GROUP COMPLETIONS AND ORBIFOLDS
OF VARIABLE NEGATIVE CURVATURE

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Abstract. W. J. Floyd's comparison of the Furstenberg maximal boundary of a noncompact, \( R \)-rank one, connected semisimple Lie group \( G \) with finite center and the group completion of a discrete, cocompact subgroup \( \Gamma \) of \( G \) is extended to a homeomorphism between the group completion of the fundamental group \( \Gamma \) of a closed Riemannian orbifold \( M = \Gamma \backslash X \) of strictly negative sectional curvatures and the sphere at infinity in the Eberlein-O'Neill compactification \( \overline{X} \) of the universal cover \( X \) of \( M \).

Two papers by Bill Floyd establish a close relationship between two distinct compactifications in the world of negative curvature [4, 5]. The first compactification is the group completion associated to the fundamental group of a compact (or sometimes geometrically finite) double coset space \( \Gamma \backslash G/K \) of a noncompact, \( R \)-rank one, connected semisimple Lie group with finite center, and the second is the Furstenberg maximal boundary of \( G \) appearing in the guise of the sphere at infinity in the disk model for \( G/K \). The present note extends these arguments to the fundamental group of any closed Riemannian orbifold of strictly negative curvature and to the Eberlein-O'Neill compactification \( \overline{X} \) of the universal covering space \( X \) of such an orbifold; this is our main result:

**Proposition.** If \( M^n = \Gamma \backslash X \) is a closed Riemannian orbifold whose sectional curvatures are bounded above by \( H < 0 \), with universal cover \( X \) and fundamental group \( \Gamma \), then there is a \( \Gamma \)-equivariant map \( \text{Completion}(\Gamma) \rightarrow \overline{X} \) that carries the completion points \( \text{Completion}(\Gamma) \backslash \Gamma \) homeomorphically to the sphere at infinity in \( \overline{X} \). □

The proposition generalizes the main theorem of [5] and is proved by Floyd's argument once the lemma below is established. Recall that Eberlein and O'Neill show [3] that a simply connected, complete Riemannian manifold \( X^n \) with sectional curvatures \( K \leq c < 0 \) has a compactification \( \overline{X} \) that is homeomorphic to the disk \( D^n \). \( \overline{X} \) is constructed from \( X \) by adding a copy of \( S^{n-1} \) that may be identified with the asymptotic classes of unit-speed geodesic rays in \( X \) and

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that is topologized by considering geodesic cones in $X$. (Two geodesic rays $r_1(t)$ and $r_2(t)$ of the same speed are asymptotic if the distance between $r_1(t)$ and $r_2(t)$ remains bounded as $t \to \infty$. The sphere at infinity may be identified with the unit sphere in the tangent space $T_pX$ for any point $p \in X$, since for each unit-speed geodesic ray $r(t)$ in $X$ there is one and only one unit-speed geodesic ray out of $p$ that is asymptotic to $r$.) Since isometries carry geodesic rays to geodesic rays, preserving speed, the action of the isometry group $\text{Isometries}(X)$ on $X$ extends to $\overline{X}$.

