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CERTAIN AVERAGES ON THE a-ADIC NUMBERS

NAKHLÉ H. ASMAR AND RADHAKRISHNAN NAIR

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ABSTRACT. For $L^p \cap L^2$ functions f, with p greater than one, defined on the $\mathbf{a}$-adic numbers $\Omega_{\mathbf{a}}$, we consider averages like

$$A_N^{(1)} f(\mathbf{x}) = \frac{1}{N} \sum_{n=1}^{N} f(\mathbf{x} + n^2 \alpha), \quad \text{and} \quad A_N^{(2)} f(\mathbf{x}) = \frac{1}{N} \sum_{n=1}^{N} f(\mathbf{x} + p_n \alpha),$$

where $\mathbf{x}$ and α are in $\Omega_{\mathbf{a}}$. Here p_n denotes the nth prime. These averages are known to converge for almost all $\mathbf{x}$. We describe explicitly these limits, which possibly contrary to expectation, turn out in general not to be the integral of f.

To fix notation and for ease of reference, we begin with some standard facts about the $\mathbf{a}$-adic numbers. Let $\mathbf{a} = (a_n)_{n=-\infty}^{\infty}$ denote a fixed doubly infinite sequence of integers greater than one. Following and using the notation of [4], we define the $\mathbf{a}$-adic numbers $\Omega_{\mathbf{a}}$ to be the space of doubly infinite sequences $\mathbf{x} = (x_n)_{n=-\infty}^{\infty}$ in $\prod_{n=-\infty}^{\infty} \{0, 1, \dots, a_n - 1\}$ with $x_n = 0$ for all $n < n_0(x)$. We can endow $\Omega_{\mathbf{a}}$ with addition "+" making it an Abelian group, and a product "$\times$" making it a ring. In the special case where, for a fixed rational prime p we set $a_n = p^n$, for each n, $\Omega_{\mathbf{a}}$ is a field. The details of all this can also be found in [4, §10].

For each integer k, let

$$\Lambda_k = \{\mathbf{x} \in \Omega_{\mathbf{a}} : x_n = 0 \text{ if } n < k\}.$$

These sets form a basis at 0 for a topology of $\Omega_{\mathbf{a}}$, with respect to which group operations are continuous. With respect to this topology, $\Omega_{\mathbf{a}}$ is locally compact, σ-compact Abelian group. For each integer k the relative topology on Λ_k is the same as the Tychonoff product topology, and hence this subgroup of $\Omega_{\mathbf{a}}$ is compact and open. We say a group is monothetic if it possesses an element, called a generator, whose orbit is dense. For each integer k, Λ_k is monothetic. An explicit example of a generator for Λ_k is $\mathbf{u}^k = (u_n^k)_{n \in \mathbf{Z}}$ where $u_k^k = 1$ and $u_n^k = 0$ when n and k are not equal. Henceforth Λ_0 is denoted $\Delta_{\mathbf{a}}$ and referred to as the $\mathbf{a}$-adic integers.

Haar measure λ on $\Omega_{\mathbf{a}}$ can be defined as follows. For each integer n, let $\lambda_n(A)$ denote the measure on $\{0, 1, \dots, a_{n-1}\}$ given by $\lambda_n(A) = \text{card}(A)/a_n$. For each integer r, let μ_r denote the corresponding product measure on

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$\prod_{n\geq r}\{0, 1, \dots, a_n - 1\}$. If A is a λ-measurable subset of Λ_n where $n < 0$, then $\lambda(A) = a_r a_{r+1} \cdots a_{-1}\mu_r(A)$. For general A we have $\lambda(A) = \lim_{r\to-\infty} a_r a_{r+1} \cdots a_{-1}\mu_r(A)$. (See [4, pp. 202–203]).

