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THE C^1 CLOSING LEMMA FOR ENDOMORPHISMS WITH FINITELY MANY SINGULARITIES

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ABSTRACT. The C^1 closing lemma for endomorphisms with finitely many singularities is obtained by combining the C^1 closing lemma for nonsingular endomorphisms together with a technique of L. S. Young.

1. INTRODUCTION

Let M be a compact Riemannian manifold without boundary, and let $F\,\mathrm{End}^1(M)$ be the set of C^1 maps of M with finitely many singularities, endowed with the C^1 topology. In this paper we prove the following:

Theorem A. *Let f be a C^1 map of M with finitely many singularities, and ω be a nonwandering point of f. Then for any C^1 neighborhood $\mathscr{U}$ of f in $F\,\mathrm{End}^1(M)$, there is a $g \in \mathscr{U}$ such that ω is a periodic point of g.*

Recall that a point x is a *singularity* of f if $T_x f$ is not injective. A point x is *nonwandering* of f if for any neighborhood U of x in M, $(f^n U) \cap U$ is nonempty for some $n \geq 1$, and *periodic* of f if $f^n x = x$ for some $n \geq 1$. The proof of Theorem A is based on the C^1 closing lemma of nonsingular endomorphisms [11] on the one hand, and a technique of L. S. Young [12] on the other.

2. PRELIMINARIES

In this section we collect from [11] some definitions and theorems needed in this paper.

By a *tree* $\mathscr{T} = (Q, f)$ we mean an infinite sequence of mutually disjoint nonempty finite sets $Q_0, Q_1, \dots, Q_n, \dots$, where Q_0 consists of a single point q_0, together with a map $f\colon Q - \{q_0\} \to Q$, where $Q = \bigcup_{n=0}^{\infty} Q_n$, such that f maps Q_n into Q_{n-1} for each $n = 1, 2, \dots$. An infinite sequence $q_0, q_1, \dots, q_n, \dots$ is called an *infinite branch* of $\mathscr{T}$ if $f(q_n) = q_{n-1}$ for each $n = 1, 2, \dots$. A finite sequence $q_0, q_1, \dots, q_k$ is called a *finite branch* of $\mathscr{T}$ if $f(q_n) = q_{n-1}$ for each $n = 1, 2, \dots, k$, and if $f^{-1}\{q_k\}$ is empty. A tree

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$\mathcal{T} = (Q, f)$ is called *complete* if f is onto. Clearly, $\mathcal{T}$ is complete iff $\mathcal{T}$ has only infinite branches. Occasionally, we may talk about *finite* trees and their branches. The definitions are obvious. Here we only mention that a finite tree is called *complete* if all its branches have the same number of terms.

By a *tree of isomorphisms* we mean a collection of linear isomorphisms parametrized by a tree $\mathcal{T}$. More precisely, this means that we associate to each $q \in Q$ an m-dimensional inner product space V_q, and to each $q \neq q_0$ a linear isomorphism $T_q\colon V_q \to V_{q_0}$.

The main result needed in this paper is the following ε-kernel avoiding transition theorem of [11]:

Theorem 2.1. *Given a complete tree of isomorphisms* $(\mathcal{T}, T_q)$ *and* $\varepsilon > 0$. *There is a number* $\rho > 2$ *and an integer* $\mu \geq 1$ *such that: For any finite ordered set* $P = \{p_0, p_1, \dots, p_t\}$ *in* V_{q_0}, *there is a point* $y \in P \cap (B(p_t, \rho|p_0 - p_t|))$ *such that for any branch* $\Sigma = \{q_0, q_1, \dots, q_n, \dots\}$ *of* $\mathcal{T}$, *there is a point* $w \in P \cap (B(p_t, \rho|p_0 - p_t|))$, *where* w *is before* y *in the order of* P, *together with* $\mu + 1$ *points* $c_0, c_1, \dots, c_\mu$ *in* $B(p_t, \rho|p_0 - p_t|)$, *not necessarily distinct, satisfying the following two conditions* (a) *and* (b).