A finitely generated group $\Gamma$ is usually topologized by a word norm with respect to a generating set $\Sigma = \{x_i : 1 \leq i \leq g\}$, in which for each $g \in \Gamma$ we define $|g|$ to be the minimal word length over all words in the $x_i$ representing $g$, where the word length of $x_1^{a_1} \cdots x_m^{a_m}$ is $\sum_{1 \leq j \leq m} |a_j|$. This defines a left-invariant metric on $\Gamma$ by $d_{\text{word}}(g, h) = |g^{-1}h|$ and is the restriction to $\Gamma$ of a unique left-invariant simplicial metric on the graph $K(\Gamma, \Sigma)$ of the group with respect to the presentation above (so group elements define vertices of this graph and two vertices $a, b$ are adjacent if and only if $a = bx_i^{\pm 1}$ for some generator $x_i$). Although the word metric depends on the choice of generators for $\Gamma$, two finite generating sets will lead to commensurable word metrics on the group and its graph. The group completion studied by Floyd is defined beginning with a monic, summable function $\sigma : \mathbb{Z}(\geq 0) \to \mathbb{R}(\geq 0)$ such that for each $k \in \mathbb{Z}(\geq 0)$ there exist positive $M, N$ such that $Mf\sigma(r) \leq \sigma(kr) \leq N\sigma(r)$ for all $r \in \mathbb{Z}(\geq 0)$. The standard example of such a function is $\sigma(r) = r^{-2}$, and we will assume $\sigma$ has this form below. Now declare two adjacent vertices $a, b \in K(\Gamma, \Sigma)$ to lie at distance $\min(\sigma(a), \sigma(b))$ and extend this to a metric on $\Gamma$ by taking shortest paths between vertices in the graph; denote the resulting metric by $d_\sigma$. The group completion $\text{Completion}(\Gamma)$ is the Cauchy completion of $\Gamma$ with respect to this metric $d_\sigma$; note that the action of $\Gamma$ on its graph does not preserve this metric, but each element of $\Gamma$ acts by a homeomorphism that is uniformly Lipschitz with respect to $d_\sigma$, so $\Gamma$ has an induced action on the completion points, $\text{Completion}(\Gamma) \setminus K(\Gamma, \Sigma)$.

Floyd's argument proceeds on the following outline. Covering space theory identifies $\Gamma$ with a group of isometries of $X$. The proof of the proposition studies the imbedding of the graph $K(\Gamma, \Sigma)$ in the manifold $X$ defined by picking a basepoint $p \in X$, sending $g \in \Gamma$ to $g(p)$, and by sending edges $[a, b]$ of the graph to the unique geodesic segment joining $a(p)$ to $b(p)$. This imbedding is a $\Gamma$-equivariant quasi-isometry with respect to the Riemannian metric on $X$ and the word metric on $\Gamma$; this imbedding is also a $\Gamma$-equivariant Lipschitz map between $K(\Gamma, \Sigma)$ in the weighted word metric $d_\sigma$ and the interior of $\overline{X}$, viewed as the disk model for $X$, in the Euclidean metric $d_{\text{Euc}}$. Proving the second of these assertions is the main technical work in the argument and depends upon the lemma generalized below. The Lipschitz map $(K(\Gamma, \Sigma), d_\sigma) \to (\text{int}(D^n), d_{\text{Euc}})$ induces a map between the completions of these metric spaces, and this is the claimed $\Gamma$-equivariant homeomorphism.

It is important to remember that the generalization in this setting of the disk model for hyperbolic space [3, Theorem 2.10, p. 54] has the following description. Given a point $p$ of $X$, let $D(p)$ be the closed unit disk in the tangent space $T_pX$, let $S(p)$ be the boundary sphere of $D(p)$, let $f : [0, 1] \to [0, \infty]$ be a homeomorphism, and define $h : \text{int}(D(p)) \to X$ by $h(v) := \exp_p(f(|v|)v)$. 

h is a homeomorphism and extends to a homeomorphism (also denoted h): $D(p) \to X$ that carries the unit vector $v$ in $S(p)$ to the asymptotic class of the unit-speed geodesic ray out of $p$ defined by $t \mapsto \exp_p(tv)$. The metrics appearing in the statement of the following lemma will be given precise definitions in the course of the proof.

**Lemma.** Let $X$ be a complete, simply connected Riemannian manifold of sectional curvatures bounded above by $H < 0$, let $p$ be a point of $X$, and let $D(p)$ be the closed unit disk in the tangent space at $p$. Given $k > 0$ there exists $K > 0$ such that if interior points $v$ and $w$ of $D(p)$ satisfy $d_X(v, w) \leq k$ and $d_X(0, v) = R$ then $d_{Euc}(v, w) \leq Ke^{-R}$.

