2) holds. We also have $H(f) \in L^p_\alpha$ for all $f \in L^p_\alpha$ if $\alpha \geq n/p$, as noted in [1], or as a consequence of Theorem A, since the Sobolev (potential) imbedding theorem implies that L^p_α is continuously imbedded into $\dot{L}^{\alpha p}_1$ whenever $\alpha p \geq n$, $\alpha > 1$ and $1 < p < \infty$. However, for $\alpha \in \mathbb{Z}^+$, Dahlberg [5] has shown that if $1 < \alpha < n/p$, $1 < p < \infty$, and $H(f) \in L^p_\alpha$ for every $f \in L^p_\alpha$, H α -admissible, then $H(t) = ct$ for some $c \in \mathbb{R}$.

In view of Dahlberg’s negative result, a natural question is to determine what extra conditions of $f \in L^p_\alpha$ guarantee that $H(f) \in L^p_\alpha$ for α -admissible H . The first result of this type is obtained from the Gagliardo–Nirenberg lemma (see [7] and Lemma 2.2 below), which implies that $H(f) \in L^p_\alpha$ for every $f \in L^p_\alpha \cap L^\infty$; see [2]. This, along with an extension of the Fefferman–Stein decomposition of BMO, was used in [2] to show that $H(f) \in L^p_\alpha$ for every $f \in L^p_\alpha \cap \text{BMO}$. This result also follows from Theorem A because of the imbedding

$$\dot{L}^p_\alpha \cap \text{BMO} \rightarrow \dot{L}^{\alpha p}_1$$

if $\alpha > 1$ and $1 < p < \infty$; see Lemma 2.2 below. It should also be noted that the methods used here to prove Theorem A are considerably simpler than those used in [2]. As for the sharpness of our main result, we also prove

Theorem B. *Let $\alpha = m \in \mathbb{Z}^+$, $m \geq 1$ and suppose that $f: \mathbb{R}^n \rightarrow \mathbb{R}$ and $H(f) \in L^p_m$ for all m -admissible H . Then $f \in L^p_m \cap \dot{L}^{mp}_1$.*

In §2 we give the necessary preliminaries, definitions, and statements for the needed lemmas. The proofs of Theorems A and B are given in §3. We discuss the proofs of the lemmas in §4.

2. PRELIMINARIES

For $\alpha > 0$, let G_α be the Bessel potential kernel of order α , i.e. the positive $L^1(\mathbb{R}^n)$ function that satisfies $\widehat{G}_\alpha(\xi) = (1 + |\xi|^2)^{-\alpha/2}$ for $\xi \in \mathbb{R}^n$, $\widehat{\cdot}$ denoting the Fourier transform on L^1 . For $1 < p < \infty$, let $L^p_\alpha = L^p_\alpha(\mathbb{R}^n)$ be the space of all $f = G_\alpha * g$, for some $g \in L^p(\mathbb{R}^n)$; in this case $\|f\|_{L^p_\alpha} = \|g\|_{L^p}$. The homogeneous space $\dot{L}^p_\alpha = \dot{L}^p_\alpha(\mathbb{R}^n)$ is the space of Riesz potentials $f = I_\alpha * g$, $g \in L^p$. Here $\widehat{I}_\alpha(\xi) = |\xi|^{-\alpha}$. However, for simplicity below, we use the

equivalent Littlewood–Paley characterization of \dot{L}^p_α for its definition; see [8, 11].

Let $\phi: \mathbb{R}^n \rightarrow \mathbb{C}$ satisfy

$$(2.1) \quad \phi \in \mathcal{S},$$

$$(2.2) \quad \text{supp } \hat{\phi} \subseteq \{\xi: 1/2 \leq |\xi| \leq 2\},$$

$$(2.3) \quad |\hat{\phi}(\xi)| \geq c > 0 \quad \text{if } 3/5 \leq |\xi| \leq 5/3,$$

and

$$(2.4) \quad \sum_{\nu \in \mathbb{Z}} \hat{\phi}(2^\nu \xi) = 1 \quad \text{for } \xi \neq 0.$$

For $\nu \in \mathbb{Z}$, set $\phi_\nu(x) = 2^{\nu n} \phi(2^\nu x)$. For $1 < p < \infty$ and $\alpha \geq 0$, let

$$\|f\|_{\dot{L}^p_\alpha} = \left\| \left(\sum_{\nu \in \mathbb{Z}} |2^{\nu \alpha} \phi_\nu * f|^2 \right)^{1/2} \right\|_{L^p}.$$

When $\alpha = 0$, $\dot{L}^p_0 = L^p$, $1 < p < \infty$ by Littlewood-Paley theory [8, 11]. With this definition, the following lemma is a simple consequence of Hölder’s inequality.

