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SECOND ORDER ERGODIC THEOREMS FOR ERGODIC TRANSFORMATIONS OF INFINITE MEASURE SPACES

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ABSTRACT. For certain pointwise dual ergodic transformations T we prove almost sure convergence of the log-averages

$$\frac{1}{\log N}\sum_{n=1}^{N}\frac{1}{na(n)}\sum_{k=1}^{n} f\circ T^k \qquad (f\in L_1)$$

and the Chung-Erdös averages

$$\frac{1}{\log a(N)}\sum_{k=1}^{N}\frac{1}{a(k)} f\circ T^k \qquad (f\in L_1^+)$$

towards $\int f$, where $a(n)$ denotes the return sequence of T.

1. INTRODUCTION

Let $(\Omega, \mathscr{F}, m, T)$ be a conservative, ergodic infinite measure preserving dynamical system, where m is a nonatomic, σ-finite, infinite measure.

For a measurable function f denote

$$S_n f = \sum_{k=1}^{n} f\circ T^k.$$

We are interested in the behaviour of sums of the form

$$\frac{1}{\log N}\sum_{n=1}^{N}\frac{S_n f}{na(n)} = \frac{1}{\log N}\sum_{n=1}^{N}\frac{1}{na(n)}\sum_{k=1}^{n} f\circ T^k \tag{1}$$

for some (and hence all) functions $f\in L_1(m)$ and

$$\frac{1}{\log a(n)}\sum_{k=1}^{n}\frac{1}{a(k)} f\circ T^k \tag{2}$$

for functions $f\in L_1^+(m)$, where $a(n)>0$ are constants. The first sums will be called the log-averages (of the normalized partial sums) and the second sums the Chung-Erdös averages (as they were first studied in [5]).

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In particular, we are interested in situations where these averages converge to $\int f$ for every $f \in L_1^+(m)$, for some sequence of normalizing constants $a(n)$. Not every conservative, ergodic measure preserving transformation has this property (see [2, §2; 8, Proposition 2.7]).

Examples where the log-averages converge have been given in [4, 8] (compare also [7]). Chung and Erdös [5, Theorem 6] proved that the Chung-Erdös averages converge for any conservative, ergodic Markov shift.

In this paper, we consider the convergence of the averages for pointwise dual ergodic transformations T [2, §1]. Denote by $\widehat{T}$ the dual operator of T : $L_\infty(m) \to L_\infty(m)$, restricted to $L_1(m)$. The assumption that T is pointwise dual ergodic means that there are normalizing constants $a(n)$ such that

$$\lim_{n\to\infty} \frac{1}{a(n)} \sum_{k=1}^{n} \widehat{T}^k f = \int_\Omega f\, dm \qquad \text{a.s.} \tag{3}$$

for every function $f \in L_1(m)$. The constants $a(n)$ are given by

$$a(n) \sim \sum_{k=1}^{n} \int_A \widehat{T}^k 1_A\, dm,$$

where $A \in \mathscr{F}$ is a set of measure one such that the convergence in (3) is a.s. and in $L_1(m_{|A})$. The sequence $a(n)$ is called a return sequence for T.

Although convergence of the log-averages is not in general equivalent to convergence of the Chung-Erdös averages, we have

Proposition 1. *Let $a(n)$ be regularly varying with index $\alpha > 0$. Then, for every positive function $f \in L_1^+(m)$, we have*

$$\lim_{N\to\infty} \frac{1}{\log a(N)} \sum_{k=1}^{N} \frac{1}{a(k)} f \circ T^k = \int f\, dm \qquad a.s.$$

if and only if

$$\lim_{N\to\infty} \frac{1}{\log N} \sum_{n=1}^{N} \frac{1}{na(n)} S_n f = \int f\, dm. \qquad a.s.$$

In general:

Proposition 2. *For any $f \in L_1^+(m)$ the intrinsic averages*

$$\frac{1}{\log a(N)} \sum_{n=1}^{N} \frac{u_n}{a(n)^2} S_n f$$

converge a.s., where $u_n = a(n+1) - a(n)$, if and only if the Chung-Erdös averages (2) *converge.*

The proofs of these propositions are elementary and left to the reader.