Again following [4], we denote the character groups of $\Omega_{\mathbf{a}}$ and $\Delta_{\mathbf{a}}$ by $\Omega_{\mathbf{a}^*}$ and $\mathbb{Z}(\mathbf{a}^\infty)$ respectively. The group $\Omega_{\mathbf{a}^*}$ is nothing other than the group of $\mathbf{a}^*$-adic numbers with $a_n^* = a_{-n}$ for each rational integer n. The group $\mathbb{Z}(\mathbf{a}^\infty)$ consists of all numbers of the form $t = l/a_0a_1\cdots a_r$ where $0 \leq l \leq a_0a_1\cdots a_r$. As in [4], to evaluate a character t in $\mathbb{Z}(\mathbf{a}^\infty)$ at $\mathbf{x}$ in $\Delta_{\mathbf{a}}$, we write $\chi_t(\mathbf{x})$ to mean $e^{2\pi i \frac{l}{a_0a_1\cdots a_r}(x_0+a_0x_1+\cdots+a_0a_1\cdots a_{r-1}x_r)}$. For $\mathbf{x}$ in $\Omega_{\mathbf{a}}$ and $\mathbf{y}$ in $\Omega_{\mathbf{a}^*}$, we write $\chi_{\mathbf{y}}(\mathbf{x})$ to denote the value of the character $\mathbf{y}$ evaluated at $\mathbf{x}$. (See [4, §25].)

Suppose G is a locally compact Abelian group, and let $\alpha_0, \alpha_1, \dots, \alpha_k$ be an arbitrary but fixed finite sequence of elements of G. The group valued polynomial

$$\rho(n) = \alpha_k n^k + \cdots + \alpha_1 n + \alpha_0,$$

is well defined for each n in $\mathbb{N}$. Let $1 \leq p \leq \infty$, and denote by $L^p(G)$ the usual space of p-summable functions on G with respect to Haar measure. For each natural number N, x in G, and f in $L^p(G)$, we define the averages

$$A_N^{(1)} f(x) = \frac{1}{N}\sum_{n=1}^{N} f(x+\rho(n)); \tag{1}$$

and for an element α of G and a polynomial ψ,

$$A_N^{(2)} f(x) = \frac{1}{\pi_N}\sum_{p\leq N} f(x+\alpha\psi(p)). \tag{2}$$

Here p runs over the rational primes and π_N denotes their number in $[1, N]$. In what follows we assume, without loss of generality, that the constant terms of ρ and ψ are zero. As a consequence of some recent results in ergodic theory [1, 2, 6, 8], it follows that when $1 < p \leq \infty$, $(A_N^{(i)} f(x))_{n=1}^{\infty}$ $(i = 1, 2)$ converges pointwise almost everywhere and in $L^p(G)$. Let $L^{(i)} f(x)$ denote this limit. In the special case where G is a compact connected monothetic group and at least one of the elements $\alpha_1, \dots, \alpha_k$ is a generator, it is shown in [7] that $L^{(i)} f(x)$ is $\int_G f\,d\lambda$. a.e.-(λ). In this note we consider $G = \Omega_{\mathbf{a}}$ and $G = \Delta_{\mathbf{a}}$, two cases not considered in [7], and show that in general $L^{(i)} f(x)$ is not constant a.e. let alone equal to $\int_G f\,d\lambda$. (Our account is independent of [7].)

For a complete description of the limit $Lf(x)$ the following notions are needed. Suppose at least one of the elements $\alpha_1, \dots, \alpha_k$ is a generator of $\Delta_{\mathbf{a}}$. Let $\delta_j\colon \mathbb{Z} \to \Delta_{\mathbf{a}}$ be the homomorphism defined by $\delta_j(m) = m\alpha_j$ $(j = 1, 2, \dots, k)$, and let $\varepsilon_j\colon \mathbb{Z}(\mathbf{a}^\infty) \to \mathbf{T}$ be the adjoint homomorphism to δ_j. Hence for all $l/a_0\cdots a_r$ in $\mathbb{Z}(\mathbf{a}^\infty)$ and all m in $\mathbb{Z}$, we have [4]

$$e^{2\pi i \varepsilon_j(\frac{l}{a_0\cdots a_r})} = \chi_{\frac{l}{a_0\cdots a_r}}(\delta_j(m)) = \chi_{\frac{l}{a_0\cdots a_r}}(m\alpha_j).$$

We can identify $\mathbb{Z}(\mathbf{a}^\infty)$ with the quotient of $\Omega_{\mathbf{a}^*}$ by $A(\Omega_{\mathbf{a}^*}, \Delta_{\mathbf{a}})$, the annihilator of $\Delta_{\mathbf{a}}$ in $\Omega_{\mathbf{a}^*}$ [4, Lemma 24.5]. Let $\Psi\colon \Omega_{\mathbf{a}^*} \to \mathbb{Z}(\mathbf{a}^\infty)$ be the associated quotient map.