Lemma 2.1. *Suppose $0 \leq \alpha_1 < \alpha_2 < \infty$, $0 < p_1, p_2 < \infty$, $0 < \theta < 1$, $\alpha = (1 - \theta)\alpha_1 + \theta\alpha_2$, and $1/p = (1 - \theta)/p_1 + \theta/p_2$. Then*

$$\|f\|_{\dot{L}^p_\alpha} \leq \|f\|_{\dot{L}^{p_1}_{\alpha_1}}^{1-\theta} \|f\|_{\dot{L}^{p_2}_{\alpha_2}}^\theta.$$

In general, \dot{L}^p_α is a space of tempered distributions modulo polynomials. However, in this paper we consider $\|f\|_{\dot{L}^p_\alpha}$ only for functions f belonging to L^q for some $q \in (1, \infty)$. This eliminates any ambiguity regarding polynomials.

The following two facts about \dot{L}^p_α are well known (see e.g. [8] or [11]). For $1 < p < \infty$ and $\alpha > 0$,

$$(2.5) \quad \|f\|_{\dot{L}^p_\alpha} \approx \|f\|_{L^p} + \|f\|_{\dot{L}^p_\alpha},$$

while for $1 \leq m \leq \alpha < m + 1$, $m \in \mathbb{Z}$,

$$(2.6) \quad \|f\|_{\dot{L}^p_\alpha} \approx \sum_{|\gamma|=m} \|D^\gamma f\|_{\dot{L}^{p-m}_\alpha}.$$

Here “ \approx ” means that the ratio of the two sides is bounded above and below by finite positive constants independent of f . We are using the standard multi-index notation: $\gamma = (\gamma_1, \dots, \gamma_n)$, $\gamma_i \geq 0$, $\gamma_i \in \mathbb{Z}$, $i = 1, \dots, n$, $|\gamma| = \gamma_1 + \dots + \gamma_n$, and $D^\gamma = (\partial/\partial x_1)^{\gamma_1} \dots (\partial/\partial x_n)^{\gamma_n}$. Obviously, (2.5) and (2.6) imply the classical result of Calderón [4]: if $1 < p < \infty$ and $1 \leq m \leq \alpha < m + 1$, $m \in \mathbb{Z}$, then

$$(2.7) \quad \|f\|_{L^p} \approx \|f\|_{L^p} + \sum_{|\gamma|=m} \|D^\gamma f\|_{\dot{L}^{p-m}_\alpha}.$$

For $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, let $\|f\|_{\text{BMO}} = \sup_Q (1/|Q|) \int_Q |f - f_Q| dx$, where $f_Q = (1/|Q|) \int_Q f$ and the supremum is over all cubes $Q \subset \mathbb{R}^n$ with sides parallel to the axes.

Lemma 2.2. *Suppose $0 < \theta < 1$, $\alpha > 0$, and $1 < p < \infty$. Then*

$$(2.8) \quad \|f\|_{\dot{L}^{p/\theta}_\alpha} \leq c \|f\|_{\text{BMO}}^{1-\theta} \cdot \|f\|_{\dot{L}^p_\alpha}.$$

Clearly $\|f\|_{\text{BMO}} \leq 2\|f\|_{L^\infty}$; using this inequality in (2.8) results in the Gagliardo–Nirenberg lemma [7].

In his study of the fractional order potential spaces, Strichartz [10] introduced the operator S_α , defined for $0 < \alpha < 1$ and $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ by

$$S_\alpha f(x) = \left(\int_0^\infty \left[\int_{|y|<1} |f(x+ry) - f(x)| dy \right]^2 r^{-1-2\alpha} dr \right)^{1/2}.$$

Polking [9] introduced a class of variants of S_α ; here we consider only the special case D_t^α , $1 \leq t < \infty$, $0 < \alpha < 1$, defined for $f \in L^t_{\text{loc}}(\mathbb{R}^n)$ by

$$D_t^\alpha f(x) = \left(\int_0^\infty \left[\int_{|y|<1} |f(x+ry) - f(x)|^t dy \right]^2 r^{-1-2t\alpha} dr \right)^{1/2t}.$$

Obviously, $D_1^\alpha f = S_\alpha f$.