In §2 we prove second order ergodic theorems for log-averages (1). The method of proof uses an estimate of the variances of (1) in terms of

$$\Phi(N, \varepsilon) = \sup\left\{ \frac{a(n)}{a(nN^\varepsilon)} : 1 \le n \le N^{1-\varepsilon} \right\} \qquad (N \ge 1,\ \varepsilon > 0). \tag{4}$$

We show in Theorem 1 that (1) converges in measure on sets of finite measure if for any $\varepsilon > 0$, $\Phi(N, \varepsilon) \to 0$ as $N \to \infty$. Moreover, imposing logarithmic rates in this convergence (specfically, condition $(\star)$ in Theorem 2) and a logarithmic asymptotic error in (3) for suitable f's, we obtain a.s. convergence for the log-averages (1) in Theorem 2.

The conditions on $\Phi(N, \varepsilon)$ are always satisfied if the return sequence $(a(n))_{n\geq 1}$ is regularly varying with positive index. The "asymptotic error" condition always holds for Markov shifts. The theorem of Chung and Erdös states that for Markov chains the averages (2) converge to $\int f\,dm$ a.s., and hence by Proposition 1, also the log-averages converge a.s., if the return sequence is regularly varying with positive index. In fact, we show in §2 that a.s. convergence holds for certain slowly varying sequences $a(n)$. This implies that there are Markov shifts for which the log-averages converge a.s., though this cannot be deduced from the convergence of the Chung-Erdös averages by Proposition 1.

As another application of Theorem 2 we obtain that number-theoretical transformations in the sense of Thaler [10] have convergent log-averages when the return sequence $a(n)$ satisfies $(\star)$.

The conclusion of Theorem 2 also applies to the shifts of Markov processes that are recurrent in the sense of Harris, and whose return sequences satisfy $(\star)$. See [2, §1, Example 2].

The method of proof for the result of Chung and Erdös is a cancellation argument. A similar argument is used in §3 to show a.s. convergence in (2) for the real restrictions of conservative, Lebesgue measure preserving, odd inner functions of the upper half-plane.

2. Log-averages

In this section let T denote a pointwise dual ergodic transformation on the nonatomic, σ-finite, infinite measure space $(\Omega, \mathscr{F}, m)$. Then there exists a set $A \in \mathscr{F}$ of measure one and a sequence $\overline{a}(n)$ such that

$$a(n) \sim \overline{a}(n) \uparrow \infty \tag{7}$$

and

$$\sum_{k=1}^{n} \widehat{T}^k 1_A(x) \leq \overline{a}(n) \tag{8}$$

for $x \in A$ and $n \geq 1$. This can be shown by a successive use of Egorov's theorem. Since $a(n)$ may be replaced by an asymptotically equivalent sequence we assume that $a(0) = 1$ and

$$a(n) = 1 + \int_A S_n 1_A\,dm. \tag{9}$$

Write

$$\frac{\overline{a}(n)}{a(n)} = 1 + \beta_n \qquad (n \geq 1). \tag{10}$$

We now state our theorems for log-averages.

Theorem 1. *Suppose that for every* $\varepsilon > 0$

$$\lim_{N\to\infty} \Phi(N, \varepsilon) = 0.$$

Then, for any $f \in L_1(m)$,

$$\lim_{N\to\infty} \frac{1}{\log N} \sum_{n=1}^{N} \frac{1}{na(n)} \sum_{k=1}^{n} f \circ T^k = \int_\Omega f\,dm$$

in measure on any set of finite measure.

Theorem 2. *Suppose that*

$$(\star) \qquad \exists \varepsilon, \gamma > 0 \ni \quad \Phi(N, (\log N)^{-\gamma}) = O\left(\frac{1}{(\log N)^\varepsilon}\right) \quad \text{as } N \to \infty.$$

Moreover, assume that β_N *defined in* (10) *satisfies*

$$\beta_N = O\left(\frac{1}{(\log N)^\gamma}\right) \quad \text{as } N \to \infty.$$

Then for every $f \in L_1(m)$

$$\lim_{N\to\infty} \frac{1}{\log N} \sum_{n=1}^{N} \frac{1}{na(n)} \sum_{k=1}^{n} f \circ T^k = \int_\Omega f\,dm \quad \text{a.s.}$$

