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ON THE STABLE RANK OF H^∞

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ABSTRACT. We prove that if f_1, f_2 are corona data and f_1 is the product of finitely many interpolating Blaschke products, then there exist corona solutions g_1, g_2 with $g_1^{-1} \in H^\infty(D)$. This provides a partial result in the direction of proving the stable rank of the algebra of bounded analytic functions on the open unit disc is one.

1. INTRODUCTION

Let A be a commutative ring with identity. An n-tuple $a \in A^n$ is said to be unimodular if there exists $b \in A^n$ such that $\sum_{i=1}^n b_i a_i = 1$. We denote the set of unimodular elements of A^n by $U_n(A)$. An element $a \in A^n$ is said to be reducible if there exists $x_1, \dots, x_{n-1} \in A$ such that

$$(a_1 + x_1 a_n, a_2 + x_2 a_n, \dots, a_{n-1} + x_{n-1} a_n) \in U_{n-1}(A).$$

We define the stable rank of A, denote by $\mathrm{sr}(A)$, to be the least $n-1$ with the property that every $a \in U_n(A)$ is reducible.

The notion of stable rank has been useful in studying problems relating to the structure of commutative Banach algebras (see [1]) and recently some work has been done on calculating the stable rank of various algebras of analytic functions. In [6] it is shown that the disc algebra has stable rank 1 and thereby answered a question raised by Rieffel in [9] in which a related concept, the topological stable rank is introduced. It is defined whenever A is a Banach algebra by $\mathrm{tsr}(A) = \min\{n : U_n(A) \text{ is dense in } A^n\}$. Rieffel leaves open whether $\mathrm{sr}(A) = \mathrm{tsr}(A)$ and points out that if A is the disc algebra then $\mathrm{tsr}(A) = 2$. However, in [4] it is shown that whenever A is a unital C^*-algebra, $\mathrm{sr}(A) = \mathrm{tsr}(A)$.

In [1] it is conjectured that the stable rank of the algebra of bounded analytic functions on the unit disc is 1; that is every $(f_1, f_2) \in U_2(H^\infty(D))$ is reducible to an element of $(H^\infty(D))^{-1}$, the invertible elements of $H^\infty(D)$. Equivalently, via the corona theorem, this means given $f_1, f_2 \in H^\infty(D)$, $|f_1| + |f_2| \geq \delta > 0$, there exists $g_1 \in (H^\infty(D))^{-1}$, $g_2 \in H^\infty(D)$ such that $f_1 g_1 + f_2 g_2 = 1$. The purpose of this paper is to provide a partial result in this direction. Before stating our theorem we require some definitions.

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We denote by H^∞ the algebra of bounded analytic functions on the unit disc $D = \{z : |z| < 1\}$ equipped with the supremum norm. $A(D)$ denotes the disc algebra, that is $A(D) = H^\infty(D) \cap C(\overline{D})$. A sequence $\{z_j\} \subseteq D$ is called a Blaschke sequence if $\sum_j (1 - |z_j|) < \infty$ and the bounded analytic function

$$B(z) = \prod_{n=1}^{\infty} \frac{\overline{z}_n}{|z_n|} \frac{z_n - z}{1 - \overline{z}_n z}$$

is called a Blaschke product. A Blaschke sequence $\{z_j\}$ for which every interpolation problem $f(z_j) = a_j$, $\{a_j\} \in l^\infty$ has a solution $f \in H^\infty$ is called an interpolating sequence and the corresponding Blaschke product is called an interpolating Blaschke product. It is not known whether the set of interpolating Blaschke products are dense in the set of Blaschke products. A positive answer to this question implies $\mathrm{sr}(H^\infty) = 1$ (see corollary below). The theorem we prove is the following:

Theorem 1. *Let* $f_1, f_2 \in H^\infty(D)$ *with* $\inf_{z \in D} \max(|f_1(z)|, |f_2(z)|) \geq \delta > 0$. *If* f_1 *is the product of finitely many interpolating Blaschke products, then there exists* $g_1 \in (H^\infty(D))^{-1}$, $g_2 \in H^\infty(D)$ *such that* $f_1 g_1 + f_2 g_2 = 1$.