Lemma 2.3. *Suppose $0 < \alpha < 1$ and $1 < p < \infty$. If $f \in L^q$ for some $q \in (1, \infty)$, then*

$$(2.9) \quad \|f\|_{\dot{L}^p_\alpha} \approx \|S_\alpha f\|_{L^p},$$

and, if $1 \leq t < p$,

$$(2.10) \quad \|D_t^\alpha f\|_{L^p} \leq c(\alpha, p, t) \|f\|_{\dot{L}^p_\alpha}.$$

In (2.9) the estimate $\|f\|_{\dot{L}^p_\alpha} \leq c(\alpha, p) \|S_\alpha f\|_{L^p}$ holds for any $f \in L^1_{\text{loc}}$; for the converse of (2.9) and (2.10), an assumption like $f \in L^q$ is needed since polynomials have norm zero in \dot{L}^p_α whereas S_α and D_t^α only annihilate constants.

For $h \in \mathbb{R}^n$, set $\Delta_h f(x) = f(x+h) - f(x)$. Following Polking [9], we easily have

$$(2.11) \quad \Delta_h(fg)(x) = f(x)\Delta_h g(x) + g(x)\Delta_h f(x) + \Delta_h f(x) \cdot \Delta_h g(x).$$

As a result, we can estimate the L^p_α norm of a product of functions in terms of their individual norms in other potential spaces. This method gives

Lemma 2.4. *Suppose $0 < \theta < 1$, $1 < p < \infty$, $k \in \mathbb{Z}$, $k \geq 2$; let $p_i \geq 1$, $\gamma_i \geq 1$, $i = 1, \dots, k$, and $1/p_j + \sum_{i \neq j} 1/\gamma_i = 1$, for $j = 1, \dots, k$. Then*

$$\left\| \prod_{i=1}^k f_i \right\|_{\dot{L}^p_\theta} \leq c \sum_{j=1}^k \|f_j\|_{\dot{L}^{pp_j}_\theta} \cdot \prod_{\substack{i=1 \\ i \neq j}}^k \|f_i\|_{L^{\gamma_i p}}.$$

3. PROOFS OF THE THEOREMS

Proof of Theorem A. Since $H(0) = 0$, we have $|H(f)(x)| = |H(f(x)) - H(0)| \leq \|H'\|_{L^\infty} \cdot |f(x)|$. Hence

$$(3.1) \quad \|H(f)\|_{L^p} \leq \|H'\|_{L^\infty} \|f\|_{L^p}, \quad 0 < p \leq \infty.$$

For $i = 1, \dots, n$, let $D_i = \partial/\partial x_i$. Then $D_i H(f) = H'(f) \cdot D_i f$, and hence by (2.6),

$$(3.2) \quad \|H(f)\|_{L^p_1} \leq c \|H'\|_{L^\infty} \|f\|_{L^p_1}, \quad 1 < p < \infty.$$

If $0 < \alpha < 1$, applying the mean value theorem to $H(f(x + ry)) - H(f(x))$ gives $S_\alpha H(f) \leq \|H'\|_{L^\infty} S_\alpha(f)$. Therefore by (2.9),

$$(3.3) \quad \|H(f)\|_{L^p_\alpha} \leq c \|H'\|_{L^\infty} \|f\|_{L^p_\alpha}.$$

(Estimates (3.1)–(3.3) give the analogue of Theorem A for $0 < \alpha \leq 1$, $1 < p < \infty$, as noted in the introduction.)

Now, suppose that $\alpha = m \in \mathbb{Z}$, $m \geq 2$. For a multi-index γ with $|\gamma| = m$ and $k \in \{1, \dots, m\}$, let $A(k, \gamma)$ be the collection of all ordered k -tuples of multi-indices $\bar{\beta} = \{\beta_1, \dots, \beta_k\}$ with $\beta_i \in (\mathbb{Z}^+)^n$, $|\beta_i| \geq 1$, and $\sum_1^k \beta_i = \gamma$. By the Leibniz rule, there exist scalars $c(\bar{\beta}, k, \gamma)$ such that

$$(3.4) \quad D^\gamma H(f) = \sum_{k=1}^m H^{(k)}(f) \sum_{\bar{\beta} \in A(k, \gamma)} c(\bar{\beta}, k, \gamma) D^{\beta_1} f \cdots D^{\beta_k} f.$$

For $\gamma_i = m/|\beta_i|$, $|\beta_1| + \dots + |\beta_k| = m$, Hölder’s inequality together with (2.6) gives

$$\left\| \prod_{i=1}^k D^{\beta_i} f \right\|_{L^p} \leq \prod_{i=1}^k \|D^{\beta_i} f\|_{L^{\gamma_i p}} \leq \prod_{i=1}^k \|f\|_{L^{\gamma_i p_{|\beta_i|}}}.$$