Corollary 1. *Let* $(\Omega, \mathscr{F}, m, T)$ *be a conservative Markov shift with return sequence* $a(n)$ *satisfying* $(\star)$. *Then for every function* $f \in L_1(m)$

$$\lim_{N\to\infty} \frac{1}{\log N} \sum_{n=1}^{N} \frac{1}{na(n)} S_n f = \int_\Omega f\,dm \quad \text{a.s.}$$

and

$$\lim_{N\to\infty} \frac{1}{\log a(N)} \sum_{k=1}^{N} \frac{1}{a(k)} f \circ T^k = \int_\Omega f\,dm \quad \text{a.s.}$$

Remark 1. Let

$$a(n) = \exp\left(\int_1^n \frac{\eta(t)}{t}\,dt\right) \qquad (n \geq 1).$$

If $\eta(t)$ is decreasing then $\Phi(N, \varepsilon) \to 0$ as $N \to \infty$ for every $\varepsilon > 0$ if and only if

$$(11) \qquad \lim_{t\to\infty} \eta(t) \log t = \infty$$

and $(\star)$ holds if and only if

$$(12) \qquad \lim_{t\to\infty} \eta(t)(\log t)^{1-\delta} = \infty$$

for some $\delta > 0$.

There are Markov shifts with slowly varying return sequences satisfying (12). For these Markov shifts, the log-averages converge a.e. by Corollary 1. On the other hand, it can be seen easily that if T is a Markov shift with return sequence $\log N$, then the log-averages do not converge in measure.

Remark 2. Suppose that T is a number-theoretical transformation in the sense of Thaler [10]. Then (see [3]) T is pointwise dual ergodic, and, indeed, there is a set A of measure one such that for all $p, n \geq 1$

$$\sum_{k=1}^{n} \widehat{T}^k 1_A \leq p + (1 + \varepsilon_p) a(n)$$

uniformly on A, where $\varepsilon_p = O(\theta^{\sqrt{p}})$ for some $\theta < 1$.

Suppose that this is the case with $\varepsilon_p = O((\log p)^{-\delta})$ and that $a(n)$ satisfies $(\star)$. Then $\log a(n) \geq (\varepsilon/2)\log n$ for n large, where ε is as in $(\star)$.

Set $p_n = a(n)(\log n)^{-\varepsilon/4}$. Then

$$\sum_{k=1}^{n} \widehat{T}^k 1_A \leq \left(1 + \frac{p_n}{a(n)} + O((\log p_n)^{-\delta})\right) a(n)$$
$$= \left(1 + (\log n)^{-\varepsilon/4} + O((\log p_n)^{-\delta})\right) a(n).$$

Now $\log p_n \geq \log a(n) - (\varepsilon/4)\log n \geq (\varepsilon/4)\log n$, and so

$$\sum_{k=1}^{n} \widehat{T}^k 1_A = (1 + O((\log n)^{-\min(\varepsilon/4,\delta)}))a(n)$$

uniformly on A and Theorem 2 applies.

The proofs of our results are based on an estimate of the variance of the log-averages. We denote $S_n = \sum_{k=1}^{n} 1_A \circ T^k$ $(n \geq 1)$, $S_0 = 0$, $T_N = \sum_{n=1}^{N}(S_n/na(n))$, and $\widehat{T}_n = \sum_{k=1}^{n} \widehat{T}^k 1_A$ $(n \geq 1)$, $\widehat{T}_0 = 0$. Finally, we shall use the expectation sign E to denote the integral with respect to the measure m over the set A.

Lemma 1. *If* $n \leq m$, *then*

$$ES_nS_m \leq \sum_{k=0}^{n}(a(m-k) + a(n-k))l_k,$$

where $l_k = \overline{a}(k) - \overline{a}(k-1)$ *and where* $\overline{a}(-1) = 0$.

Proof. Using the duality of T and $\widehat{T}$, we have

$$\begin{aligned}
ES_nS_m &= \sum_{k=1}^{n}\sum_{l=1}^{m}\int_A 1_A \circ T^k 1_A \circ T^l \, dm \\
&= \sum_{k=1}^{n}\int_A (1_A S_{m-k}) \circ T^k \, dm + \sum_{k=1}^{n-1}\int_A (1_A S_{n-k}) \circ T^k \, dm + a(n) - 1 \\
&= \sum_{k=1}^{n}\int_A \widehat{T}^k 1_A S_{m-k} \, dm + \sum_{k=1}^{n-1}\int_A \widehat{T}^k 1_A S_{n-k} \, dm + a(n) - 1 \\
&= \sum_{k=1}^{n}\int_A (\widehat{T}_k - \widehat{T}_{k-1}) S_{m-k} \, dm + \sum_{k=1}^{n-1}\int_A (\widehat{T}_k - \widehat{T}_{k-1}) S_{n-k} \, dm + a(n) - 1 \\
&= \sum_{k=1}^{n}\int_A \widehat{T}_k (S_{m-k} - S_{m-k-1}) \, dm + \sum_{k=1}^{n-1}\int_A \widehat{T}_k (S_{n-k} - S_{n-k-1}) \, dm + a(n) - 1 \\
&\quad + \int_A \widehat{T}_n S_{m-n-1} \, dm.
\end{aligned}$$

