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SPECTRAL ASYMPTOTICS FOR TOEPLITZ MATRICES GENERATED BY THE POISSON-CHARLIER POLYNOMIALS

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ABSTRACT. A conjecture of Grenander and Szegö for the traces of Toeplitz matrices generated by the Poisson-Charlier polynomials is proved.

The Poisson-Charlier polynomials $\{p_m(x)\}$, $m \geq 0$, are defined on the set X of all nonnegative integers [6, p. 34]. This is a complete system of orthonormal polynomials in the space $l^2 = l^2(X, j(x))$, where $j(x) = e^{-a}a^x(x!)^{-1}$, $a > 0$, is the Poisson weight: $\sum_{x=0}^{\infty} p_k(x)p_m(x)j(x) = \delta_{km}$.

Let $q(x)$ be a real almost periodic function (in the Bohr sense) on X. Then the matrix

$$M_n(q) = \left\{\sum_{x=0}^{\infty} p_k(x)p_m(x)q(x)j(x);\ k, m = 0, 1, \dots, n-1\right\}$$

is called the Toeplitz matrix, generated by the polynomials $\{p_m\}$ and the function q.

Here we prove the following conjecture of Grenander and Szegö [2, p. 174]:

$$(1) \qquad \lim_{n\to+\infty} \frac{1}{n}\text{trace } [M_n(q)]^k = \lim_{n\to+\infty} \frac{1}{n}\sum_{x=0}^{n-1}[q(x)]^k, \qquad k = 0, 1, \dots.$$

As a consequence one obtains the asymptotic distribution of the spectrum of the matrix $M_n(q)$ as $n \to +\infty$. Namely, let $N(\alpha, \beta, n)$ be the number of all eigenvalues of $M_n(q)$ lying on the segment $[\alpha, \beta]$. Define the distribution function $D(\alpha)$ by the lower limit: $D(\alpha) = \underline{\lim}_{n\to+\infty} \frac{1}{n}\sum_{x=0}^{n-1}\chi(\alpha - q(x))$ where χ is the characteristic function of the interval $(0, \infty)$. Then we have

$$(2) \qquad \lim_{n\to+\infty} \frac{1}{n}N(\alpha, \beta, n) = D(\beta) - D(\alpha)$$

if α and β are points of continuity for D.

Proof of (1). Let E_n be the operator in l^2 with kernel

$$(3) \qquad e(n, x, y) = \sum_{m=0}^{n-1} p_m(x)p_m(y)$$

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and let Q be the operator of multiplication by q. Then we have

$$\text{trace}\,[M_n(q)]^k = \text{trace}\, Q_n^k \tag{4}$$

where $Q_n = E_n Q E_n$. Indeed, if T is the infinite matrix $M_\infty(q)$, considered as an operator in l^2, then it is not hard to see that

$$\text{trace}\,[M_n(q)]^k = \text{trace}\, T_n^k \tag{5}$$

where $T_n = P_n T P_n$ and P_n is the operator of multiplication by the characteristic function of the set $\{0, 1, \dots, n-1\}$. On the other hand, the operators T_n and Q_n are unitary equivalent: $Q_n = F^{-1} T_n F$ where F is the Fourier transform with respect to the complete system $\{p_m\}$, that is, $(Fu)(m) = \sum_{x=0}^{\infty} p_m(x) u(x) j(x)$. Thus (4) follows from (5).

Further, the operator Q_n has a kernel

$$Q_n(x, y) = \sum_{z=0}^{\infty} e(n, x, z) e(n, z, y) q(z) j(z)$$

and E_n is an orthogonal projection. Therefore

$$\text{trace}\, Q_n = \sum_{x=0}^{\infty} e(n, x, x) q(x) j(x).$$

Since $Q_n^2 = E_n Q^2 E_n - S_n^* S_n$ where $S_n = (\text{id} - E_n) Q E_n$, it follows that

$$\text{trace}\, Q_n^2 = \sum_{x=0}^{\infty} e(n, x, x) q^2(x) j(x) - \|S_n\|_2^2$$

where $\|\cdot\|_2$ stands for the Hilbert-Schmidt norm [1]. Analogously, $Q_n^k = E_n Q^k E_n + S_{k,n}$ where $S_{k,n}$ is a sum of $2^{k-1} - 1$ terms, each of them containing as a cofactor S_n^* and S_n. Therefore

