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GROUP COMPLETIONS AND ORBIFOLDS OF VARIABLE NEGATIVE CURVATURE

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ABSTRACT. W. J. Floyd's comparison of the Furstenberg maximal boundary of a noncompact, **R**-rank one, connected semisimple Lie group G with finite center and the group completion of a discrete, cocompact subgroup Γ of G is extended to a homeomorphism between the group completion of the fundamental group Γ of a closed Riemannian orbifold $M = \Gamma\backslash X$ of strictly negative sectional curvatures and the sphere at infinity in the Eberlein-O'Neill compactification $\overline{X}$ of the universal cover X of M.

Two papers by Bill Floyd establish a close relationship between two distinct compactifications in the world of negative curvature [4, 5]. The first compactification is the group completion associated to the fundamental group of a compact (or sometimes geometrically finite) double coset space $\Gamma\backslash G/K$ of a noncompact, **R**-rank one, connected semisimple Lie group with finite center, and the second is the Furstenberg maximal boundary of G appearing in the guise of the sphere at infinity in the disk model for G/K. The present note extends these arguments to the fundamental group of any closed Riemannian orbifold of strictly negative curvature and to the Eberlein-O'Neill compactification $\overline{X}$ of the universal covering space X of such an orbifold; this is our main result:

Proposition. *If $M^n = \Gamma\backslash X$ is a closed Riemannian orbifold whose sectional curvatures are bounded above by $H < 0$, with universal cover X and fundamental group Γ, then there is a Γ-equivariant map* Completion(Γ) $\to \overline{X}$ *that carries the completion points* Completion(Γ)$\backslash\Gamma$ *homeomorphically to the sphere at infinity in $\overline{X}$.* □

The proposition generalizes the main theorem of [5] and is proved by Floyd's argument once the lemma below is established. Recall that Eberlein and O'Neill show [3] that a simply connected, complete Riemannian manifold X^n with sectional curvatures $K \leq c < 0$ has a compactification $\overline{X}$ that is homeomorphic to the disk D^n. $\overline{X}$ is constructed from X by adding a copy of S^{n-1} that may be identified with the asymptotic classes of unit-speed geodesic rays in X and

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that is topologized by considering geodesic cones in X. (Two geodesic rays $r_1(t)$ and $r_2(t)$ of the same speed are asymptotic if the distance between $r_1(t)$ and $r_2(t)$ remains bounded as $t \to \infty$. The sphere at infinity may be identified with the unit sphere in the tangent space T_pX for any point $p \in X$, since for each unit-speed geodesic ray $r(t)$ in X there is one and only one unit-speed geodesic ray out of p that is asymptotic to r.) Since isometries carry geodesic rays to geodesic rays, preserving speed, the action of the isometry group $\mathrm{Isometries}(X)$ on X extends to $\overline{X}$.

A finitely generated group Γ is usually topologized by a word norm with respect to a generating set $\Sigma = \{x_i : 1 \le i \le g\}$, in which for each $g \in \Gamma$ we define $|g|$ to be the minimal word length over all words in the x_i representing g, where the word length of $x_{i_1}^{a_1} \cdots x_{i_m}^{a_m}$ is $\sum_{1 \le j \le m} |a_j|$. This defines a left-invariant metric on Γ by $d_{\mathrm{word}}(g, h) = |g^{-1}h|$ and is the restriction to Γ of a unique left-invariant simplicial metric on the graph $K(\Gamma, \Sigma)$ of the group with respect to the presentation above (so group elements define vertices of this graph and two vertices a, b are adjacent if and only if $a = bx_i^{\pm 1}$ for some generator x_i). Although the word metric depends on the choice of generators for Γ, two finite generating sets will lead to commensurable word metrics on the group and its graph. The group completion studied by Floyd is defined beginning with a monic, summable function $\sigma: \mathbf{Z}(\ge 0) \to \mathbf{R}(\ge 0)$ such that for each $k \in \mathbf{Z}(\ge 0)$ there exist positive M, N such that $Mf\sigma(r) \le \sigma(kr) \le N\sigma(r)$ for all $r \in \mathbf{Z}(\ge 0)$. The standard example of such a function is $\sigma(r) = r^{-2}$, and we will assume σ has this form below. Now declare two adjacent vertices $a, b \in K(\Gamma, \Sigma)$ to lie at distance $\min(\sigma(a), \sigma(b))$ and extend this to a metric on Γ by taking shortest paths between vertices in the graph; denote the resulting metric by d_σ. The group completion $\mathrm{Completion}(\Gamma)$ is the Cauchy completion of Γ with respect to this metric d_σ; note that the action of Γ on its graph does not preserve this metric, but each element of Γ acts by a homeomorphism that is uniformly Lipschitz with respect to d_σ, so Γ has an induced action on the completion points, $\mathrm{Completion}(\Gamma)\backslash K(\Gamma, \Sigma)$.

Floyd's argument proceeds on the following outline. Covering space theory identifies Γ with a group of isometries of X. The proof of the proposition studies the imbedding of the graph $K(\Gamma, \Sigma)$ in the manifold X defined by picking a basepoint $p \in X$, sending $g \in \Gamma$ to $g(p)$, and by sending edges $[a, b]$ of the graph to the unique geodesic segment joining $a(p)$ to $b(p)$. This imbedding is a Γ-equivariant quasi-isometry with respect to the Riemannian metric on X and the word metric on Γ; this imbedding is also a Γ-equivariant Lipschitz map between $K(\Gamma, \Sigma)$ in the weighted word metric d_σ and the interior of $\overline{X}$, viewed as the disk model for X, in the Euclidean metric d_{Euc}. Proving the second of these assertions is the main technical work in the argument and depends upon the lemma generalized below. The Lipschitz map $(K(\Gamma, \Sigma),\ d_\sigma) \to (\mathrm{int}(D^n),\ d_{\mathrm{Euc}})$ induces a map between the completions of these metric spaces, and this is the claimed Γ-equivariant homeomorphism.