By (8) it follows that

$$\begin{aligned} ES_nS_m &\le \sum_{k=1}^{n} \bar{a}(k)(a(m-k) - a(m-k-1)) \\ &\quad + \sum_{k=1}^{n-1} \bar{a}(k)(a(n-k) - a(n-k-1)) + a(n) + \bar{a}(n)a(m-n-1) \\ &\le \sum_{k=0}^{n} (a(m-k) + a(n-k))(\bar{a}(k) - \bar{a}(k-1)). \end{aligned}$$

Lemma 2. *For every* $\varepsilon > 0$, $N \ge 2$, *and* $N \ge n \ge 1$

$$\sum_{m=n}^{N} \frac{1}{ma(m)} \le (\varepsilon + \Phi(N, \varepsilon) + (\log N)^{-1}) \frac{\log N}{a(n)}.$$

Proof. Fix $N \ge 2$. For $n \ge N^{1-\varepsilon}$ it follows that

$$\sum_{m=n}^{N} \frac{1}{ma(m)} \le \frac{1}{a(n)}(1 + \log N - \log N^{1-\varepsilon}) = \frac{1}{a(n)}(\varepsilon + (\log N)^{-1}) \log N.$$

If $n \le N^{1-\varepsilon}$, then

$$\begin{aligned} \sum_{m=n}^{N} \frac{1}{ma(m)} &\le \frac{1}{a(n)} \left(\Phi(N, \varepsilon) \sum_{m=[nN^{\varepsilon}]+1}^{N} \frac{1}{m} + \sum_{m=n}^{[nN^{\varepsilon}]} \frac{1}{m} \right) \\ &= [\varepsilon + \Phi(N, \varepsilon) + (\log N)^{-1}] \frac{\log N}{a(n)}. \end{aligned}$$

Lemma 3. *For* $N \ge 2$

$$\begin{aligned} (13) \qquad & \sum_{n=1}^{N} \frac{1}{na(n)} \sum_{m=n}^{N} \frac{1}{ma(m)} \sum_{k=0}^{n} l_k a(n-k) \\ & \le (1 + \sup_{n \ge 1} \beta_n))(\varepsilon + \Phi(N, \varepsilon) + (\log N)^{-1})(1 + \log N) \log N. \end{aligned}$$

Proof. Since $a(n-k) \le a(n)$ for $k \le n$ the left hand side in (13) is bounded by

$$\sum_{n=1}^{N} \sum_{m=n}^{N} \frac{\bar{a}(n)a(n)}{nma(n)a(m)} = \sum_{n=1}^{N} \sum_{m=n}^{N} \frac{(1+\beta_n)a(n)}{nma(m)}.$$

By Lemma 2 it follows that

$$\sum_{n=1}^{N} \sum_{m=n}^{N} \frac{(1+\beta_n)a(n)}{nma(m)} \le (1 + \sup_{n \ge 1} \beta_n)(\varepsilon + \Phi(N, \varepsilon) + (\log N)^{-1})(1 + \log N) \log N.$$

Lemma 4. *For any* $\varepsilon > 0$ *and* $N \ge 2$

$$2 \sum_{n=1}^{N} \frac{1}{na(n)} \sum_{m=n+1}^{N} \frac{1}{ma(m)} \sum_{k=0}^{n} l_k a(m-k) \le (1 + \log N)^2 + 2 \sum_{n=1}^{N} \frac{\beta_n}{n} \log N.$$

Proof. Write

$$2\sum_{n=1}^{N}\sum_{m=n+1}^{N}\frac{1}{nma(n)a(m)}\sum_{k=0}^{n}a(m-k)l_k$$
$$\leq 2\sum_{n=1}^{N}\sum_{m=n+1}^{N}\frac{1+\beta_n}{nm}=(1+\log N)^2+2\sum_{n=1}^{N}\frac{\beta_n}{n}\log N.$$

Lemma 5. *For any* $\varepsilon>0$ *and* $N\geq 2$

$$\sum_{k=1}^{N}\frac{\beta_k}{k}\leq(\gamma(1)\varepsilon+\gamma(N^{\varepsilon}))(1+\log N),$$

where

$$\gamma(N)=\sup\{\beta_k:\ N\leq k\}.$$

Proof. Obvious.