$$\text{trace}\, Q_n^k = \sum_{x=0}^{\infty} e(n, x, x) q^k(x) j(x) + R_{n,k}, \qquad k \geq 1, \tag{6}$$

$$|R_{n,k}| \leq (2^{k-1} - 1) \|q\|^{k-2} \|S_n\|_2^2, \qquad R_{n,1} = 0 \tag{7}$$

where $\|q\| = \sup_{x \in X} |q(x)|$. Further, the operator S_n has a kernel

$$S_n(x, y) = \sum_{z=0}^{\infty} (q(x) - q(z)) e(n, x, z) e(n, z, y) j(z),$$

hence

$$\|S_n\|_2^2 = \frac{1}{2} \sum_{x=0}^{\infty} \sum_{y=0}^{\infty} (q(x) - q(y))^2 |e(n, x, y)|^2 j(x) j(y). \tag{8}$$

Thus, the formulas (6)–(8) show that it suffices to know the asymptotics of the function $e(n, x, x)$ as $n \to +\infty$ and an estimate of $e(n, x, y)$ if $x \neq y$ and $n \to +\infty$. We shall prove the following uniform asymptotics and estimates, which are sufficient for our purposes:

Case 1. $0 \leq x \leq n(1 - n^{-\delta})$, where $0 < \delta < 1/4$.

$$e(n, x, x) j(x) = 1 + e^{-\sqrt{n}} O(1), \qquad n \to +\infty; \tag{9}$$

Case 2: $x \geq n(1+n^{-\delta})$.

$$e(n, x, x)j(x) = n^{\delta} \exp\left(\frac{n}{1+n^{\delta}} - \frac{xn^{\delta}}{(1+n^{\delta})^2}\right), \qquad n \to +\infty; \tag{10}$$

Case 3. $0 \leq y < x \leq n(1-n^{-\delta})$.

$$((x-y)e(n, x, y))^2 j(x)j(y) = ne^{-\sqrt{n}}O(1), \qquad n \to +\infty; \tag{11}$$

Case 4. For all x and n

$$e(n, x, x)j(x) < 1. \tag{12}$$

Before proving these properties, we shall show that (4) and (6)–(12) imply (1). Namely, it is not hard to see that

$$\sum_{x=0}^{\infty} e(n, x, x)q^k(x)j(x) = \sum_{x=0}^{n-1} q^k(x) + \|q\|^k o(n), \qquad n \to +\infty. \tag{13}$$

On the other hand,

$$\|S_n\|_2^2 \leq \sum_{x=1}^{n(1-n^{-\delta})} \sum_{y=0}^{x-1} (q(x)-q(y))^2 |e(n, x, y)|^2 j(x)j(y) + 4\|q\|^2 \sum_{x=n(1-n^{-\delta})}^{\infty} e(n, x, x)j(x).$$

Therefore, if q is a Lipschitz function with a norm

$$\|q\|_1 = \sup_{x, y} |q(x)-q(y)|\,|x-y|^{-1} + \|q\|,$$

we obtain

$$\|S_n\|_2^2 = \|q\|_1^2 o(n), \qquad n \to +\infty. \tag{14}$$

Thus (6)–(8), (13), and (14) yield the asymptotics

$$\operatorname{trace} Q_n^k = \sum_{x=0}^{n-1} q^k(x) + 2^k \|q\|^{k-2} \|q\|_1^2 o(n), \qquad n \to +\infty. \tag{15}$$

Hence, (15) and (4) imply (1) if q is a Lipschitz function. It remains to notice that the Lipschitz class is a dense set in the space of all almost periodic functions with respect to the supremum norm $\|q\|$. Thus (1) is proved.