The existence of $g_1, g_2 \in H^\infty(D)$ with $f_1 g_1 + f_2 g_2 = 1$ follows from Carleson's corona theorem (see [3, Chapter VIII]). However the proofs of the corona theorem do not give $g_1 \in (H^\infty(D))^{-1}$. If f_1 is the finite product of interpolating Blaschke products, it is not too difficult to obtain $g_1 \in H^\infty(D)$, $g_2 \in (H^\infty(D))^{-1}$. See [7, Corollary 3.5]. Theorem 1 is proved in [1, 6] under the assumption that $f_1, f_2 \in A(D)$. This has been extended to the case $f_1 \in A(D)$, $f_2 \in H^\infty(D)$ in [2] by first showing that it suffices to assume $f_1(z) = z$. We note that this is then a special case of Theorem 1. More generally, Laroco [8] has shown that $\mathrm{sr}(H^\infty) = 1$ if $\log f_1$ can be boundedly analytically defined on $\{z : |f_2(z)| < \varepsilon\}$ for some $\varepsilon > 0$. It is also shown in [8] (Theorem 3.6) that in proving the reducibility of a general corona pair (f_1, f_2) we can assume f_1 is a Blaschke product. Combining Theorem 1 with some of the results in [8], we have the following corollary, which was shown to me by L. Laroco.

Corollary. *If every Blaschke product can be uniformly approximated by interpolating Blaschke products, then* $\mathrm{sr}(H^\infty(D)) = 1$.

Proof of Corollary. Let $(f_1, f_2) \in U_2(H^\infty(D))$, which we require to show is reducible. The hypothesis of the corollary and the proof of Theorem 1.1 in [8] show that the set $\{Bh : B$ is an interpolating Blaschke product, $h \in (H^\infty(D))^{-1}\}$ is dense in $H^\infty(D)$. Consequently, by Corollary 1.2 in [8] there exists interpolating Blaschke products B_1, B_2 and $h_1, h_2 \in (H^\infty(D))^{-1}$ such that

$$f_1 B_1 h_1 + f_2 B_2 h_2 = 1. \tag{1}$$

We now utilize the proof of Theorem 3.6 in [8]. Equation (1) implies $(B_1, f_2) \in U_2(H^\infty(D))$ and so by Theorem 1 is reducible to $(H^\infty(D))^{-1}$; that is there exists $g_1 \in (H^\infty(D))^{-1}$, $g_2 \in H^\infty(D)$ such that $B_1 + g_2 f_2 = g_1$. Substituting into (1) gives

$$f_1(g_1 - g_2 f_2) h_1 + f_2 B_2 h_2 = f_1 g_1 h_1 + f_1(B_2 h_2 - f_1 g_2 h_1) = 1$$

and $g_1 h_1 \in (H^\infty(D))^{-1}$ as required. □

2. Proof of Theorem 1

In proving Theorem 1 it is convenient to work in the upper halfplane $\mathbf{R}_2^+$. We denote by $H^\infty(\mathbf{R}_2^+)$, or simply H^∞, the algebra of bounded analytic functions on $\mathbf{R}_2^+$. In $\mathbf{R}_2^+$ the Blaschke product with zeros $\{z_n\}$ is

$$B(z) = \prod_{n=0}^{\infty} \alpha_n \left(\frac{z - z_n}{z - \overline{z}_n}\right) \quad \text{where } \alpha_n = \frac{|1 + z_n^2|}{1 + z_n^2}.$$

Denote by

$$\delta(B) = \inf_n \prod_{k \neq n} \left|\frac{z_k - z_n}{z_k - \overline{z}_n}\right|$$

and by $\rho(z, w) = |(z - w)/(z - \overline{w})|$ the pseudo-hyperbolic distance between $z, w \in \mathbf{R}_2^+$. If $z \in \mathbf{R}_2^+$, $r > 0$ let $D(z, r) = \{w \in \mathbf{R}_2^+ : \rho(z, w) < r\}$. A Carleson cube in $\mathbf{R}_2^+$ is a cube of the form $Q = \{(x, y) : x_0 < x < x_0 + h, 0 < y < h\}$, and we denote its length by $l(Q)$. A measure μ on $\mathbf{R}_2^+$ is called a Carleson measure if $|\mu|(Q) \leq Cl(Q)$ for all Carleson cubes Q. Carleson's interpolation theorem states that the following are equivalent:

(1) $\{z_j\}$ is an interpolating sequence in $\mathbf{R}_2^+$;

(2) there exists $\eta > 0$ such that $\inf_k \prod_{j \neq k} \rho(z_j, z_k) \geq \eta > 0$;

(3) there exists $a > 0$ such that for all $j \neq k$, $\rho(z_j, z_k) \geq a$, and the measure $d\mu = \sum_j (1 - |z_j|)\delta_{z_j}$, is a Carleson measure where δ_z denotes the Dirac measure at z (see [3, Chapter VII]).

Finally, if $E \subseteq \mathbf{R}_2^+$ we denote by E^* the set $\{x : x + iy \in E \text{ for some } y\}$.