(a) $c_0 = w$, $c_\mu = y$, *and*

(b) $|T_{q_n}^{-1}(c_n) - T_{q_n}^{-1}(c_{n+1})| \leq \varepsilon d(T_{q_n}^{-1}(c_{n+1}), T_{q_n}^{-1}(A))$ *for* $n = 0, 1, \dots, \mu - 1$, *where* T_{q_0} *stands for the identity,* $A = (P(w, y)) \cup (\partial B(p_t, \rho|p_0 - p_t|))$, $P(w, y) = \{p \in P | p$ *is after* w *and before* $y\}$, *and* d *is the distance on* V_{q_0}. Q.E.D.

We also need the following two basic perturbation lemmas that deal with the ε-kernel lifts and the local linearizations for C^1 maps. They are essentially the same as Lemmas 4.1 and 4.2 of [11] with some minor changes. Let $C^1(M)$ be the set of C^1 maps of M into itself. For simplicity we assume that M is Riemann embedded into some R^d. Then $C^1(M)$ has a C^1 metric d_1 inherited from $C^1(M, R^d)$ compatible with its C^1 topology. Fix a $\zeta > 0$ such that $\exp_p$ embeds $\{u \in T_pM | |u| \leq \zeta\}$ into M for each $p \in M$.

Lemma 2.2. *For any* $\eta > 0$, *there is an* $\varepsilon > 0$ *such that for any* $f \in C^1(M)$, *any* $p \in M$, *and any two points* $v_1, v_2 \in T_pM$ *with* $B(v_2, |v_1 - v_2|/\varepsilon) \subset \{u \in T_pM | |u| \leq \zeta\}$, *there is a diffeomorphism* $h = h_{p,\varepsilon,v_1,v_2}\colon M \to M$, *called an* ε*-kernel lift, such that:*

(1) $h(\exp_p(v_2)) = \exp_p(v_1)$;

(2) $\mathrm{supp}(h) \subset \exp_p(B(v_2, |v_1 - v_2|/\varepsilon))$, *here the support means the closure of the set where* h *differs from the identity*;

(3) $d_1(hf, f) < \eta$. Q.E.D.

Now let $f \in C^1(M)$ be given. Before stating Lemma 2.3, recall that the *negative orbit* of $p \in M$ under f is defined as $\mathrm{Orb}^-(p) = \mathrm{Orb}_f^-(p) = \bigcup_{n=1}^{\infty} f^{-n}\{p\}$, where $f^{-n}\{p\}$ denotes the preimage $(f^n)^{-1}\{p\}$. Given an integer $\mu \geq 1$, if $\bigcup_{n=1}^{\mu} f^{-n}\{p\}$ contains no singularities of f, then f is a local diffeomorphism near each $q \in \bigcup_{n=1}^{\mu} f^{-n}\{p\}$. Thus $f^{-n}\{p\}$ is finite for each $n = 1, 2, \dots, \mu$, as M is compact. Assume further that all terms in $\bigcup_{n=0}^{\mu} f^{-n}\{p\}$ are distinct. In this case we may find a neighborhood W of p in M, called a *μ-dynamical neighborhood* of p, such that each connected component U of

$\bigcup_{n=0}^{\mu} f^{-n}(W)$ is a neighborhood of a unique point $q \in \bigcup_{n=0}^{\mu} f^{-n}\{p\}$, denoted as $U = W(q) = W_f(q)$ and called the *W-component* at q, and that f^n maps $W(q)$ onto W whenever $f^n(q) = p$, $n = 1, 2, \dots, \mu$. Notice that $\bigcup_{n=0}^{\mu} f^{-n}\{p\}$ forms in this case a finite tree, which may not be complete.