It is important to remember that the generalization in this setting of the disk model for hyperbolic space [3, Theorem 2.10, p. 54] has the following description. Given a point p of X, let $D(p)$ be the closed unit disk in the tangent space T_pX, let $S(p)$ be the boundary sphere of $D(p)$, let $f: [0, 1] \to [0, \infty]$ be a homeomorphism, and define $h: \mathrm{int}(D(p)) \to X$ by $h(v) := \exp_p(f(|v|)v)$.

h is a homeomorphism and extends to a homeomorphism (also denoted h): $D(p) \to \overline{X}$ that carries the unit vector v in $S(p)$ to the asymptotic class of the unit-speed geodesic ray out of p defined by $t \mapsto \exp_p(tv)$. The metrics appearing in the statement of the following lemma will be given precise definitions in the course of the proof.

Lemma. *Let X be a complete, simply connected Riemannian manifold of sectional curvatures bounded above by $H < 0$, let p be a point of X, and let $D(p)$ be the closed unit disk in the tangent space at p. Given $k > 0$ there exists $K > 0$ such that if interior points v and w of $D(p)$ satisfy $d_X(v, w) \le k$ and $d_X(0, v) = R$ then $d_{\text{Euc}}(v, w) \le Ke^{-R}$.*

Proof. Fix a homeomorphism f as above. The main element of this proof is a comparison of the unit disk models for X^n and for Y^n, where Y^n is the simply connected, complete Riemannian manifold of constant sectional curvature H. Select p in X and q in Y, let $D(p)$ be the closed unit disk about 0 in T_pX and let $D(q)$ be the closed unit disk about 0 in T_pY, and form

$$\alpha: D(p) \to \overline{X}, \qquad v \mapsto \exp_p(f(|v|)v)$$

and

$$\beta: D(q) \to \overline{Y}, \qquad w \mapsto \exp_q(f(|w|)w).$$

If u and v belong to $D(p)$ then let $d_{\text{Euc}}(u, v) := |u - v|$; abuse notation if u and v lie in the interior of $D(p)$ to write $d_X(u, v) := d(\alpha(u), \alpha(v))$, where d is the metric on X. Similarly define d_{Euc} on $D(q)$ and d_Y on $\operatorname{int}(D(q))$. In each case d_{Euc} is the Euclidean metric on the unit disk and the other metric (d_X or d_Y) is a metric on the interior of the unit disk with negative sectional curvatures.

Let U: $T_pX \to T_qY$ be an inner-product preserving linear map and consider the square

$$\begin{array}{ccc} D(p) & \xrightarrow{U} & D(q) \\ {\scriptstyle\alpha}\big\downarrow & & \big\downarrow{\scriptstyle\beta} \\ \overline{X} & \xrightarrow{u} & \overline{Y}, \end{array}$$

where the lower horizontal arrow $u: \overline{X} \to \overline{Y}$ is the composite $\beta \circ U \circ \alpha^{-1}$, carrying the geodesic spray at p to the geodesic spray at q. Observe that u is given more efficiently if x is a point of X by the formula $u(x) = \exp_q(U(\exp_p^{-1}(x)))$.

The Rauch Comparison Theorem implies that $u|$: $X \to Y$ is length-reducing, in the sense that if c: $[0, 1] \to X$ is a C^1 curve then $\text{Length}(c) \ge \text{Length}(u(c))$ [2, Corollary 1.30, p. 30]. It follows that $u|$: $X \to Y$ is distance-reducing in the sense that $d_X(w, x) \ge d_Y(u(w), u(x))$ for all $w, x \in X$.

The calculations done by Floyd [5, pp. 1018–1019] in real hyperbolic space $\mathbf{R}H^n$ and the disk model for $\mathbf{R}H^n$ apply in Y and $D(q)$ as well: given $k > 0$ there exists a $K > 0$ such that if v, w are points of $D(q)$ with $d_Y(v, w) \le k$, and $d_Y(0, v) = R$, then $d_{\text{Euc}}(v, w) \le Ke^{-r}$. Observe now that if v belongs to $D(p)$ then

$$d_X(0, v) = d(p, \alpha(v)) = |f(|v|)v| = f(|v|)|v|,$$

while

$$d_Y(0, U(v)) = d(q, \beta(U(v))) = |f(|U(v)|)U(v)|$$
$$= f(|U(v)|)|U(v)| = f(|v|)|v| = d_X(0, v).$$

If $v, w \in D(p)$ and satisfy $d_X(v, w) \le k$ and $d_X(0, v) = R$, then

$$d_Y(U(v), U(w)) \le d_X(v, w) \le k \quad \text{and} \quad d_Y(0, U(v)) = d_X(0, v) = R,$$

so $d_{\text{Euc}}(v, w) = d_{\text{Euc}}(U(v), U(w)) \le Ke^{-R}$. $\square$

Floyd proves a version of the main result above for geometrically finite Kleinian groups in [4], using facts on points of approximation. The generalization of that argument to higher dimensions should follow from extensions to variable negative curvature of results in the paper of Tukia [6] and the work of Apanasov cited there. A technique for comparing the group completion directly to the Gromov construction of a completion [1, §3] (which coincides in the manifold case with the Eberlein-O'Neill compactification) would also be of interest.

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