Lemma 1. *Let $f_1, \dots, f_n, g \in H^\infty$. If $(f_i, g) \in U_2(H^\infty)$, $1 \leq i \leq N$ and each (f_i, g) is reducible to $(H^\infty)^{-1}$ then $(\prod_{i=1}^n f_i, g)$ is reducible to $(H^\infty)^{-1}$.*

Proof. The hypothesis of the lemma implies there exist $k_i \in H^\infty$, $h_i \in (H^\infty)^{-1}$ such that $f_i + k_i g = h_i$. Then $\prod_{i=1}^n (f_i + k_i g) = \prod_{i=1}^n h_i$, which implies $(\prod_{i=1}^n f_i) + kg = \prod_{i=1}^n h_i$ for some $k \in H^\infty$ and $\prod_{i=1}^n h_i \in (H^\infty)^{-1}$. □

A consequence of Lemma 1 is that we need only prove Theorem 1 for interpolating Blaschke products.

Lemma 2. *Let $0 < \eta_0 < 1$. Then there exists $\mu_0 = \mu_0(\eta_0)$, $0 < \eta_0 < 1$ such that for all $0 < \mu \leq \mu_0$ there exists $\lambda = \lambda(\mu)$ such that if $B(z)$ is an interpolating Blaschke product with zeros $\{z_k\}$ and $\inf_n \prod_{k \neq n} \rho(z_k, z_n) \geq \eta_0$, then*

(i) *$\{z : |B(z)| < \mu\} \subseteq \bigcup_n D(z_n, \lambda)$ and*

(ii) *$D(z_n, \lambda) \cap D(z_k, \lambda) = \varnothing$ for all $k \neq n$.*

Furthermore $\lambda(\mu)$ may be chosen so that $\lim_{\mu \to 0} \lambda(\mu) = 0$.

Proof. The proof of this result for the unit disc is contained in the proof of Lemma 4.2 in [5]. The result for $\mathbf{R}_2^+$ follows by a conformal map. □

Lemma 3. *Let $\{z_k\}_{k \geq 1}$ be an interpolating sequence with $\inf_n \prod_{k \neq n} \rho(z_k, z_n) = \eta > 0$. If $\rho(z, z_k) < \eta/3$, then $\prod_{n \neq k} \rho(z, z_n) > \eta/3$.*

Proof. The triangle inequality for the metric ρ implies

$$\prod_{n \neq k} \rho(z, z_n) \geq \prod_{n \neq k} \frac{\rho(z_k, z_n) - \rho(z, z_k)}{1 - \rho(z, z_k)\rho(z_n, z_k)}. \tag{2}$$

Let $B(z)$ be the Blaschke product with zeros $\{\rho(z_k, z_n)\}_{n\neq k}$. Note that the right-hand side of (2) is $B(\rho(z, z_k))$ while $B(0) = \prod_{n\neq k}\rho(z_k, z_n)$. Schwarz's lemma implies

$$\frac{|B(\rho(z, z_k)) - B(0)|}{|1 - B(\rho(z, z_k))\overline{B(0)}|} \leq \rho(z, z_k)$$

and hence,

$$|B(\rho(z, z_k))| \geq \frac{|B(0)| - \rho(z, z_k)}{1 + \rho(z, z_k)} > \frac{\eta}{3}$$

Hence by (2) we have $\prod_{n\neq k}\rho(z, z_n) > \eta/3$. □

Lemma 4. *Let $B(z)$ be an interpolating Blaschke product with zeros $\{z_k\}$ and let $\delta(B) = \inf_n \prod_{k\neq n}\rho(z_k, z_n)$. Then B has a factorization $B = B_1B_2$ such that $\delta(B_j) \geq \delta(B)^{1/2}$, $j = 1, 2$.*

Proof. See Corollary 1.6 in Chapter X of [3]. □

Lemma 5. *Let $B(z)$ be an interpolating Blaschke product with zeros $\{z_k\}$ and suppose $\delta(B) > 0$. Then if Q is a Carleson cube and $z_k \in Q$, $\Im z_k > \frac{1}{2}l(Q)$, then*

$$\sum_{\substack{z_j \in Q \\ j\neq k}} y_j \leq 5\left(\log\frac{1}{\delta(B)}\right) l(Q).$$

Proof. This result is contained in the proof of Lemma 2 [7, p. 267]. □

We also need a version of Theorem 1.1 in Chapter VIII of [3].