Consider $G^{(1)}\colon \mathbb{Z}(\mathbf{a}^\infty) \to \mathbb{C}$ defined by

$$G^{(1)}(\chi_{\frac{l}{a_0\cdots a_r}}) = G^{(1)}_{\alpha_1,\dots,\alpha_k}(\chi_{\frac{l}{a_0\cdots a_r}}) = \frac{1}{D_r^{(1)}}\sum_{m=1}^{D_r^{(1)}} e^{2\pi i\gamma^{(1)}(m)/D_r^{(1)}},$$

for all $l/a_0\cdots a_r$ in $\mathbb{Z}(\mathbf{a}^\infty)$. Here the positive integer $D_r^{(1)}$ and the polynomial $\gamma^{(1)}$ of degree k with nonnegative integer coefficients are described as follows. Let

$$l_j = l(\alpha_j(0)+\alpha_j(1)a_0+\alpha_j(2)a_0a_1+\cdots+\alpha_j(r-1)a_0\cdots a_{r-2}) \qquad (j=1,\dots,k).$$

Here $\alpha_j(r)$ denotes the rth term of α_j viewed as a sequence. Let m_j/B_j with $(m_j, B_j)=1$ denote $l_j/a_0\cdots a_r$ in reduced form. We use $D_r^{(1)}$ to denote the least common multiple of $B_1,\dots,B_k$ and define $\gamma^{(1)}(x)$ by

$$\frac{\gamma^{(1)}(x)}{D_r^{(1)}} = \frac{m_k}{B_k}x^k+\cdots+\frac{m_1}{B_1}x.$$

We define $G^{(1)}\colon \mathbb{Z}(\mathbf{a}^\infty)\to\mathbb{C}$ by

$$G^{(1)}(\chi_{\frac{l}{a_0\cdots a_r}}) = G^{(1)}_{\alpha_1,\dots,\alpha_k}(\chi_{\frac{l}{a_0\cdots a_r}}) = \frac{1}{D_r^{(1)}}\sum_{m=1}^{D_r^{(1)}} e^{2\pi i\frac{\gamma^{(1)}(m)}{D_r^{(1)}}},$$

for all $l/a_0\cdots a_r$ in $\mathbb{Z}(\mathbf{a}^\infty)$.

Now consider $G^{(2)}\colon \mathbb{Z}(\mathbf{a}^\infty)\to\mathbb{C}$ defined by

$$G^{(2)}(\chi_{\frac{l}{a_0\cdots a_r}}) = G^{(2)}_{\alpha}(\chi_{\frac{l}{a_0\cdots a_r}}) = \frac{1}{\phi(D_r^{(2)})}\sum_{\substack{m=1\\(D_r^{(2)},m)=1}}^{D_r^{(2)}} e^{2\pi i\frac{\gamma^{(2)}(m)}{D_r^{(2)}}},$$

where ϕ denotes Euler's totient function. The positive integer $D_r^{(2)}$ and the polynomial $\gamma^{(2)}$ are described as follows. Suppose

$$\psi(x) = \frac{d_k}{c_k}x^k+\cdots+\frac{d_1}{c_1}x.$$

(It is a straightforward consequence of the Lagrange interpolation formula that any polynomial mapping $\mathbb{N}$ to itself must have nonnegative rational coefficients.) Let f_j/g_j with $(f_j, g_j)=1$ denote $(l/a_0\cdots a_r)\cdot(d_j/c_j)$ in reduced form. We set $D_r^{(2)}$ to be the least common multiple of the integers $g_0,\dots,g_k$ and define $\gamma^{(2)}$ by

$$\frac{\gamma^{(2)}(x)}{D_r^{(2)}} = \frac{f_k}{g_k}x^k+\cdots+\frac{f_1}{g_1}x.$$

Define $m^{(i)}\colon \Omega_{\mathbf{a}^*}\to\mathbb{C}$ by $m^{(i)}(\chi) = G^{(i)}(\Psi(\chi))$ for all χ in $\Omega_{\mathbf{a}^*}$ $(i=1,2)$.

Here and henceforth $\hat{f}$ denotes the Fourier transform of f.