By Lemma 2.1, with $\theta_i = (|\beta_i| - 1)/(\alpha - 1)$, $i = 1, \dots, k$,

$$\|f\|_{L^{\gamma_i p_{|\beta_i|}}} \leq \|f\|_{L_1^{m p}}^{1-\theta_i} \|f\|_{L_m^p}^{\theta_i}.$$

Summing $1 - \theta_i$ and θ_i , respectively, over $i = 1, \dots, k$ gives

$$(3.5) \quad \left\| \prod_{i=1}^k D^{\beta_i} f \right\|_{L^p} \leq c \|f\|_{L_1^{m p}}^{m(k-1)/(m-1)} \|f\|_{L_m^p}^{(m-k)/(m-1)}.$$

The terms on the right side of (3.5) for $k = 1, \dots, m$ are dominated by the term for either $k = 1$ or $k = m$ depending on whether $\|f\|_{L_1^{m p}}^{m/(m-1)}$ exceeds $\|f\|_{L_m^p}^{1/(m-1)}$ or vice-versa. Thus by (2.6), (3.4), and (3.5),

$$(3.6) \quad \|H(f)\|_{L^p_m} \leq c M (\|f\|_{L^p_m} + \|f\|_{L_1^{m p}}^m).$$

This with (2.5) and (3.1) establishes the desired result for $\alpha \in \mathbb{Z}^+$.

Now suppose that $1 \leq m < \alpha < m + 1$, $m \in \mathbb{Z}$. In view of (2.7), (3.1), and (3.4), fix γ with $|\gamma| = m$, $k \in \{1, \dots, m\}$, and $\bar{\beta} \in A(k, \gamma)$. By Lemma 2.3, or more precisely, by the remark immediately following its statement,

$$\|H^{(k)}(f) D^{\beta_1} f \cdots D^{\beta_k} f\|_{L^p_{\alpha-m}} \leq c \|S_{\alpha-m}(H^{(k)}(f) \cdot D^{\beta_1} f \cdots D^{\beta_k} f)\|_{L^p}.$$

As noted by Polking [9], the identity

$$\Delta_h(FG)(x) = F(x + h)\Delta_h G(x) + G(x)\Delta_h F(x)$$

yields

$$S_\theta(FG) \leq \|F\|_{L^\infty} S_\theta G + |G| S_\theta F$$

for $0 < \theta < 1$. Hence by (3.4)

$$(3.7) \quad \|D^\gamma H(f)\|_{\dot{L}^p_{\alpha-m}} \leq c \sum_{k=1}^m \sum_{\bar{\beta} \in A(k, \gamma)} (\|I\|_{L^p} + \|II\|_{L^p}),$$

where

$$I = D^{\beta_1} f \dots D^{\beta_k} f \cdot S_{\alpha-m}(H^{(k)}(f))$$

and

$$II = \|H^{(k)}\|_{L^\infty} S_{\alpha-m}(D^{\beta_1} f \dots D^{\beta_k} f).$$

To estimate I , let $r = \alpha/(\alpha - m)$ and $\gamma_i = \alpha/|\beta_i|$, $i = 1, \dots, k$. Hölder's inequality gives

$$\|I\|_{L^p} \leq \|D^{\beta_1} f\|_{L^{\gamma_1 p}} \dots \|D^{\beta_k} f\|_{L^{\gamma_k p}} \|S_{\alpha-m}(H^{(k)}(f))\|_{L^p},$$

since $|\beta_1| + \dots + |\beta_k| = m$. Let $\theta_i = (|\beta_i| - 1)/(\alpha - 1)$ for $i = 1, \dots, k$. Then by (2.6) and Lemma 2.1,

$$(3.8) \quad \|D^{\beta_i} f\|_{L^{\gamma_i p}} \leq c \|f\|_{\dot{L}^p_{|\beta_i|}} \leq c \|f\|_{\dot{L}^p_1}^{1-\theta_i} \|f\|_{\dot{L}^p_\alpha}^{\theta_i}.$$

Summing on i yields

$$\prod_{i=1}^k \|D^{\beta_i} f\|_{L^{\gamma_i p}} \leq c \|f\|_{\dot{L}^p_1}^{(k\alpha-m)/(\alpha-1)} \|f\|_{\dot{L}^p_\alpha}^{(m-k)/(\alpha-1)}.$$