Lemma 6. *Let $d\mu = h\,dx\,dy$ be a Carleson measure on $\mathbf{R}_2^+$, where $h \in C^\infty(\mathbf{R}_2^+)$ and $\operatorname{supp} h \cap \overline{\mathbf{R}}_2^+ \subseteq \{z\colon |z| \leq R\}$ for some $R > 0$. Then there exists $u \in C(\overline{\mathbf{R}}_2^+) \cap C^1(\mathbf{R}_2^+)$ such that $\bar\partial u = h$ and $\sup_{x\in\mathbf{R}}|u(x)| \leq C$ where C depends only on $\sup_Q |\mu|(Q)/|Q|$.*

Proof. For $z \in \overline{\mathbf{R}}_2^+$, define

$$F(z) = \frac{1}{\pi}\int\int_{\mathbf{R}_2^+}\frac{h(\zeta)}{\zeta - z}du\,dv, \qquad \zeta = u + iv.$$

Then since h has compact support, $F \in C(\overline{\mathbf{R}}_2^+)$. Also, $F \in C^1(\mathbf{R}_2^+)$ since F is convolution of a function in $C^\infty(\mathbf{R}^2\backslash\{0\})$ with a function in $C^\infty(\mathbf{R}_2^+)$. Also note that $F \in C_0 = \{f \in C(\mathbf{R})\colon \lim_{x\to\pm\infty} f(x) = 0\}$. The argument that $\bar\partial F = h$ is essentially the same as the corresponding argument in [3, p. 319]. To obtain the solution u, use the duality argument in [3, p. 321], observing that the dual of $C_0/C_0 \cap H^\infty$ is H^1 (see [7, p. 193]). □

We prove Theorem 1 with $f_1(z) = B(z)$, where $B(z)$ is an interpolating Blaschke product with zeros $\{z_k\}$. Let $\delta_0 = \inf_{z\in\mathbf{R}_2^+}\max(|B(z)|, |f_2(z)|)$ and we can assume without loss of generality that $\|f_2\|_\infty \leq 1$. With $\eta_0 = \inf_n \prod_{k\neq n}\rho(z_k, z_n)$ in Lemma 2, choose $0 < \mu_1 < \delta_0$ so that $\lambda_1 = \lambda_1(\mu_1) < \delta_0/8$. Also choose $\eta \geq \eta_0$ so that $100 \log 1/\eta < 1/32$ and note that $\eta > 3/4$. Now choose $\mu_2 \leq \mu_1$ so that $\lambda_2 = \lambda_2(\mu_2) \leq \min(\mu_1/2, \lambda_1/2, \eta/20)$. The constants η, μ_1, μ_2, λ_1, λ_2 depend only on δ_0 and η_0.

Using Lemma 4, factor B into interpolating Blaschke products $B_1, \ldots, B_n$ such that for each B_j, $\delta(B_j) \geq \eta$. By Lemma 1, it suffices to prove Theorem 1 for each of the factors $B_1, \ldots, B_n$. Select one factor and denote it by B and its zeros by $\{z_k\}$. Observe that $\inf_{z\in\mathbf{R}_2^+} \max(|B(z)|, |f_2(z)|) \geq \delta_0$. Now let $r > 0$ be small. Let $B_r(z)$ denote the finite Blaschke product with zeros $\{z_j\}$ where $\Im z_j > (4/\delta_0)r$ and $|z_j| \leq 1/r$. Denote these zeros by $z_1, \ldots, z_N$. Also let $f_{2,r}(z) = f_2(z + ir)$. Then $B_r(z), f_{2,r}(z) \in H^\infty(\{y > -r\})$ and $\lim_{r\to 0} B_r(z) = B(z)$, $\lim_{r\to 0} f_{2,r}(z) = f_2(z)$ uniformly on compact subsets of $\mathbf{R}_2^+$. We claim that

$$\inf_{z\in\mathbf{R}_2^+} \max(|B_r(z)|, |f_{2,r}(z)|) \geq \mu_1 .$$

Indeed, if $|f_{2,r}(z)| < \mu_1 < \delta_0$, then $|B(z + ir)| \geq \delta_0$. This implies that $\rho(z + ir, z_n) \geq \delta_0$, $1 \leq i \leq N$. Now $D(z_n, \delta_0)$ is the Euclidean disc $\{z : |z - c| < R\}$ where

$$c = x_n + i\left(\frac{1+\delta_0^2}{1-\delta_0^2}\right) y_n \quad \text{and} \quad R = \frac{2\delta_0}{1-\delta_0^2} y_n .$$

A calculation then shows that $\operatorname{dist}(\partial D(z_n, \delta_0), \partial D(z_n, \delta_0/2)) > (\delta_0/3)y_n > r$. Hence $\rho(z + ir, z_n) \geq \delta_0$ implies $\rho(z, z_n) > \delta_0/2 > \lambda_1$, $1 \leq n \leq N$. By Lemma 2 this implies $|B_r(z)| \geq \mu_1$. Suppose now $|B_r(z)| < \mu_1$. Then $\rho(z + ir, z_n) < \lambda_1 < \delta_0/2$ for some z_n. A similar calculation to the above shows then that $z + ir \in D(z_n, \delta_0)$ and hence $|B(z + ir, z_n)| < \delta_0$. Thus $|f_2(z + ir)| \geq \delta_0 > \mu_1$. We now denote B_r, $f_{2,r}$ by B and f_2 respectively. By using a normal families argument it suffices to prove Theorem 1 for B and f_2 provided we show that the upper and lower bounds for g_1 and the upper bound for g_2 depend only on δ_0 and η_0.