Theorem 1. *Suppose $p\in(1,\infty)$ and $f\in L^p\cap L^2(\Omega_{\mathbf{a}})$. Let $(A_n^{(i)}f(x))_{N=1}^{\infty}$ $(i=1,2)$ be defined by* (1) *and* (2). *Then if in addition $p\ge 2$ when both $i=2$ and the degree of ψ is greater than one, $(A_N^{(i)}f(x))_{n=1}^{\infty}$ converges almost everywhere and in $L^p(\Omega_{\mathbf{a}})$ to a function $L^{(i)}f\in L^p(\Omega_{\mathbf{a}})$ where*

$$(L^{(i)}f)^{\wedge}(\chi) = m^{(i)}(\chi)\hat{f}(\chi) \qquad (i=1,2) \text{ a.e. in } \Omega_{\mathbf{a}^*}.$$

We have the following version of Theorem 1 on the **a**-adic integers.

Theorem 2. *Suppose* $p \in (1, \infty)$ *and* $f \in L^P(\Delta_{\mathbf{a}})$. *Let* $(A_N^{(i)} f(x))_{N=1}^{\infty}$ $(i = 1, 2)$ *be defined by* (1) *and* (2). *Then if in addition* $p \geq 2$ *when both* $i = 2$ *and the degree of* ψ *is greater than one,* $(A_N^{(i)} f(x))_{n=1}^{\infty}$ *converges almost everywhere and in* $L^p(\Delta_{\mathbf{a}})$ *to a function* $L^{(i)} f \in L^p(\Delta_{\mathbf{a}})$ *where*

$$(L^{(i)} f)^{\wedge}(\chi) = m^{(i)}(\chi) \hat{f}(\chi) \quad (i = 1, 2)$$

for all $\chi \in \mathbb{Z}(\mathbf{a}^{\infty})$. *Here we identify* $\mathbb{Z}(\mathbf{a}^{\infty})$ *with the quotient of the group* $\Omega_{\mathbf{a}^*}$ *by* $A(\Omega_{\mathbf{a}^*}, \Delta_{\mathbf{a}})$.

The reason we use $L^p \cap L^2$ instead of L^p in Theorem 1 is because of the difficulty of defining the Fourier transform in the noncompact setting. This difficulty does not exist in the context of Theorem 2. Note also that Ψ is the identity in the context of Theorem 2. We need two lemmas.

Lemma 3. *Let* $M^{(i)} f = \sup_{N \geq 1} |A_N^{(i)} f|$ $(i = 1, 2)$. *Then there exist absolute constants* C_1 *and* C_2 *such that*

$$\|M^{(i)} f\|_p \leq C_i \|f\|_p,$$

where $p > 1$ *if* $i = 1$, $p > 1$ *if* $i = 2$ *with the degree of* ψ *equal one, and* $p \geq 2$ *if* $i = 2$ *and the degree of* ψ *is greater than one. Under the same conditions on* p, *the sequences* $(A_N^{(i)} f(x))_{N=1}^{\infty}$ $(i = 1, 2)$ *converge almost everywhere.*

The case $i = 1$ of Lemma 3 is proved in [2] when $l = 1$ and for all l when $p \geq 2$ in [1]. The argument in [1] may readily be combined with that in [2] to prove Lemma 3 $(i = 1)$ for general l. Lemma 3 $(i = 2)$ in the case $\psi(x) = x$ is proved in [8] and for $\psi(x) = x^k$ $(k > 1)$ in [6]. The arguments needed there may readily be adapted to show Lemma 3 $(i = 2)$ for general ψ and the details are forgone in [6] only to keep technicalities to a minimum.

Lemma 4. *For all* $\chi_{\frac{l}{a_0 \cdots a_r}}$, *we have*

$$G^{(1)}(\chi_{\frac{l}{a_0 \cdots a_r}}) = \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \chi_{\frac{l}{a_0 \cdots a_r}}(\rho(n)) \tag{3}$$

and

$$G^{(2)}(\chi_{\frac{l}{a_0 \cdots a_r}}) = \lim_{N \to \infty} \frac{1}{\pi_N} \sum_{1 \leq p \leq N} \chi_{\frac{l}{a_0 \cdots a_r}}(\alpha \psi(p)). \tag{4}$$

Further, for all χ *in* $\Omega_{\mathbf{a}^*}$, *we have*

$$m^{(1)}(\chi) = \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \chi(\rho(n)) \tag{5}$$

and

$$m^{(2)}(\chi) = \lim_{N \to \infty} \frac{1}{\pi_N} \sum_{1 \leq p \leq N} \chi(\alpha \psi(p)). \tag{6}$$