Proposition 3. *For every* $\varepsilon>0$ *and* $N\geq 2$

$$\operatorname{Var}\left(\frac{1}{\log N}T_N\right)\leq\left[2\gamma(N^{\varepsilon})+(5+4\gamma(1))\left(\varepsilon+\Phi(N,\varepsilon)+(\log N)^{-1}\right)\right]\left(\frac{1+\log N}{\log N}\right).$$

Proof. Since $E(T_N)=\log N$ the statement follows immediately from Lemmas 1, 3, 4, and 5.

Proof of Theorem 1. By Proposition 3 and Chebychev's inequality we obtain that

$$\frac{1}{\log N}\sum_{n=1}^{N}\frac{1}{na(n)}S_n\ \to 1 \tag{14}$$

in probability with respect to m restricted to the set A.

The theorem then follows from general arguments. The convergence of (14) in probability on A is equivalent to the fact that every subsequence contains a further subsequence N_j so that (14) converges along this subsequence a.s. on A. But the set of convergence is T-invariant as is the limit, and so by ergodicity this convergence is to 1 a.s. with respect to m, and hence the convergence in (14) is in measure on any set of finite measure. In order to obtain the statement for arbitrary $f\in L_1(m)$ apply Hopf's Ergodic Theorem to every a.s. convergent subsequence.

Proof of Theorem 2. The proof is similar to that one of Theorem 1.

Under the assumptions of the theorem

$$E(T_N-E(T_N))^2=O((\log N)^{2-\delta}),$$

where $\delta=\min(\varepsilon,\gamma)$. This follows from Proposition 3. Choose r so large that $r\delta>1$. By the Borel-Cantelli Lemma it follows that

$$\lim_{p\to\infty}\frac{1}{p^r\log 2}T_{2^{p^r}}\ =\ 1$$

almost surely. The theorem then follows from the monotonicity of T_N and the asymptotic equivalence $\lim_{p\to\infty}p^r(p+1)^{-r}=1$.

Proof of Corollary 1. If T is a Markov shift, then we may take $A = \{(x_i)_{i\geq 1} : x_1 = a\}$ where a denotes some fixed state. It is well known that the corresponding $\beta_n = 0$, where β_n is defined by (10).

3. Chung-Erdös averages for inner functions

Let $T : (\mathbb{R}^2)^+ \to (\mathbb{R}^2)^+$ be an analytic endomorphism of the upper half-plane $(\mathbb{R}^2)^+$. It is well known that T has a representation

$$T(z) = \alpha z + \beta + \int_{\mathbb{R}} \frac{1+tz}{t-z}\,\mu(dt), \tag{15}$$

where $\alpha \geq 0$, $\beta \in \mathbb{R}$, and μ denotes some positive measure on the real line. T is called an inner function if for a.e. $x \in \mathbb{R}$

$$\lim_{y\to 0} T(x+iy) =: T(x) \in \mathbb{R},$$

i.e., μ is singular with respect to the one-dimensional Lebesgue measure. An inner function is called odd if $T(-x) = -T(x)$ $(x \in \mathbb{R})$, or equivalently: $\beta = 0$ and μ is symmetric, i.e., $\mu(A) = \mu(-A)$ for $A \subset \mathbb{R}$, see [1, §2]. In this section we shall assume that T is odd and we shall consider the dynamical system defined by T on the real line $\mathbb{R}$. Moreover, we shall assume that $\alpha = 1$, since in this case the Lebesgue measure λ on $\mathbb{R}$ is an invariant measure (see [9]).