In [11, Lemma 4.2], where f was a nonsingular endomorphism, we did a local linearization along a set of the form $\bigcup_{n=0}^{\mu} f^{-n}\{p\}$, which was a finite complete tree. In the present paper, as noticed above, $\bigcup_{n=0}^{\mu} f^{-n}\{p\}$ is still a finite tree, but may not be complete. Although a local linearization can also be made along a finite noncomplete tree in general, for our purpose, we do a local linearization along a subset of it, which does form a finite complete tree. This is the following

Lemma 2.3. *Let $f \in C^1(M)$, $p \in M$, and an integer $\mu \geq 1$ be given. Assume that all terms of $\bigcup_{n=0}^{\mu+1} f^{-n}\{p\}$ are distinct and are not singularities of f. Let $Q_0, Q_1, \dots, Q_{\mu+1}$ be nonempty sets in M such that $Q_0 = \{p\}$, and that $f(Q_n) = Q_{n-1}$ for each $n = 1, 2, \dots, \mu+1$. Then for any $\eta > 0$, there is a $\lambda > 0$, and a map $f_1 \in C^1(M)$, called a local linearization of f, with the following properties* (1)–(5).

Write $W' = \{u \in T_pM \mid |u| \leq \lambda\}$, $V' = \{u \in T_pM \mid |u| \leq \lambda/4\}$, $W = \exp_p(W')$, $V = \exp_p(V')$.

(1) *W is $(\mu+1)$-dynamical for both f and f_1, and the W-component for f and for f_1 are the same, i.e. $W_f(q) = W_{f_1}(q)$, for each $q \in \bigcup_{n=1}^{\mu+1} Q_n$;*

(2) *$f_1 = \exp_{f(q)}(T_q f) \exp_q^{-1}$ on $V_{f_1}(q)$ if $q \in \bigcup_{n=1}^{\mu} Q_n$;*

(3) *$f_1^{\mu+1} = f^{\mu+1}$ on $W(q)$ if $q \in Q_{\mu+1}$. In particular, if $q \in Q_{\mu+1}$ then $f_1 = \exp_{f(q)}(T_{f(q)} f^{\mu})^{-1} \exp_p^{-1} f^{\mu+1}$ on $V(q)$. Note that $V_f(q) = V_{f_1}(q)$ here and we have written both of them as $V(q)$;*

(4) *$f_1 = f$ on $M - \bigcup\{W(q) \mid q \in \bigcup_{n=1}^{\mu+1} Q_n\}$;*

(5) *$d_1(f_1, f) < \eta$.* Q.E.D.

Roughly, f_1 near $q \in \bigcup_{n=1}^{\mu} Q_n$ is just $T_q f$ module those exp, and f_1 near $q \in Q_{\mu+1}$ cancels out these linearizations. The following corollary is clear.

Corollary 2.4. *If $x \in W$, $f^k x \in W(q)$ for some $q \in f^{-\mu-1}\{p\}$, and if the orbit $fx, f^2x, \dots, f^{k-1}x$ never meets $W(q)$ for all $q \in \bigcup_{n=1}^{\mu+1}(f^{-n}\{p\} - Q_n)$, then $f^k x = f_1^k x$.* Q.E.D.

Roughly, this is because whenever the orbit $fx, f^2x, \dots, f^{k-1}x$ meets the area of changes, they meet at a whole $\mu+1$ successive iterates from some $W(q)$, $q \in Q_{\mu+1}$, to $W(p)$. Thus those changes cancel out.

3. Proof of Theorem A

First we prove an easy lemma that is used in Case 1 of the proof of Theorem A. Let $\Omega(f)$ and $P(f)$ denote the sets of nonwandering points and periodic points of f respectively.