The proof of Theorem 1 consists of constructing to each z_j, $1 \leq j \leq N$ regions T_j, $\widetilde{T}_j$, $T_j \subseteq \widetilde{T}_j \subseteq \overline{\mathbf{R}}_2^+$ satisfying the following properties:

(i) $D(z_j, \lambda_2) \subseteq T_j$ and $\widetilde{T}_j^* = D(z_j, 2\lambda_2)^*$.

(ii) The region $\overline{\mathbf{R}}_2^+ \backslash T_j$ is simply connected and for each $z \in \overline{\mathbf{R}}_2^+ \backslash T_j$

$$\left|\arg\left(\alpha_j\left(\frac{z - z_j}{z - \overline{z}_j}\right)\right)\right| \leq C,$$

where C is independent of j.

(iii) There exists $\varepsilon = \varepsilon(\delta_0, \eta_0)$, $0 < \varepsilon < 1$ such that $\{z \in \overline{\mathbf{R}}_2^+ : |f_2(z)| < \varepsilon\} \cap \widetilde{T}_j = \varnothing$, $1 \leq j \leq N$.

(iv) $\widetilde{T}_j \cap \widetilde{T}_k = \varnothing$, $j \neq k$ and

$$\operatorname{dist}(\widetilde{T}_j \backslash D(z_j, 2\lambda_2), \widetilde{T}_k \backslash D(z_k, 2\lambda_2)) \geq C(\delta_0, \eta_0) \min(y_j, y_k).$$

(v) There exists $\phi_j \in C^\infty(\overline{\mathbf{R}}_2^+)$, $0 \leq \phi_j \leq 1$, $\phi_j = 1$ on T_j, $\phi_j = 0$ on $\widetilde{T}_j^c$, $\overline{\partial}\phi_j \in L^\infty(\overline{\mathbf{R}}_2^+)$, and such that for any Carleson cube Q,

$$\int\int_Q |\overline{\partial}\phi_j|\, dx\, dy \leq C \min(l(Q), y_j)$$

for some absolute constant C.

Before constructing the above regions, we show how Theorem 1 is established. Properties (i) and (ii) imply that on $\overline{\mathbf{R}}_2^+\setminus T_j$ we can define an analytic branch of $\log(\alpha_j((z-z_j)/(z-\overline{z}_j)))$ with

$$\left|\log\left(\alpha_j\left(\frac{z-z_j}{z-\overline{z}_j}\right)\right)\right| \le C. \tag{3}$$

Denote this branch by $\log_j(\alpha_j((z-z_j)/(z-\overline{z}_j)))$. Define

$$F = \exp\left(\sum_{j=1}^N \phi_j \log_j\left(\alpha_j\left(\frac{z-z_j}{z-\overline{z}_j}\right)\right)\right)$$

and note that $F \in C(\overline{\mathbf{R}}_2^+)$, $\|F\|_\infty \le C$. Now let

$$h = -\frac{1}{f_2 F}\overline{\partial} F = -\frac{1}{f_2}\sum_{j=1}^N \overline{\partial}\phi_j \log_j\left(\alpha_j\left(\frac{z-z_j}{z-\overline{z}_j}\right)\right).$$

Then $h \in C^\infty(\overline{\mathbf{R}}_2^+)$ since $\overline{\partial}\phi_j \ne 0$ only if $z \in \bigcup(\widetilde{T}_j\setminus T_j)$ and in this case by (iii), $|f_2(z)| \ge \varepsilon$. Since there are finitely many zeros, we see also that $\operatorname{supp} h \cap \overline{\mathbf{R}}_2^+ \subseteq \{z: |z| < R\}$ for some $R > 0$. Also, the measure $|h|\,dx\,dy$ is a Carleson measure on $\mathbf{R}_2^+$. To see this, let $Q = \{(x, y): x_0 < x < x_0 + l,\ 0 < y < l\}$ be any Carleson cube in $\mathbf{R}_2^+$. If $\Im z_j \ge 3l$ then $D(z_j, 2\lambda_2)\cap Q = \varnothing$. Consequently, by (iv) there are at most C $\widetilde{T}_j$'s corresponding to points z_j, $\Im z_j \ge 3l$ for which $\widetilde{T}_j \cap Q \ne \varnothing$. If $\Im z_j < 3l$ and $\widetilde{T}_j\cap Q \ne \varnothing$, then z_j is contained in the cube $Q' = \{(x, y): x_0 - 2l < x < x_0 + 3l,\ 0 < y < 5l\}$. Thus