Proof of (9)–(12). We shall use the formula

$$e(\lambda, x, y) = \frac{1}{2\pi i} \int_{\varepsilon - i\pi}^{\varepsilon + i\pi} e^{\lambda w} U(w, x, y) H(\lambda, w)\, dw, \qquad \varepsilon > 0, \tag{16}$$

where $e(\lambda, x, y)$ is the step function: $e(\lambda, x, y) = e(n, x, y)$ if $n \leq \lambda < n+1$, $n = 1, 2, \ldots$, and $e(\lambda, x, y) = 0$ if $\lambda < 1$. Here

$$U(w, x, y) = \int_0^{\infty} e^{-\lambda w} de(\lambda, x, y) = \sum_{n=0}^{\infty} e^{-w(n+1)} p_n(x) p_n(y)$$

is an entire $2i\pi$-periodic function with respect to w, and moreover (see [5]) for $y \geq x$ we have:

$$U(w, x, y) = \exp(ae^{-w}) \sum_{k=0}^{x} \binom{x}{k} \binom{y}{k} k! a^{-k} e^{-wk} (1 - e^{-w})^{x+y-2k}.$$

In particular, $\sum_{n=0}^{\infty} p_n^2(x) = U(0, x, x) = 1/j(x)$, whence the estimate (12) follows immediately. Finally, the function $s \to H(s, w)$ is 1-periodic and $H(s, w) = \frac{1}{2}(\mathrm{sh}\frac{w}{2})^{-1} \exp(\frac{1}{2} - s)w$ if $0 \leq s < 1$.

For proving (16), we use the relation

$$(17) \qquad w^{-1} U(w, x, y) = \int_0^{\infty} e^{-\lambda w} e(\lambda, x, y)\, d\lambda.$$

Since the function $\lambda \to e(\lambda, x, y)$ is continuous only from the right, we pass to the average:

$$e_h(\lambda, x, y) = \frac{1}{h} \int_0^h e(\lambda + \mu, x, y)\, d\mu, \qquad h > 0.$$

Notice that $e_h(\lambda, x, y) \to e(\lambda, x, y)$ as $h \to +0$ for every (λ, x, y). From (17) it follows that

$$\frac{e^{hw} - 1}{h} \frac{U(w, x, y)}{w^2} = \int_0^{\infty} e^{-\lambda w} e_h(\lambda, x, y)\, d\lambda, \qquad \mathrm{Re}\, w > 0,$$

so the inverse Laplace formula gives the equality

$$e_h(\lambda, x, y) = \frac{1}{2\pi i} \int_{\varepsilon - i\infty}^{\varepsilon + i\infty} e^{\lambda w} \frac{e^{hw} - 1}{h} \frac{U(w, x, y)}{w^2}\, dw, \qquad \varepsilon > 0.$$

Taking into account the periodicity of the function $w \to U(w, x, y)$, we obtain

$$(18) \qquad e_h(\lambda, x, y) = \frac{1}{2\pi i} \int_{\varepsilon - i\pi}^{\varepsilon + i\pi} e^{\lambda w} U(w, x, y) \frac{g(h, w) - g(0, w)}{h}\, dw$$

where $g(s, w) = e^{ws} f(\lambda + s, w)$ and $f(s, w) = \sum_{k=-\infty}^{\infty} (e^{i2sk\pi}/(w + 2ik\pi)^2)$. It is clear that the function $s \to f(s, w)$ is 1-periodic and

$$f(s, w) = \frac{1}{4} \left(\mathrm{sh}\frac{w}{2}\right)^{-1} \left(\mathrm{cth}\frac{w}{2} - 1 + 2s\right) \exp\left(\frac{1}{2} - s\right) w \quad \text{if } 0 \leq s < 1.$$

Therefore, $\lim_{h \to 0} h^{-1}(g(h, w) - g(0, w)) = H(\lambda, w)$, and the Lebesgue limit theorem is applicable. So (16) follows from (18).

Further, we have the estimate

$$(19) \qquad |j(x) U(w, x, x)| \leq \exp(4\sqrt{ax}|\mathrm{sh}\tfrac{w}{2}| - x\mathrm{Re}\, w) \exp(ae^{-\mathrm{Re}\, w} - a).$$

Indeed, it is obvious that

$$U(w, x, x) = \exp(ae^{-w}) a^{-x} x! e^{-wx} \sum_{k=0}^{x} \binom{x}{k} \frac{1}{k!} \left(2\sqrt{a}\, \mathrm{sh}\frac{w}{2}\right)^{2k}.$$

Since $\binom{x}{k} \leq x^k (k!)^{-1}$ and $\sum_{k=0}^{x} (k!)^{-2} (2\sqrt{ax}|\mathrm{sh}\frac{w}{2}|)^{2k} \leq \exp(4\sqrt{ax}|\mathrm{sh}\,\frac{w}{2}|)$, the estimate (19) holds.