Lemma 3.1. *Let $f \in C^1(M)$, $\sigma \in \Omega(f) - P(f)$. If $\mathrm{Orb}^-(\sigma)$ contains no singularities of f, then $(\mathrm{Orb}^-(\sigma) \cap \Omega(f), f)$ forms a complete tree.*

Proof. First we remark that if $p \in \Omega(f)$, and if $f^{-1}\{p\}$ contains no singularities of f, then $f^{-1}\{p\} \cap \Omega(f)$ is nonempty. Actually, since p is nonwandering, there is a sequence $\{x_k\}$ in M and a sequence of positive integers

$\{n_k\}$ such that $\{x_k\}$ and $\{f^{n_k}(x_k)\}$ both converge to p. The set $\{f^{n_k-1}(x_k)\}$ has some limit point q since M is compact. Clearly, $q \in f^{-1}\{p\}$. Since q is not a singularity of f, f is a local diffeomorphism near q. It follows that $q \in \Omega(f)$. This proves the remark.

Now let $Q_n = f^{-n}\{\sigma\} \cap \Omega(f)$, $n = 0, 1, 2, \ldots$. It follows from the above remark that Q_n is nonempty for each $n \geq 0$. It also follows that f maps Q_n onto Q_{n-1} for each $n \geq 1$. Now σ is not periodic of f, hence all terms in $\mathrm{Orb}^-(\sigma)$ are distinct. Therefore $(\mathrm{Orb}^-(\sigma) \cap \Omega(f), f)$ forms a complete tree. Q.E.D.

Now we prove Theorem A.

Proof of Theorem A. Let U be a small neighborhood of ω in M. It suffices to find a $g \in \mathscr{U}$, such that g has a periodic point in U, since another perturbation allows us to push this periodic point onto ω (see [2, 5]). We assume that ω is not periodic already of f, and divide the proof into two cases.

Case 1. There is a $\sigma \in \mathrm{Orb}^-(\omega) \cap \Omega(f)$ such that $\mathrm{Orb}^-(\sigma)$ does not contain singularities of f.

Assume that $f^s(\sigma) = \omega$. Let N be a neighborhood of σ such that $f^s(N) \subset U$. Since ω is nonperiodic of f, so is σ. Then by Lemma 3.1, $(\mathrm{Orb}^-(\sigma) \cap \Omega(f), f)$ forms a complete tree. The proof in Case 1 is very much similar to the proof of Theorem B of [11]. The difference is that $\mathrm{Orb}^-(\sigma)$ this time is just a tree that may not be complete. For explicitly we write down the proof in all details as follows.

Take any $\eta > 0$ such that the η-ball of f in $F\,\mathrm{End}^1(M)$ is contained in $\mathscr{U}$. By Lemma 2.2, there is an $\varepsilon > 0$ such that

$$d_1(hf, f) < \eta/2$$

for any $f \in F\,\mathrm{End}^1(M)$, where h is any ε-kernel lift.

Denote by $\mathscr{T}$ the complete tree $(\mathrm{Orb}^-(\sigma) \cap \Omega(f), f)$, and denote by $(\mathscr{T}, T_q)$ the tree of isomorphisms, where $q \in \mathrm{Orb}^-(\sigma) \cap \Omega(f) - \{\sigma\}$, and $T_q = T_q f^n$ if $f^n q = \sigma$.

Let $\rho > 2$, $\mu \geq 1$ be the two numbers guaranteed by Theorem 2.1 respecting $\{\mathscr{T}, T_q\}$ and $\varepsilon > 0$. For the f, σ, μ, there is by Lemma 2.3 a $\lambda > 0$ and a local linearization f_1 with the following properties (1)–(5). Write $W' = \{u \in T_\sigma M \mid |u| \leq \lambda\}$, $V' = \{u \in T_\sigma M \mid |u| \leq \lambda/4\}$, $W = \exp_\sigma(W')$, $V = \exp_\sigma(V')$.