Each point $z = a + ib \in (\mathbb{R}^2)^+$ defines a Cauchy density function

$$\phi_z(t) = \frac{b}{\pi[(x-a)^2 + b^2]}. \tag{16}$$

Define

$$a(k) = \sum_{j=1}^{k} u_j, \tag{17}$$

where

$$u_j = \int_{\mathbb{R}} \phi_i \circ T^j \cdot \phi_i \, d\lambda. \tag{18}$$

In the subsequent proof we shall make use of the following fact about the relation of T and ϕ_z.

Lemma 6 [9]. *Denote by P_z the Cauchy distribution with density ϕ_z. Then*

$$P_z \circ T^{-1} = P_{T(z)} \tag{19}$$

and

$$\widehat{T}\phi_z = \phi_{T(z)}. \tag{20}$$

Using Lemma 6, it is shown in Corollary 3.5 in [1] that T is conservative if and only if $a(n) \to \infty$ as $n \to \infty$ and in this case, T is pointwise dual ergodic with return sequence $a(n)$. The main result in this section is

Theorem 3. *Let T be a conservative odd inner function of the upper half-plane restricted to the real line. Then*

$$\lim_{N\to\infty} \frac{1}{\log a(N)} \sum_{k=1}^{N} \frac{1}{a(k)} \phi_i \circ T^k = 1 \tag{21}$$

and for every function $f \in L_1^+(\lambda)$,

$$\lim_{N\to\infty} \frac{1}{\log a(N)} \sum_{k=1}^{N} \frac{1}{a(k)} f \circ T^k = \int_{\mathbb{R}} f\, d\lambda \tag{22}$$

almost surely.

Proof. It suffices to prove the statement (21), as (22) follows from (21), by Proposition 2 and Hopf's Ergodic Theorem.

Next observe that it suffices to show (21) P_i-almost surely, since λ and the Cauchy distribution P_i are equivalent.

We denote by E the expectation with respect to the measure P_i and by

$$T_N = \sum_{n=1}^{N} \frac{1}{a(n)} \phi_i \circ T^n.$$

Then

$$E(T_N^2) = \sum_{n=1}^{N} \frac{1}{a(n)^2} \int \phi_i^2 \circ T^n \cdot \phi_i\, d\lambda + 2 \sum_{n=1}^{N-1} \sum_{m=n+1}^{N} \frac{1}{a(n)a(m)} \int \phi_i \circ T^n \cdot \phi_i \circ T^m \cdot \phi_i\, d\lambda$$

and

$$(E(T_N))^2 = \sum_{n=1}^{N} \frac{u_n^2}{a(n)^2} + \sum_{n=1}^{N-1} \sum_{m=n+1}^{N} \frac{u_n u_m}{a(n)a(m)}.$$

Note that $T(i\mathbb{R}_+) \subset i\mathbb{R}_+$. Therefore we can define c_n by $T^n(i) = c_n i$. It is not hard to see that $c_n \uparrow \infty$ and indeed that $c_n >> \sqrt{n}$ (see [1]). Then by (19) and (20)

$$\begin{aligned} \int \phi_i \cdot \phi_i \circ T^n \cdot \phi_i \circ T^{n+k}\, d\lambda &= \int \phi_{T^n(i)} \cdot \phi_i \cdot \phi_i \circ T^k\, d\lambda \\ &= \int \widehat{T}^k (\phi_i \cdot \phi_{c_n i}) \cdot \phi_i\, d\lambda. \end{aligned} \tag{23}$$

For any $b \in \mathbb{R}_+$ we have

$$\begin{aligned} \pi^2 \phi_i \cdot \phi_{bi}(x) &= \frac{b}{[x^2+1][x^2+b^2]} = \frac{b}{[x^2+1][b^2-1]} - \frac{b}{[x^2+b^2][b^2-1]} \\ &= \frac{b\pi}{b^2-1}\phi_i - \frac{\pi}{b^2-1}\phi_{bi} \end{aligned}$$

and

$$\pi^2 \int \phi_i \cdot \phi_{bi}\, d\lambda = \frac{\pi b}{b^2-1} - \frac{\pi}{b^2-1} = \frac{\pi}{b+1}. \tag{24}$$