**Proof.** Fix a homeomorphism $f$ as above. The main element of this proof is a comparison of the unit disk models for $X^n$ and for $Y^n$, where $Y^n$ is the simply connected, complete Riemannian manifold of constant sectional curvature $H$. Select $p$ in $X$ and $q$ in $Y$, let $D(p)$ be the closed unit disk about 0 in $T_pX$ and let $D(q)$ be the closed unit disk about 0 in $T_qY$, and form

$$\alpha: D(p) \to X, \quad v \mapsto \exp_p(f(|v|)v)$$

and

$$\beta: D(q) \to Y, \quad w \mapsto \exp_q(f(|w|)w).$$

If $u$ and $v$ belong to $D(p)$ then let $d_{Euc}(u, v) := |u - v|$; abuse notation if $u$ and $v$ lie in the interior of $D(p)$ to write $d_X(u, v) := d(\alpha(u), \alpha(v))$, where $d$ is the metric on $X$. Similarly define $d_{Euc}$ on $D(q)$ and $d_Y$ on $\text{int}(D(q))$. In each case $d_{Euc}$ is the Euclidean metric on the unit disk and the other metric ($d_X$ or $d_Y$) is a metric on the interior of the unit disk with negative sectional curvatures.

Let $U: T_pX \to T_qY$ be an inner-product preserving linear map and consider the square

$$\begin{array}{ccc}
  D(p) & \xrightarrow{U} & D(q) \\
  \downarrow{\alpha} & & \downarrow{\beta} \\
  \overline{X} & \xrightarrow{u} & \overline{Y}
\end{array}$$

where the lower horizontal arrow $u: \overline{X} \to \overline{Y}$ is the composite $\beta \circ U \circ \alpha^{-1}$, carrying the geodesic spray at $p$ to the geodesic spray at $q$. Observe that $u$ is given more efficiently if $x$ is a point of $X$ by the formula $u(x) = \exp_q(U(\exp^{-1}_p(x)))$.

The Rauch Comparison Theorem implies that $u|: X \to Y$ is length-reducing, in the sense that if $c: [0, 1] \to X$ is a $C^1$ curve then $\text{Length}(c) \geq \text{Length}(uc(c))$ [2, Corollary 1.30, p. 30]. It follows that $u|: X \to Y$ is distance-reducing in the sense that $d_X(w, x) \geq d_Y(u(w), u(x))$ for all $w, x \in X$.

The calculations done by Floyd [5, pp. 1018–1019] in real hyperbolic space $\mathbb{R}H^n$ and the disk model for $\mathbb{R}H^n$ apply in $Y$ and $D(q)$ as well: given $k > 0$ there exists a $K > 0$ such that if $v, w$ are points of $D(q)$ with $d_Y(v, w) \leq k$, and $d_Y(0, v) = R$, then $d_{Euc}(v, w) \leq Ke^{-R}$. Observe now that if $v$ belongs to $D(p)$ then

$$d_X(0, v) = d(p, \alpha(v)) = |f(|v|)v| = f(|v|)|v|.$$
while
\[ d_Y(0, U(v)) = d(q, \beta(U(v))) = |f(|U(v)|)U(v)| = f(|U(v)|)|U(v)| = f(|v||v| = d_X(0, v). \]

If \( v, w \in D(p) \) and satisfy \( d_X(v, w) \leq k \) and \( d_X(0, v) = R \), then
\[ d_Y(U(v), U(w)) \leq d_X(v, w) \leq k \quad \text{and} \quad d_Y(0, U(v)) = d_X(0, v) = R, \]
so \( d_{\text{Euc}}(v, w) = d_{\text{Euc}}(U(v), U(w)) \leq K e^{-R}. \)

Floyd proves a version of the main result above for geometrically finite Kleinian groups in [4], using facts on points of approximation. The generalization of that argument to higher dimensions should follow from extensions to variable negative curvature of results in the paper of Tukia [6] and the work of Apanasov cited there. A technique for comparing the group completion directly to the Gromov construction of a completion [1, §3] (which coincides in the manifold case with the Eberlein-O'Neill compactification) would also be of interest.

References