(1) W is $(\mu+1)$-dynamical for both f and f_1, and $W_f(q) = W_{f_1}(q)$ for each $q \in \bigcup_{n=1}^{\mu+1} f^{-n}\{\sigma\} \cap \Omega(f)$;

(2) $f_1 = \exp_{f(q)}(T_q f)\exp_q^{-1}$ on $V_{f_1}(q)$ if $q \in \bigcup_{n=1}^{\mu} f^{-n}\{\sigma\} \cap \Omega(f)$;

(3) $f_1^{\mu+1} = f^{\mu+1}$ on $W(q)$ if $q \in f^{-\mu-1}\{\sigma\} \cap \Omega(f)$. In particular, if $q \in f^{-\mu-1}\{\sigma\} \cap \Omega(f)$, then $f_1 = \exp_{f(q)}(T_{f(q)} f^{\mu})^{-1} \exp_\sigma^{-1} f^{\mu+1}$ on $V(q)$;

(4) $f_1 = f$ on $M - \bigcup\{W(q) \mid q \in \bigcup_{n=1}^{\mu+1} f^{-n}\{\sigma\} \cap \Omega(f)\}$;

(5) $d_1(f_1, f) < \eta/2$.

Clearly, $f_1 \in F\,\mathrm{End}^1(M)$.

By shrinking W if necessary, we assume that $W \subset N$, and that $f^k(W(q)) \cap (W(q)) = \varnothing$ for any $k \geq 1$ and any $q \in \bigcup_{n=1}^{\mu+1} f^{-n}\{\sigma\} - \Omega(f)$. Put a metric d' on W by defining

$$d'(p, q) = |u - v|,$$

where $p, q \in W$, $u = \exp_\sigma^{-1}(p)$, $v = \exp_\sigma^{-1}(q)$. Since σ is nonwandering of f, there are two points p and $f^\psi(p)$, where $\psi \geq 1$ is an integer, such that the ball $B(f^\psi(p), \rho d'(p, f^\psi(p)); d')$ is contained in V. Let $P = \{p, fp, \dots, f^\psi p\} \cap V$. Say, $P = \{p_0, p_1, \dots, p_t\}$. Note that $p_0 = p$, $p_t = f^\psi p$. Hence $B(p_t, \rho d'(p_0, p_t); d') \subset V$. Let $P' = \exp_\sigma^{-1}(P)$, $p_i' = \exp_\sigma^{-1}(p_i)$. Then $P' = \{p_0', p_1', \dots, p_t'\}$.

By Theorem 2.1, there is a point $y' \in P' \cap B(p_t', \rho|p_0' - p_t'|)$ such that for any branch $\Sigma = \{q_0, q_1, \dots, q_n, \dots\}$ of $\mathrm{Orb}_f^-(\sigma) \cap \Omega(f)$, there is a point $w'(\Sigma) \in P' \cap B(p_t', \rho|p_0' - p_t'|)$, where $w'(\Sigma)$ is before y' in P', together with $\mu + 1$ points $c_0'(\Sigma), c_1'(\Sigma), \dots, c_\mu'(\Sigma)$ in $B(p_t', \rho|p_0' - p_t'|)$, not necessarily distinct, satisfying the following two conditions (a) and (b).

(a) $c_0'(\Sigma) = w'(\Sigma)$, $c_\mu'(\Sigma) = y'$; and

(b)

$$\begin{aligned}|(T_{q_n} f^n)^{-1}(c_n'(\Sigma)) - (T_{q_n} f^n)^{-1}(c_{n+1}'(\Sigma))| \\ \leq \varepsilon d((T_{q_n} f^n)^{-1}(c_{n+1}'(\Sigma)), (T_{q_n} f^n)^{-1}(A))\end{aligned}$$

for $n = 0, 1, 2, \dots, \mu - 1$, where $A = (P'(w'(\Sigma), y')) \cup \partial B(p_t', \rho|p_0' - p_t'|)$, and $P'(w'(\Sigma), y') = \{p \in P' | p \text{ is before } y' \text{ and after } w'(\Sigma)\}$.