By (2.9), Lemma 2.2, and the trivial inequality $\|F\|_{\text{BMO}} \leq 2\|F\|_{L^\infty}$, we have

$$\begin{aligned} \|S_{\alpha-m}(H^{(k)}(f))\|_{L^p} &\approx \|H^{(k)}(f)\|_{\dot{L}^p_{\alpha-m}} \\ &\leq c \|H^{(k)}(f)\|_{L^\infty}^{m+1-\alpha} \cdot \|H^{(k)}(f)\|_{\dot{L}^p_\alpha}^{\alpha-m}. \end{aligned}$$

Note that the equivalence above follows since constants are annihilated by both the $S_{\alpha-m}$ operator and the $\dot{L}^p_{\alpha-m}$ norm, hence we can apply (2.9) to $(H^{(k)}(f) - H^{(k)}(0))$, which is in L^p , since $|H^{(k)}(f) - H^{(k)}(0)| \leq \|H^{(k+1)}\|_{L^\infty} \cdot |f|$. So applying (3.2) to $H^{(k)}(f)$ gives

$$\|S_{\alpha-m}(H^{(k)}(f))\|_{L^p} \leq cM \|f\|_{\dot{L}^p_\alpha}^{\alpha-m},$$

and then we have

$$(3.9) \quad \|I\|_{L^p} \leq cM \cdot \|f\|_{\dot{L}^p_\alpha}^{\alpha(1-\lambda)} \|f\|_{\dot{L}^p_\alpha}^\lambda,$$

where $\lambda = (m - k)/(\alpha - 1)$.

To estimate II , we claim that we can apply (2.9) and conclude that

$$(3.10) \quad \|II\|_{L^p} \leq cM \cdot \|D^{\beta_1} f \dots D^{\beta_k} f\|_{\dot{L}^p_{\alpha-m}}.$$

To justify this, note first that $f \in L^p_\alpha \subset L^p_m \subset \dot{L}^p_m$, since $\alpha > m$. Also note that $D_i f \in L^{\alpha p} \cap L^p$, $i = 1, \dots, n$, since we are assuming $f \in \dot{L}^{\alpha p}_1 \cap L^p_\alpha$ and $\alpha > 1$. Thus it follows that $D_i f \in L^{mp}$ since $1 < m < \alpha$, or $f \in \dot{L}^{mp}_1$. Now applying (3.5), we have $\prod_{i=1}^k D^{\beta_i} f \in L^p$.

Now if $k = 1$ in (3.10), then $|\beta_1| = m$ and (2.6) implies $\|D^{\beta_1} f\|_{\dot{L}^p_{\alpha-m}} \leq c\|f\|_{\dot{L}^p_{\alpha}}$. If $2 \leq k \leq m$, we apply Lemma 2.4 with $\gamma_i = \alpha/|\beta_i|$ and $p_i = \alpha/(\alpha - m + |\beta_i|)$, $i = 1, \dots, k$. Notice that

$$1/p_j + \sum_{i \neq j} 1/\gamma_i = \left(\alpha - m + \sum_{i=1}^k |\beta_i| \right) / \alpha = 1.$$

Hence

$$\left\| \prod_{i=1}^k D^{\beta_i} f \right\|_{\dot{L}^p_{\alpha-m}} \leq c_k \sum_{j=1}^k \|D^{\beta_j} f\|_{\dot{L}^{pp_j}_{\alpha-m}} \cdot \prod_{\substack{i=1 \\ i \neq j}}^k \|D^{\beta_i} f\|_{L^{\gamma_i p}}.$$

By (2.6) and Lemma 2.1,

$$\|D^{\beta_j} f\|_{\dot{L}^{pp_j}_{\alpha-m}} \leq c\|f\|_{\dot{L}^{pp_j}_{\alpha-m+|\beta_j|}} \leq c\|f\|_{\dot{L}^{\alpha p}_{1}}^{1-\lambda_j} \|f\|_{\dot{L}^p_{\alpha}}^{\lambda_j},$$

where $\lambda_j = 1 - (m - |\beta_j|)/(\alpha - 1)$. We finally estimate $\|D^{\beta_i} f\|_{L^{\gamma_i p}}$ by (3.8) with the same values of θ_i . Altogether, this gives

$$(3.11) \quad \|II\|_{L^p} \leq cM\|f\|_{\dot{L}^{\alpha p}_{1}}^{\alpha(1-\mu)} \cdot \|f\|_{\dot{L}^p_{\alpha}}^{\mu}$$

with $\mu = (\alpha - k)/(\alpha - 1)$, which as noted above, is even correct when $k = 1$.