$$\begin{aligned}
\int\!\!\int_Q |h|\,dx\,dy &\le \frac{C}{\varepsilon}\int\!\!\int_Q \sum_{j=1}^N |\overline{\partial}\phi_j|\,dx\,dy \\
&\le C \sum_{\substack{\Im z_j < 3l \\ \widetilde{T}_j\cap Q\ne\varnothing}} \int\!\!\int_Q |\overline{\partial}\phi_j|\,dx\,dy + C \sum_{\substack{\Im z_j \ge 3l \\ \widetilde{T}_j\cap Q\ne\varnothing}} \int\!\!\int_Q |\overline{\partial}\phi_j|\,dx\,dy \\
&\le C\sum_{z_j\in Q'} y_j + C \sum_{\substack{\Im z_j \ge 3l \\ \widetilde{T}_j\cap Q\ne\varnothing}} l(Q), \quad \text{by (v)} \\
&\le Cl(Q') + Cl(Q) \\
&\le Cl(Q),
\end{aligned}$$

where C depends only on δ_0 and η_0. Hence by Lemma 6, there exists $u \in C(\overline{\mathbf{R}}_2^+) \cap C^1(\mathbf{R}_2^+)$ such that

$$\overline{\partial} u = h = -\frac{1}{f_2 F}\overline{\partial} F \tag{4}$$

and $\sup_{x\in\mathbf{R}} |u(x)| \le C(\delta_0, \eta_0)$. Now define

$$g_1 = \frac{F}{B} e^{u f_2}.$$

We first note that $e^{\pm u f_2} \in C(\overline{\mathbf{R}}_2^+)$ and on $\{x = 0\}$, $|e^{\pm u f_2}| \le C(\delta_0, \eta_0)$. Now if $z \in D(z_j, \lambda_2)$ for some z_j then $F(z) = \alpha_j((z - z_j)/(z - \overline{z}_j))$ and by Lemma 3, since $\rho(z, z_j) < \lambda_2 < \eta/3$, we have

$$1 \le \frac{\rho(z, z_j)}{|B(z)|} \le \frac{1}{\prod_{k \ne j} \rho(z, z_k)} \le \frac{3}{\eta}$$

and hence,

$$\frac{1}{C(\delta_0, \eta_0)} \le \frac{|F(z)|}{|B(z)|} \le C(\delta_0, \eta_0). \tag{5}$$

If $z \notin \bigcup D(z_j, \lambda_2)$, then by Lemma 2, $|B(z)| \ge \mu_2$ and by (3), $|F(z)| \ge C$, and so we again have (5) for $z \notin \bigcup D(z_j, \lambda_2)$. (5) now extends by continuity to $\overline{\mathbf{R}}_2^+$. Thus $g_1, g_1^{-1} \in C(\overline{\mathbf{R}}_2^+)$, g_1 is bounded on $\mathbf{R}_2^+$ and so on $\{x = 0\}$,

$$1/C(\delta_0, \eta_0) \le |g_1(x)| \le C(\delta_0, \eta_0). \tag{6}$$

Also by (4), g_1 is analytic on $\mathbf{R}_2^+$, and so by (6) and the maximum principle applied to g_1, g_1^{-1}, we have for all $z \in \mathbf{R}_2^+$

$$1/C(\delta_0, \eta_0) \le |g_1(z)| \le C(\delta_0, \eta_0).$$

Now define

$$g_2 = \frac{1}{f_2}(1 - F e^{u f_2})$$

and note that $g_1 B + g_2 f_2 = 1$ for all $z \in \mathbf{R}_2^+$. If $|f_2(z)| \ge \varepsilon$, g_2 is clearly bounded while if $0 < |f_2(z)| < \varepsilon$, then by (iii) $z \notin \bigcup \widetilde{T}_j$, which implies $F = 1$ and so

$$g_2 = \frac{1}{f_2}(1 - e^{u f_2}) \to -u \quad \text{as } f_2 \to 0.$$

Hence $g_2 \in C(\overline{\mathbf{R}}_2^+)$, $\max_{x \in \mathbf{R}} |g_2(x)| \le C(\delta_0, \eta_0)$, and by (4), g_2 is analytic on $\mathbf{R}_2^+$. Also g_2 is bounded on $\mathbf{R}_2^+$ so applying the maximum principle gives $\|g_2\|_\infty \le C(\delta_0, \eta_0)$. Theorem 1 now follows from a normal families argument.