Let $w(\Sigma) = \exp_\sigma(w'(\Sigma))$, $y = \exp_\sigma(y')$. Then $w(\Sigma)$ and y are both in P. Hence there is an integer $\phi(\Sigma) \geq 1$ such that $f^{\phi(\Sigma)}(w(\Sigma)) = y$. Note that the orbit $f(w(\Sigma)), f^2(w(\Sigma)), \dots, f^{\phi(\Sigma)}(w(\Sigma))$ never meets $W(q)$ for $q \in \bigcup_{n=1}^{\mu+1} f^{-n}\{\sigma\} - \Omega(f)$ since, for those q, $W(q)$ is wandering (i.e. $f^k(W(q)) \cap W(q) = \varnothing$ for any $k \geq 1$). Thus it must meet a $W(q)$ for some $q \in f^{-\mu-1}\{\sigma\} \cap \Omega(f)$, since $\mathrm{Orb}^-(\sigma) \cap \Omega(f)$ is a complete tree. Hence $\phi(\Sigma) > \mu + 1$. Let $z(\Sigma) = f^{\phi(\Sigma)-\mu-1}(w(\Sigma))$. It is ready to see that $z(\Sigma)$ actually does not depend on Σ, since $w(\Sigma)$ and y are both in P, and $f^{\phi(\Sigma)}(w(\Sigma)) = y$, and since μ and y do not depend on Σ. Thus we simply write

$$z = f^{\phi(\Sigma)-\mu-1}(w(\Sigma))$$

for any branch Σ. Clearly, $f^{\mu+1}(z) = y$.

Since $y \in V$, there is a unique point $\sigma_{\mu+1}$, here $\sigma_{\mu+1}$ is in $f^{-\mu-1}\{\sigma\} \cap \Omega(f)$ as just noticed above, such that

$$z \in V(\sigma_{\mu+1}).$$

Let Γ be any branch of $\mathrm{Orb}_f^-(\sigma) \cap \Omega(f)$ that contains $\sigma_{\mu+1}$. Say $\Gamma = \{\sigma_0, \sigma_1, \dots, \sigma_n, \dots\}$. Let w', together with $c_0', c_1', \dots, c_\mu'$, be guaranteed by Theorem 2.1 respecting Γ, and let $\phi > \mu + 1$ be the integer such that $f^\phi(w) = y$, where $w = \exp_\sigma(w')$. Note that $w \in N$.

For each σ_n, $n = 0, 1, \dots, \mu + 1$, let h_{σ_n} be the ε-kernel lift obtained by treating in Lemma 2.2 $p = \sigma_n$, $v_1 = (T_{\sigma_n} f^n)^{-1}(c_n')$, and $v_2 = (T_{\sigma_n} f^n)^{-1}(c_{n+1}')$. Define a map g by

$$g = \begin{cases} h_{\sigma_n} \circ f_1 & \text{on } W(\sigma_{n+1}),\ n = 0, 1, \dots, \mu - 1, \\ f_1 & \text{on the rest of } M. \end{cases}$$

Then $q \in F\,\mathrm{End}^1(M)$ and $d_1(g, f_1) < \eta/2$. Hence

$$d_1(g, f) < \eta.$$

We now verify that w is periodic of g. It suffices to verify that $g^{\phi-\mu-1}(w) = z$ and $g^{\mu+1}(z) = w$. By the condition (b) above, the g orbit from w to z never touches the supports of these lifts. Then $g^{\phi-\mu-1}(w) = f_1^{\phi-\mu-1}(w)$. But $f_1^{\phi-\mu-1}(w) = f^{\phi-\mu-1}(w)$ by Corollary 2.4. Therefore,

$$g^{\phi-\mu-1}(w) = f^{\phi-\mu-1}(w) = z.$$

It remains to verify that $g^{\mu+1}(z) = w$. By the condition (3) above,

$$g(z) = f_1(z) = \exp_{\sigma_\mu}(T_{\sigma_\mu} f^{\mu})^{-1} \exp_{\sigma}^{-1} f^{\mu+1}(z)$$

because $z \in V(\sigma_{\mu+1})$. Hence

$$g(z) = \exp_{\sigma_\mu}(T_{\sigma_\mu} f^{\mu})^{-1}(y')$$

because $f^{\mu+1}(z) = y$. Thus these lifts $h_{\sigma_{\mu-1}}, h_{\sigma_{\mu-2}}, \dots, h_{\sigma_0}$ give rise to

$$g^{\mu}(g(z)) = w$$

by condition (2) above. This verifies that w is periodic of g.