Putting estimates (3.9) and (3.11) into (3.7), we see that the term $cM\|f\|_{\dot{L}^{\alpha p}_{1}}^{\alpha}$ arises in (3.9) when $k = m$, while $cM\|f\|_{\dot{L}^p_{\alpha}}$ arises in (3.11) when $k = 1$. The other terms for $k \in \{1, \dots, m\}$ in (3.9) and (3.11) are dominated by one or the other of these extreme terms, so that

$$\|D^{\gamma} H(f)\|_{\dot{L}^p_{\alpha-m}} \leq cM(\|f\|_{\dot{L}^p_{\alpha}} + \|f\|_{\dot{L}^{\alpha p}_{1}}^{\alpha}),$$

for $|\gamma| = m$. With (2.5) and (3.1), the proof of Theorem A is complete. \square

Proof of Theorem B. First notice that $f \in L^p_m$ since $H(t) = t$ is m -admissible for any m . Let $\chi \in C^{\infty}(\mathbb{R}^n)$ satisfy $\text{supp } \chi \subseteq [-1, 1]$ and $\chi(t) = 1$ for $-2/3 \leq t \leq 2/3$. For $j = 1, \dots, m$, let

$$H_j(t) = \sum_{l \in \mathbb{Z}} (t - 2l)^j \chi(t - 2l)$$

and $A = \cup_{l \in \mathbb{Z}} (-2/3 + 2l, 2/3 + 2l)$. Also let $\tilde{H}_j(t) = H_j(t - 1)$ and $\tilde{A} = \cup_{l \in \mathbb{Z}} (1/3 + 2l, 5/3 + 2l)$. Clearly $A \cup \tilde{A} = \mathbb{R}$, and

$$(3.12) \quad H_j^{(j)}(t) = j! \quad \text{for } t \in A, \quad \text{and} \quad \tilde{H}_j^{(j)}(t) = j! \quad \text{for } t \in \tilde{A}.$$

Obviously, by periodicity both H_j and \tilde{H}_j are m -admissible for all j and m .

Now fix an $i \in \{1, \dots, n\}$. By the chain rule

$$(3.13) \quad D_i^{(m)} H(f) = \sum_{k=1}^m H^{(k)}(f) F_k,$$

where $F_k = \sum c_{m_1, \dots, m_k} D_i^{(m_1)} f \dots D_i^{(m_k)} f$, with the sum extending over all m_1, \dots, m_k such that $\sum_{j=1}^k m_j = m$, $m_j \geq 1$ for each j . (Recall $D_i = \partial/\partial x_i$). Note that F_k depends on f and i , but not on H . By assumption,

$D_i^{(m)}H(f) \in L^p$ for $H = H_j$ or $H = \tilde{H}_j$, $j = 1, \dots, m$. We show by induction that this implies that $F_j \in L^p$, $j = 1, \dots, m$. First, since we have noted that $f \in L_m^p$, we have $F_1 = D_i^{(m)}f \in L^p$. Now suppose $F_1, \dots, F_{j-1} \in L^p$. Let H be H_j in (3.13). By (3.12), which obviously implies that $H_j^{(k)}(t) = 0$ for $t \in A$ and $k \geq j + 1$, we have

$$D_i^{(m)}H_j(f)(x) = j!F_j(x) + \sum_{k=1}^{j-1} H_j^{(k)}(f(x)) \cdot F_k(x)$$

for $x \in f^{-1}(A)$. Since $H_j^{(k)} \in L^\infty$ and $F_1, \dots, F_{j-1} \in L^p$, it follows that $F_j \in L^p(f^{-1}(A))$. By the same argument with \tilde{H}_j in place of H_j , $F_j \in L^p(f^{-1}(\tilde{A}))$; hence $F_j \in L^p(\mathbb{R}^n)$. This completes the induction; after finitely many steps, we obtain that $F_m = (D_i f)^m \in L^p$, or $D_i f \in L^{mp}$. Since $i \in \{1, \dots, n\}$ is arbitrary, we have $f \in \dot{L}_1^{mp}$. \square

The analogue of Theorem B for $0 < \alpha \notin \mathbb{Z}^+$ remains open.

4. PROOF OF THE LEMMAS

Proof of Lemma 2.1. Applying Hölder’s inequality with exponents $1/(1 - \theta)$ and $1/\theta$ gives

$$\begin{aligned} \|f\|_{L_\alpha^p} &= \left\| \left[\sum_{\nu \in \mathbb{Z}} (2^{\nu(1-\theta)\alpha_1} |\phi_\nu * f|^{1-\theta} 2^{\nu\theta\alpha_2} |\phi_\nu * f|^\theta)^2 \right]^{1/2} \right\|_{L^p} \\ &\leq \left\| \left[\sum_{\nu \in \mathbb{Z}} (2^{\nu\alpha_1} |\phi_\nu * f|)^2 \right]^{(1-\theta)/2} \left[\sum_{\nu \in \mathbb{Z}} (2^{\nu\alpha_2} |\phi_\nu * f|)^2 \right]^{\theta/2} \right\|_{L^p}. \end{aligned}$$