Proof. We first prove (3). Note that

$$\begin{aligned}\chi_{\frac{l}{a_0\cdots a_r}}(\rho(n)) &= \chi_{\frac{l}{a_0\cdots a_r}}(\alpha_0+\alpha_1 n+\cdots+\alpha_k n^k)\\ &= \prod_{j=1}^{k}(\chi_{\frac{l}{a_0\cdots a_r}}(\alpha_j))^{n^j}\\ &= \prod_{j=1}^{k}(e^{2\pi i\frac{l}{a_0\cdots a_r}n^j(\alpha_j(0)+\alpha_j(1)a_0+\cdots+\alpha_j(r-1)a_0\cdots a_{r-2})})\\ &= \prod_{j=1}^{k}(e^{2\pi i\frac{l_j}{a_0\cdots a_r}n^j}) = \prod_{j=1}^{k}(e^{2\pi i\frac{m_j}{B_j}n^j}) = e^{2\pi i\frac{\gamma^{(1)}(n)}{D_r^{(1)}}}\end{aligned}$$

(see [4, (25.2), (2) pp. 402–403]). If we now partition the integers into their residue classes modulo $D_r^{(1)}$, it follows immediately that

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\chi_{\frac{l}{a_0\cdots a_r}}(\rho(n)) = \frac{1}{D_r^{(1)}}\sum_{m=1}^{D_r^{(1)}}e^{2\pi i\frac{\gamma^{(1)}(m)}{D_r^{(1)}}},$$

proving (3). We now show (4). Arguing as in (3)

$$(7)\qquad \frac{1}{\pi_N}\sum_{1\le p\le N}\chi_{\frac{l}{a_0\cdots a_r}}(\alpha\psi(p)) = \frac{1}{\pi_N}\sum_{1\le p\le N}e^{2\pi i\frac{\gamma^{(2)}(p)}{D_r^{(2)}}}.$$

Let $\Lambda\colon \mathbb{N}\to\mathbb{R}$ be the von Mangoldt function, defined by $\Lambda(n)=\log_e p$ if $n=p^l$ for some prime p and positive integer l, and zero otherwise. Using partial summation we readily see that

$$(8)\qquad \frac{1}{\pi_N}\sum_{1\le p\le N}e^{2\pi i\frac{\gamma^{(2)}(p)}{D_r^{(2)}}} = \frac{1}{N}\sum_{1\le n\le N}\Lambda(n)e^{2\pi i\frac{\gamma^{(2)}(n)}{D_r^{(2)}}} + O((\log N)^{-1}).$$

The Siegel–Walfish prime number theorem for arithmetic progressions [3, p. 133] says that for fixed positive u, if $1\le D_r^{(2)}\le(\log N)^u$ and $(m, D_r^{(2)})=1$, then for some $C>0$,

$$\sum_{\substack{1\le n\le N\\ n\equiv m \pmod{D_r^{(2)}}}}\Lambda(n) = \frac{N}{\phi(D_r^{(2)})} + o(Ne^{-C(\log N)^{1/2}}).$$

Now note that

$$\sum_{n=1}^{N}\Lambda(n)e^{2\pi i\frac{\gamma^{(2)}(n)}{D_r^{(2)}}} = \left(\sum_{\substack{m=1\\(m,D_r^{(2)})=1}}^{D_r^{(2)}}e^{2\pi i\frac{\gamma^{(2)}(m)}{D_r^{(2)}}}\right)\left(\sum_{\substack{1\le n\le N\\ n\equiv m \pmod{D_r^{(2)}}}}\Lambda(n)\right) + O\left(\sum_{p^l\le N;\,p\mid D_r^{(2)}}\Lambda(p^l)e^{2\pi i\frac{\gamma^{(2)}(p^l)}{D_r^{(2)}}}\right).$$

Using the fact that the third sum on the right is $O((\log N)(\log\log N))$ and the Siegel–Walfish theorem,

(9)

$$\frac{1}{N}\sum_{n=1}^{N}\Lambda(n)e^{2\pi i\frac{\gamma^{(2)}(n)}{D_r^{(2)}}} = \left(\frac{1}{\phi(D_r^{(2)})}\sum_{\substack{m=1\\(m,D_r^{(2)})=1}}^{D_r^{(2)}} e^{2\pi i\frac{\gamma^{(2)}(m)}{D^{(2)}}}\right) + O\left(\frac{(\log N)(\log\log N)}{N}\right).$$

By combining (7), (8), and (9) and letting N tend to infinity (4) is proved. Statements (5) and (6) of Lemma 4 follow from the fact that for all $\mathbf{x}$ in $\Delta_{\mathbf{a}}$ and all χ in $\Omega_{\mathbf{a}^*}$, we have $\chi(\mathbf{x}) = \Psi(\chi)(\mathbf{x})$ [4, Lemma 24.5]. □

We prove Theorem 1 only in the case $i = 1$, the modifications needed to prove the case $i = 2$ being minor. Henceforth the superscript $i = 1$ is dropped.