Before constructing the regions T_j, $\widetilde{T}_j$ we require the following lemma, the proof of which is given in Theorem 3.2 in Chapter VII of [3].

Lemma 7. *Let $f(z)$ be a bounded analytic function on $\{y > -r\}$ with $\|f\|_\infty \le 1$. For $0 < \beta < 1$, $0 < \gamma < 1$ there exists $\alpha = \alpha(\beta, \gamma)$, $0 < \alpha < 1$ such that for any cube Q in $\{y > -r\}$ with base on $\{y = -r\}$, $\sup_{T(Q)} |f(z)| > \beta$ implies $|\{z \in Q : |f(z)| \le \alpha\}^*| < \gamma l(Q)$. Here $T(Q)$ denotes the top half of Q, that is $T(Q) = \{z \in Q : \Im z > \frac{1}{2} l(Q) - r\}$.*

We first construct the regions T_j, $\widetilde{T}_j$ corresponding to a fixed zero z_j of B, and without loss of generality, we assume z_j belongs to the top half of the unit cube $Q_0 = \{(x, y) : 0 < x < 1, -r < y < 1 - r\}$, which we think of as being dyadic. The construction we use is very similar to the first part of the corona construction described in Chapter VIII of [3]; the assumption $z_j \in T(Q_0)$ implies $\sup_{T(Q_0)} |f_2(z)| > \mu_1$ and so Q_0 is a case I cube in the construction described in [3].

Using Lemma 7, choose $N \in \mathbf{N}$ so that whenever Q is a cube in $\{y \ge -r\}$ with base on $\{y = -r\}$ and $\sup_{T(Q)} |f_2(z)| > \mu_1$, we have

$$|\{z \in Q : |f_2(z)| < 2^{-(N-3)}\}^*| < (\lambda_2/16) l(Q). \tag{7}$$

We can assume N is sufficiently large so that $2^{-(N-3)} < \mu_1$. For each dyadic cube $Q \subseteq Q_0$ with base on $\{y = -r\}$, partition $T(Q)$ into 2^{2N-1} dyadic squares S_j. Let

$$\mathscr{R}(Q_0) = \bigcup\{S_j \subseteq Q_0 \colon S_j \cap \overline{\mathbf{R}}_2^+ \neq \varnothing,\ \inf_{z \in S_j} |f_2(z)| \leq 2^{-N}\}.$$

Then if $\varepsilon = 2^{-(N+1)}$,

$$\{z \in Q_0 \colon |f_2(z)| < 2\varepsilon\} \subseteq \mathscr{R}(Q_0),$$

while by Schwarz's lemma, if $z, w \in S_j \subseteq \mathscr{R}(Q_0)$ then

$$|f_2(z) - f_2(w)| \leq 6\, 2^{-N},$$

which implies

$$\sup_{z \in \mathscr{R}(Q_0)} |f_2(z)| < 2^{-(N-3)},$$

and hence by (7)

$$|\mathscr{R}(Q_0)^*| \leq \lambda_2/16. \tag{8}$$

Now let Q_1, Q_2 denote the two cubes in $\{y \geq -r\}$, base on $\{y = -r\}$, and each having a common vertex at z_j. Note that $1/2 \leq l(Q_i) \leq 1$, $i = 1, 2$ and since $\eta > 3/4$, Q_1 and Q_2 contain no other zeros in their respective top-halves. Now by Lemma 5,

$$\sum_{\substack{z_k \in Q_1 \\ k \neq j}} |D(z_k, 4\lambda_2)^*| \leq 20 \sum_{\substack{z_k \in Q_1 \\ k \neq j}} \lambda_2 y_k \leq 100\lambda_2 \log\frac{1}{\eta} < \frac{\lambda_2}{32}.$$

Similarly

$$\sum_{\substack{z_k \in Q_2 \\ k \neq j}} |D(z_k, 4\lambda_2)^*| < \frac{\lambda_2}{32}$$

and hence,

$$\sum_{\substack{z_k \in Q_1 \cup Q_2 \\ k \neq j}} |D(z_k, 4\lambda_2)^*| < \frac{\lambda_2}{16}, \tag{9}$$