Hölder’s inequality again, this time with conjugate exponents $p_1/(1 - \theta)p$ and $p_2/\theta p$, yields the result. \square

Proof of Lemma 2.2. This can be derived from the complex interpolation result $[\text{BMO}, \dot{L}_\alpha^p]_\theta \approx \dot{L}_{\alpha\theta}^{p/\theta}$. Also we could give a proof based on the theory of the “ ϕ -transform” S_ϕ in [6] and the Calderón product (see [6, Theorems 2.2, 9.2, 8.2]). However, we omit the proof since all that is used in the proof of Theorem A is the well-known Gagliardo–Nirenberg inequality [7], i.e. (2.8) with L^∞ in place of BMO. \square

Proof of Lemma 2.3. The inhomogeneous analogues of (2.9) and (2.10), i.e.

$$\|f\|_{L_\alpha^p} \approx \|f\|_{L^p} + \|S_\alpha\|_{L^p} \quad \text{and} \quad \|D_t^\alpha f\|_{L^p} \leq c\|f\|_{L_\alpha^p},$$

are proved in Strichartz [10], and Polking [9], respectively. Undoubtedly, (2.9) and (2.10) are implicit in their arguments. However, we want to note another proof based on the Littlewood–Paley definition of \dot{L}_α^p given in §2. The analogue of (2.10) in the inhomogeneous case for $t = 1$ is done in Triebel [11, Theorem 2.5.11]. With a few modifications, the same argument gives (2.10). The assumption $f \in L^q$ is used to prove the pointwise a.e. equality of f and $\sum_\nu \phi_\nu * f$ under assumptions (2.1)–(2.4). We omit the details.

We do, however, give a proof of the lower bound of $S_\alpha f$ in (2.9). This proof appears to be more direct than the corresponding proofs contained in the

literature. For $r > 0$, let

$$\begin{aligned} A_{r,\alpha}f(x) &= r^{-\alpha} \int_{|y|<1} |f(x+ry) - f(x)| dy \\ &= r^{-\alpha-n} \int_{|y|<r} |f(x+y) - f(x)| dy. \end{aligned}$$

It is easy to see that if $2^\mu \leq r \leq 2^{\mu+1}$, then

$$2^{-\alpha-n} A_{2^\mu,\alpha}f(x) \leq A_{r,\alpha}f(x) \leq 2^{\alpha+n} A_{2^{\mu+1},\alpha}f(x).$$

Thus a simple discretization of the integral from 0 to ∞ shows that

$$(4.1) \quad S_\alpha f(x) \approx \left(\sum_{\mu \in \mathbb{Z}} |A_{2^\mu,\alpha}f(x)|^2 \right)^{1/2},$$

with constants depending only on n and α .

By (2.2), $\int \phi = 0$, so for $\nu \in \mathbb{Z}$,

$$\begin{aligned} \phi_\nu * f(x) &= \int [f(x-y) - f(x)]\phi_\nu(y) dy \\ &= \int [f(x-2^{-\nu}y) - f(x)]\phi(y) dy. \end{aligned}$$

For $\mu \in \mathbb{Z}$, let $R_\mu = \{x \in \mathbb{R}^n : 2^{\mu-1} \leq |x| \leq 2^\mu\}$ and $B_\mu = \{x \in \mathbb{R}^n : |x| \leq 2^\mu\}$. Selecting $M > n + \alpha$, we have

$$\begin{aligned} 2^{\nu\alpha} |\phi_\nu * f(x)| &= 2^{\nu\alpha} \sum_{\mu \in \mathbb{Z}} \int_{R_\mu} |\phi(y)| \cdot |f(x-2^{-\nu}y) - f(x)| dy \\ &\leq 2^{\nu\alpha} \sum_{\mu \in \mathbb{Z}} \sup_{R_\mu} |\phi| \cdot \int_{B_\mu} |f(x-2^{-\nu}y) - f(x)| dy \\ &\leq c_M 2^{\nu\alpha} \sum_{\mu \in \mathbb{Z}} (1+2^\mu)^{-M} 2^{\nu n} \int_{B_{\mu-\nu}} |f(x+y) - f(x)| dy \\ &= c_M \sum_{\mu \in \mathbb{Z}} 2^{\mu(n+\alpha)} (1+2^\mu)^{-M} \cdot A_{2^{\mu-\nu},\alpha}f(x). \end{aligned}$$