Proof of Theorem 1 *in the case* $i = 1$. Fix $p \in (1, \infty)$, let $f \in L^p(\Omega_{\mathbf{a}})$, and suppose the support of f is contained in Λ_k for some nonpositive integer k. This means $f = fI_{\Lambda_k}$, where for a set A, we let I_A denote its indicator function. A simple application of the Stone–Weierstrass theorem shows that f can be approximated in the L^p-norm by functions of the form

$$I_{\Lambda_k}\sum_{j=1}^{\nu} b_j\chi_j, \tag{10}$$

with each b_j in $\mathbb{C}$ and χ in $\Omega_{\mathbf{a}^*}$. (Use [5, Lemma 31.4, p. 211] with $d\mu = I_{\Lambda_k}\,d\lambda$). Because compactly supported functions are dense in $L^p(\Omega_{\mathbf{a}})$, it follows that functions of the form (10) are also dense in $L^p(\Omega_{\mathbf{a}})$. It is clear that for each positive integer N, A_N commutes with translations on $\Omega_{\mathbf{a}}$. We know from Lemma 3 that $Lf(\mathbf{x})$ exists λ a.e. In addition, as a consequence of Lemma 3 and the Lebesgue dominated convergence theorem, it follows that the functions $(A_N f(\mathbf{x}))_{N=1}^{\infty}$ also converge in L^p to the same limit. Obviously L commutes with translations on $\Omega_{\mathbf{a}}$. It follows that the clearly linear operator $f \to Lf$ is an $L^p(\Omega_{\mathbf{a}})$ multiplier. Hence there is a bounded measurable function m on $\Omega_{\mathbf{a}^*}$ such that for all $f \in L^p \cap L^2(\Omega_{\mathbf{a}})$ we have

$$(Lf)\hat{\ }(\chi) = m(\chi)\hat{f}(\chi),$$

almost everywhere in $\Omega_{\mathbf{a}^*}$. To identify m we evaluate L on functions of the form (10). We first note that

$$\begin{aligned}\hat{f} &= \sum_{j=1}^{\nu} b_j(\chi_j I_{\Lambda_k})\hat{\ } = \sum_{j=1}^{\nu} b_j(I_{\Lambda_k})\hat{\ }(\cdot + \chi_j)\\ &= \lambda(\Lambda_k)\sum_{j=1}^{\nu} b_j I_{\chi_j + A(\Omega_{\mathbf{a}^*}, \Lambda_k)},\end{aligned} \tag{11}$$

where $A(\Omega_{\mathbf{a}^*}, \Lambda_k)$ denotes the annihilator in $\Omega_{\mathbf{a}^*}$ of Λ_k. The penultimate identity follows from the fact that multiplication by a character shifts the Fourier transform. The last identity follows from the identity

$$\left(\frac{1}{\lambda(\Lambda_k)}I_{\Lambda_k}\right)\hat{\ } = I_{A(\Omega_{\mathbf{a}^*}, \Lambda_k)}. \tag{12}$$

Indeed if $\chi \in \Omega_{\mathbf{a}^*}$ the restriction χ to Λ_k is plainly a continuous character of Λ_k and (12) follows since the Haar integral on Λ_k of any nontrivial character of Λ_k is zero [4, Lemma 23.19, p. 363]. For f as in (10) we have

$$A_N f(\mathbf{x}) = \frac{1}{N}\sum_{n=1}^{N} I_{\Lambda_k}(\mathbf{x}+\rho(n))\sum_{j=1}^{\nu} b_j\chi_j(\mathbf{x}+\rho(n)).$$