Also if $z_k \notin Q_1 \cup Q_2$, $\Im z_k \leq 1$, then

$$D(z_j, 4\lambda_2)^* \cap D(z_k, 4\lambda_2)^* = \varnothing. \tag{10}$$

Now since $|(\partial D(z_k, \lambda_2) \cap Q_0)^*| \geq \lambda_2/4$, (8)–(10) imply there exists $w_0 = x_0 + iy_0 \in \partial D(z_j, \lambda_2) \cap Q_0$ such that the vertical line $x = x_0$ is disjoint from both $\mathscr{R}(Q_0)$ and $\bigcup\{D(z_k, 4\lambda_2) \colon \Im z_k \leq 1\}$ and such that w_0 lies above no other point on $\partial D(z_j, \lambda_2) \cup Q_0$ with this property. In particular, the line $x = x_0$ is contained in cubes S_j not contained in $\mathscr{R}(Q_0)$. Since $r > 0$ we have

$$\min\{l(S_j) \colon S_j \cap \overline{\mathbf{R}}_2^+ \neq \varnothing,\ S_j \not\subseteq \mathscr{R}(Q_0)\} > 0$$

and hence, there exists $x_1 \neq x_0$ on $\{y = 0\} \cap \partial Q_0$ such that if x_2 is between x_0 and x_1 then the vertical line $x = x_1$ is disjoint from $\mathscr{R}(Q_0)$. Let

$$\rho_0 = \tfrac{1}{2}\min\{2|x_0 - x_1|, \lambda_2 y_1, \ldots, \lambda_2 y_N\}$$

and choose x_3 between x_0 and x_1 so that $|x_0 - x_3| = \rho_0$. Assume without loss of generality that $x_0 < x_3$. Define

$$\widetilde{T}_j = \{z \in \overline{\mathbf{R}}_2^+ : x_0 < \Re(z) < x_3,\ \text{there exists } w_1 \in D(z_j, 2\lambda_2) \text{ such that } \Im(z) \leq \Im(w_1)\} \cup D(z_j, 2\lambda_2)$$

and

$$T_j = \{z \in \widetilde{T}_j : \operatorname{dist}(z, \partial\widetilde{T}_j \backslash \{x = 0\} \cap \partial\widetilde{T}_j) \geq \tfrac{1}{4}\rho_0\}.$$

Standard arguments give $\phi_j \in C_0^\infty(\overline{\mathbf{R}}_2^+)$, $0 \leq \phi_j \leq 1$, $\phi_j = 1$ on T_j, $\phi_j = 0$ on $\widetilde{T}_j^c$, $|\overline{\partial}\phi_j| \leq C/\rho_0$, and such that if Q is any Carleson cube in $\mathbf{R}_2^+$, then

$$\int\int_Q |\overline{\partial}\phi_j|\, dx\, dy \leq C \min(l(Q), y_j)$$

for some absolute constant C.

Repeat the construction for each of the remaining zeros z_j, and it remains only to verify properties (iii) and (iv). In verifying (iii) we can assume z_j is the zero considered above. If $z \in \widetilde{T}_j \backslash D(z_j, 2\lambda_2)$ then $z \notin \mathscr{R}(Q_0)$ and hence $|f_2(z)| \geq \varepsilon$. If $z \in D(z_j, 2\lambda_2)$ then $|B(z)| \leq 2\lambda_2 < \mu_1$. This implies $|f_2(z)| \geq \mu_1 > 2^{-(N-3)} > \varepsilon$, and (iii) follows. To prove (iv), suppose that $\widetilde{T}_j$, $\widetilde{T}_k$ are the regions corresponding to z_j and z_k, and we can assume $y_j \geq y_k$. For convenience we assume z_j is the zero considered in the construction above. First, $\lambda_2 \leq \frac{1}{2}\lambda_1$ implies $D(z_j, 2\lambda_2) \cap D(z_k, 2\lambda_2) = \varnothing$ by Lemma 2. $D(z_j, 2\lambda_2)$ is a Euclidean disc with center

$$c_j = x_j + i\left(\frac{1 + 4\lambda_2^2}{1 - 4\lambda_2^2}\right) y_j.$$

Hence $y_j \geq y_k$ implies $\Im c_j \geq \Im c_k$. Since $\widetilde{T}_k^* = D(z_k, 2\lambda_2)^*$, this implies $D(z_j, 2\lambda_2) \cap \widetilde{T}_k = \varnothing$. Thus $\widetilde{T}_j \cap \widetilde{T}_k = \varnothing$ follows from the second part of (iv), which we now prove. If $z \in \widetilde{T}_j \backslash D(z_j, 2\lambda_2)$ then z is at a distance $\leq \frac{1}{2}\lambda_2 y_k$ from a vertical line that is disjoint from $D(z_k, 4\lambda_2)$. Together with the fact that $\widetilde{T}_k^* = D(z_k, 2\lambda_2)$, this implies $\operatorname{dist}(z, \widetilde{T}_k) \geq C(\delta_0, \eta_0) y_k$ and (iv) now follows.

This completes the proof of Theorem 1.

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