This last sum is a discrete convolution and $\{2^{\mu(\alpha+n)}(1+2^\mu)^{-M}\} \in l^1(\mathbb{Z})$, so Young's inequality gives

$$\left(\sum_{\nu \in \mathbb{Z}} |2^{\nu\alpha} \phi_\nu * f(x)|^2 \right)^{1/2} \leq c_{\mu,\alpha} \left(\sum_{\mu \in \mathbb{Z}} |A_{2^\mu,\alpha}f(x)|^2 \right)^{1/2}.$$

Taking L^p -norms and using (4.1) gives $\|f\|_{L^p_\theta} \leq c \|S_\alpha f\|_{L^p}$. \square

Proof of Lemma 2.4. We first prove the case $k = 2$. Using (2.11) and applying the Cauchy-Schwarz inequality twice to the final term, gives

$$S_\theta(f_1 f_2)(x) \leq |f_1(x)| S_\theta f_2(x) + |f_2(x)| S_\theta f_1(x) + D_2^{\theta/2} f_1(x) D_2^{\theta/2} f_2(x).$$

(This is a special case of Polking's formula in [9].) Hence by (2.9) and the remark following Lemma 2.3,

$$\|f_1 f_2\|_{L^p_\theta} \leq c \|f_1 S_\theta f_2\|_{L^p} + c \|f_2 S_\theta f_1\|_{L^p} + c \|D_2^{\theta/2} f_1 D_2^{\theta/2} f_2\|_{L^p}.$$

Since $1/p_1 + 1/\gamma_2 = 1$, Hölder's inequality and (2.9) imply

$$\|f_2 S_\theta f_1\|_{L^p} \leq \|S_\theta f_1\|_{L^{p_1 p}} \|f_2\|_{L^{\gamma_2 p}} \leq c \|f_1\|_{\dot{L}_\theta^{p_1 p}} \|f_2\|_{L^{\gamma_2 p}}.$$

(Note that (2.9) and (2.10) apply to f_1 and f_2 since $f_j \in L^{\gamma_j p}$, $j = 1, 2$, else there is nothing to prove.) Similarly,

$$\|f_1 S_\theta f_2\|_{L^p} \leq c \|f_2\|_{\dot{L}_\theta^{p_2 p}} \|f_1\|_{L^{\gamma_1 p}}.$$

Define conjugate indices s and s' so that $1/s = (1/\gamma_1 + 1/p_1)/2$ and $1/s' = (1/\gamma_2 + 1/p_2)/2$. By Hölder's inequality and (2.10),

$$\|D_2^{\theta/2} f_1 D_2^{\theta/2} f_2\|_{L^p} \|D_2^{\theta/2} f_1\|_{L^{sp}} \|D_2^{\theta/2} f_2\|_{L^{s'p}} \leq c \|f_1\|_{\dot{L}_\theta^{sp}} \|f_2\|_{\dot{L}_\theta^{s'p}}.$$

By Lemma 2.1, we have

$$\|f_1\|_{\dot{L}_\theta^{sp}} \leq \|f_1\|_{L^{\gamma_1 p}}^{1/2} \|f_1\|_{\dot{L}_\theta^{p_1 p}}^{1/2}$$

and

$$\|f_2\|_{\dot{L}_\theta^{s'p}} \leq \|f_2\|_{L^{\gamma_2 p}}^{1/2} \|f_2\|_{\dot{L}_\theta^{p_2 p}}^{1/2}.$$

Thus since $|abcd|^{1/2} \leq |ab| + |cd|$,

$$\|f_1 f_2\|_{\dot{L}_\theta^p} \leq c(\|f_1\|_{\dot{L}_\theta^{p_1 p}} \|f_2\|_{L^{\gamma_2 p}} + \|f_2\|_{\dot{L}_\theta^{p_2 p}} \|f_1\|_{L^{\gamma_1 p}}),$$

as desired in the case $k = 2$. The general case follows by induction on k . We only sketch the details. Apply the case $k = 2$ with p_1 , γ_2 , p_2 , γ_1 , f_1 , f_2 replaced by $\gamma'_k = \gamma_k/(\gamma_k - 1)$, γ_k , p_k , $p'_k = p_k/(p_k - 1)$, $\prod_{i=1}^{k-1} f_i$, and f_k , respectively. This yields an estimate for $\|\prod_{i=1}^k f_i\|_{\dot{L}_\theta^p}$ involving two products of the form $\prod_{i=1}^{k-1}$. Using the $k - 1$ case for one, and Hölder's inequality for the other, yields the result. \square

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