If $\mathbf{x}$ is not in Λ_k then $\mathbf{x}+\rho(n)$ is not in Λ_k because $\rho(n)$ is in Λ_k. This means that for all $\mathbf{x}$ in $\Omega_{\mathbf{a}}$ we have

$$A_N f(\mathbf{x}) = A_N f(\mathbf{x}) I_{\Lambda_k}(\mathbf{x}). \tag{13}$$

For $\mathbf{x}$ in Λ_k we have

$$\begin{aligned} A_N f(\mathbf{x}) &= \frac{1}{N}\sum_{n=1}^{N}\sum_{j=1}^{\nu} b_j\chi_j(\mathbf{x}+\rho(n)) \\ &= \sum_{j=1}^{\nu}\chi_j(\mathbf{x})\frac{1}{N}\sum_{n=1}^{N}\chi_j(\rho(n)). \end{aligned}$$

By Lemma 4 and (13), for all $\mathbf{x} \in \Omega_{\mathbf{a}}$, we get

$$\begin{aligned} Lf(\mathbf{x}) &= \lim_{N\to\infty} I_{\Lambda_k}(\mathbf{x}) A_N f(\mathbf{x}) = I_{\Lambda_k}(\mathbf{x})\sum_{j=1}^{\nu} b_j G(\Psi(\chi_j))\chi_j(\mathbf{x}) \\ &= I_{\Lambda_k}(\mathbf{x})\sum_{j=1}^{\nu} b_j m(\chi_j)\chi_j(\mathbf{x}). \end{aligned} \tag{14}$$

A computation similar to (11) and (14) shows that for all $\chi \in \Omega_{\mathbf{a}^*}$

$$(Lf)^\frown(\chi) = \lambda(\Lambda_k)\sum_{j=1}^{\nu} b_j G(\Psi(\chi_j)) I_{\lambda_j + A(\Omega_{\mathbf{a}^*}, \Lambda_k)}(\chi). \tag{15}$$

If $\chi \in \chi_j + A(\Omega_{\mathbf{a}^*}, \Lambda_k)$, then $\chi = \chi_j + \chi'$ where χ' is in $A(\Omega_{\mathbf{a}^*}, \Lambda_k)$. Consequently, $\Psi(\chi) = \Psi(\chi_j) + \Psi(\chi')$. Recall that $\Psi\colon \Omega_{\mathbf{a}^*} \to \Omega_{\mathbf{a}^*}/A(\Omega_{\mathbf{a}^*}, \Delta_{\mathbf{a}})$ and χ' is in $A(\Omega_{\mathbf{a}^*}, \Lambda_k)$, which is a subset of $A(\Omega_{\mathbf{a}^*}, \Delta_{\mathbf{a}})$ because $\Delta_{\mathbf{a}}$ is a subset of Λ_k. Hence, if $\chi \in \chi_j + A(\Omega_{\mathbf{a}^*}, \Delta_{\mathbf{a}})$, then $\Psi(\chi) = \Psi(\chi_j)$. Using this observation, (11), and (13), we find that for all $\mathbf{x}$ in $\Omega_{\mathbf{a}^*}$

$$(Lf)^\frown(\mathbf{x})(\chi) = m(\chi)\hat{f}(\chi), \tag{16}$$

establishing the theorem for all f of the form (10). To prove the theorem in general, given $f \in L^p \cap L^2$, let $(f_n)_{n=1}^{\infty}$ be a sequence of functions of the form (10), which converge to f in L^2. Because L is continuous in L^2, if a sequence $(f_n)_{n=1}^{\infty}$ converges to f in L^2 then $(Lf_n)_{n=1}^{\infty}$ converges to Lf in L^2, and hence a subsequence of $((Lf_n)^\frown)_{n=1}^{\infty}$ converges to $(Lf)^\frown$ a.e. on $\Omega_{\mathbf{a}^*}$. The desired conclusion follows from the fact that $(Lf_n)^\frown(\chi) = m(\chi)\hat{f}_n(\chi)$ and a subsequence of $(\hat{f}_n)_{n=1}^{\infty}$ converges to $\hat{f}$ a.e. □

To prove Theorem 2, simply note that a function $f \in L^p(\Delta_{\mathbf{a}})$ may be approximated by functions in $L^p(\Omega_{\mathbf{a}})$ of the form (10) with $k = 0$. Theorem 2 thus follows from (16).

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MISSOURI-COLUMBIA, COLUMBIA, MISSOURI 65211

DEPARTMENT OF PURE MATHEMATICS, UNIVERSITY OF LIVERPOOL, LIVERPOOL L69 3BX, UNITED KINGDOM