Now since σ is not periodic of f, we may take W so small in advance that $f^n(W)$ does not intersect $\bigcup\{W(p) | p \in \bigcup_{n=1}^{\mu+1} f^{-n}\{\sigma\}\}$ for $n = 1, 2, \dots, s$, here s is the same integer that appeared in the beginning of Case 1 such that $f^s(\sigma) = \omega$. Thus $g^s(w) = f^s(w)$ is a periodic point of g in U. This proves Theorem A for Case 1.

Case 2. No such σ exists.

First we note that there is an $n_0 \geq 1$ such that $f^{-n}\{\omega\}$ does not contain singularities of f for $n \geq n_0$. This is because otherwise there would be a singularity p of f and two integers $1 \leq n_1 < n_2$ such that

$$p \in (f^{-n_1}\{\omega\}) \cap (f^{-n_2}\{\omega\})$$

because f has only finitely many singularities. Then

$$f^{n_2-n_1}(\omega) = f^{n_2-n_1}(f^{n_1}(p)) = f^{n_2}(p) = \omega,$$

contradicting that ω is not periodic of f.

Thus by the assumption of Case 2, there is a $k \geq 1$, such that $(f^{-k}\{\omega\}) \cap (\Omega(f)) = \varnothing$ but $(f^{-k+1}\{\omega\}) \cap (\Omega(f)) \neq \varnothing$. Then we can take a $q \in (f^{-k+1}\{\omega\}) \cap \Omega(f)$ such that $(f^{-1}\{q\}) \cap \Omega(f) = \varnothing$. We now adopt a technique of [12] to handle this case. Also see [9] for this technique.

Because q is nonwandering of f, there is a sequence of points $x_1, x_2, \dots, x_i, \dots$ in M that converges to q, and a sequence of positive integers $1 < j_1 < j_2 < \cdots < j_i < \cdots$ such that the sequence $f^{j_i}(x_i)$ also converges to q. Let p be a limit point of the sequence

$$f^{j_1-1}(x_1), f^{j_2-1}(x_2), \dots, f^{j_i-1}(x_i), \dots.$$

Then $p \in f^{-1}\{q\}$. Thus $p \notin \Omega(f)$.

Fix a ball B around p such that $(f^n B) \cap B = \varnothing$ for all $n \geq 1$. Since $q \in \Omega(f)$ and $f(\Omega(f)) \subset \Omega(f)$, the positive orbit of q never enters B. Take a neighborhood V_j of $f^j(q)$ for each $j = 0, 1, \dots, k-2$, such that $V_0, V_1, \dots, V_{k-2}$ are all disjoint from B, that $f(V_j) \subset V_{j+1}$ for $j = 0, 1, \dots, k-3$, and that $f(V_{k-2}) \subset U$. Arbitrarily near p and q, there are two points $f^{j_i-1}(x_i)$ and x_i for large i. Hence there is a C^1 small perturbation g of

f (here g is f composed with a lift. This lift can be even C^∞ close to the identity), supported on B, that takes $f^{j_i-1}(x_i)$ onto x_i. Note that the g-orbit from x_i to $g^{j_i-1}(x_i)$ are the same as the f-orbit from x_i to $f^{j_i-1}(x_i)$ since the latter intersects B only at the last point $f^{j_i-1}(x_i)$ by the way of choosing B. Hence x_i is periodic of g. Since $f = g$ on $V, V_1, \dots, V_{k-2}$, it follows that $g^{k-1}(x_i) = f^{k-1}(x_i) \in U$. Hence g has a periodic point $g^{k-1}(x_i)$ in U. This proves Theorem A for Case 2, and hence proves the whole Theorem A. Q.E.D.

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