To prove the asymptotic bound (10), we use formula (16) with $\varepsilon = (1+n^{\delta})^{-1}$ where $0 < \delta < 1/4$. Since $x > n(1-\varepsilon)^{-1}$, we have $x > 16a\varepsilon^{-4}\mathrm{ch}^2\frac{\varepsilon}{2}$ if $n^{1-4\delta} > (1+n^{-\delta})^3 16a\,\mathrm{ch}^2\frac{\varepsilon}{2}$, therefore (19) yields

$$(20)\qquad |j(x)U(w, x, x)| \le e^{-\varepsilon(1-\varepsilon)x} \quad \text{if } x > n(1+n^{-\delta}),\ n > C(\delta, a).$$

Taking into account the bound $|H(n, w)| \le C\varepsilon^{-1}$ if $\mathrm{Re}\, w = \varepsilon$, we obtain (10) from (16) and (20).

For proving (9), we notice that the function $w \to e^{\lambda w}U(w, x, y)H(\lambda, w)$ is $2i\pi$-periodic, so the Cauchy formula and (16) imply

$$(21)\quad e(n, x, x) = U(0, x, x) + \frac{1}{2\pi i}\int_{-1-i\pi}^{-1+i\pi} e^{nw}U(w, x, x)H(n, w)\,dw.$$

Since $x \le n(1-n^{-\delta})$, $0 < \delta < 1/4$, we have $4\sqrt{ax}|\mathrm{sh}\frac{w}{2}| + x - n \le -\sqrt{n}$ if $n > C(\delta, a)$, $\mathrm{Re}\, w = -1$, so the estimate (19) with $\mathrm{Re}\, w = -1$ shows that

$$|j(x)U(w, x, x)| \le \exp(ae - a)e^{-\sqrt{n}} \quad \text{if } x \le n(1-n^{-\delta}).$$

Thus (9) follows from (21), since $j(x)U(0, x, x) = 1$.

The proof of the estimate (11) is based on the Christoffel-Darboux formula:

$$(22)\qquad (x-y)e(n, x, y) = \sqrt{an}(p_n(x)p_{n-1}(y) - p_{n-1}(x)p_n(y)),$$

which follows from the recurrence relation [5]:

$$\lambda_n p_{n+1}(x) = (x - a - n)p_n(x) - \lambda_{n-1}p_{n-1}(x), \qquad \lambda_{n-1} = \sqrt{an}.$$

According to (9) we have

(23)

$$j(x)p_n^2(x) = e^{-\sqrt{n}}O(1), \qquad n \to +\infty \text{ if } 0 \le x \le n(1-n^{-\delta}),\ 0 < \delta < \tfrac{1}{4}.$$

Therefore the estimate (11) is a consequence of (22) and (23).

Proof of (2). It is sufficient to consider the points α, β on the segment $[m, M]$, where $m = \inf q(x)$, $M = \sup q(x)$. For the sequence of distribution functions $D_n(\alpha) = \frac{1}{n}\sum_{x=0}^{n-1}\chi(\alpha - q(x))$ the inequalities $0 \le D_n(\alpha) \le 1$ hold, hence there exists a limit $D_1(\alpha) = \lim_{j\to\infty} D_{n_j}(\alpha)$. Then formula (1) can be written in the form:

$$\lim_{n\to\infty} \tfrac{1}{n}\,\mathrm{trace}\,[M_n(q)]^k = \int_m^M \alpha^k\,dD_1(\alpha).$$

Therefore, we can apply the method of Grenander and Szegö [2, p. 129] and conclude that

$$(24)\qquad \lim_{n\to\infty} \tfrac{1}{n}N(\alpha, \beta, n) = D_1(\beta) - D_1(\alpha)$$

if α and β are points of continuity for $D_1(\alpha)$. Since the functions $D_1(\alpha)$ and $D(\alpha)$ coincide on the set of the points $\{\alpha\}$ of continuity [4], formula (2) follows from (24).

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