By (20)

$$\widehat{T}^k(\phi_i \cdot \phi_{c_n i}) = \widehat{T}^k\left(\frac{c_n}{\pi(c_n^2-1)}\phi_i - \frac{1}{\pi(c_n^2-1)}\phi_{c_n i}\right)$$

$$= \frac{c_n}{\pi(c_n^2-1)}\phi_{c_k i} - \frac{1}{\pi(c_n^2-1)}\phi_{c_{n+k} i} \tag{25}$$

and integrating with respect to P_i yields (use (23), (24), and (25))

$$\begin{aligned}\int \phi_i \cdot \phi_i \circ T^n \cdot \phi_i \circ T^{n+k}\, d\lambda &= \int \widehat{T}^k(\phi_i \cdot \phi_{c_n i}) \cdot \phi_i \, d\lambda \\ &= \frac{1}{\pi(c_n^2-1)}\left[c_n \int \phi_i \cdot \phi_{c_k i}\, d\lambda - \int \phi_i \cdot \phi_{c_{n+k} i}\, d\lambda\right] \\ &= \frac{1}{\pi^2(c_n^2-1)}\left[\frac{c_n}{c_k+1} - \frac{1}{c_{n+k}+1}\right].\end{aligned}$$

It follows from this and (24) that

$$\begin{aligned} &E(T_N^2) - (E(T_N))^2 \\ &= \sum_{n=1}^{N} \frac{1}{a(n)^2}\left[\int \phi_i^2 \cdot \phi_{c_n i}\, d\lambda - \frac{1}{\pi^2(c_n+1)^2}\right] + 2\sum_{n=1}^{N-1}\frac{1}{\pi^2 a(n)}\mathscr{D}(N, n)\end{aligned} \tag{26}$$

where

$$\begin{aligned}\mathscr{D}(N, n) = &\sum_{k=1}^{N-n} \frac{1}{a(n+k)(c_n^2-1)}\left(\frac{c_n}{c_k+1} - \frac{1}{c_{n+k}+1}\right) \\ &- \sum_{k=n+1}^{N} \frac{1}{a(k)(c_n+1)(c_k+1)}.\end{aligned}$$

Now

$$\sum_{n=1}^{N} \frac{1}{a(n)^2} \int \phi_i^2 \cdot \phi_{c_n i}\, d\lambda \le \sum_{n=1}^{N} \frac{u_n}{\pi a(n)^2} = O(\log a(N))$$

and

$$\begin{aligned}&\sum_{k=n+1}^{N-n} \frac{1}{a(n+k)(c_n^2-1)}\left(\frac{c_n}{c_k+1} - \frac{1}{c_{n+k}+1}\right) - \frac{1}{a(k)(c_n+1)(c_k+1)} \\ &\le \sum_{k=n+1}^{N-n} \frac{1}{a(k)(c_k+1)}\left(\frac{c_n-1}{c_n^2-1} - \frac{1}{c_n+1}\right) \\ &\quad + \sum_{k=n+1}^{N-n} \frac{1}{a(n+k)(c_n^2-1)}\left(\frac{1}{c_k+1} - \frac{1}{c_{n+k}+1}\right) \\ &= O\left(\sum_{k=n+1}^{N-n} \frac{1}{a(k)(c_n+1)^2(c_k+1)}\right) \\ &= O\left(\sum_{k=n+1}^{N-n} \frac{u_n^2 u_k}{a(k)}\right) = O(u_n^2 \log a(N)).\end{aligned}$$

Moreover

$$\sum_{k=1}^{n} \frac{1}{a(n+k)(c_n^2-1)}\left[\frac{c_n}{c_k+1}-\frac{1}{c_{n+k}+1}\right] = O\left(\sum_{k=1}^{n} \frac{u_n u_k}{a(n+k)}\right)$$
$$= O\left(\frac{u_n}{a(n)}\sum_{k=1}^{n} u_k\right) = O(u_n).$$

Using these estimates in (26) we arrive at

$$E(T_N)^2 - (E(T_N))^2 = O\Big(\big(1+\sum_{n=1}^{N-1}\frac{u_n^2}{a(n)}\big)\log a(N)\Big) = O(\log a(N)),$$

since $\sum_{n\geq 1} u_n^2(a(n))^{-1} < \infty$. This is because $c_n \geq d\sqrt{n}$ for some constant $d > 0$, and $u_n \downarrow$, hence $u_n \leq n^{-1}a(n)$ and

$$\sum_{k=1}^{\infty}\frac{u_k^2}{a(k)} \leq \frac{1}{d}\sum_{k=1}^{\infty} k^{-3/2} < \infty.$$

It follows that

$$\operatorname{Var}\Big(\sum_{k=1}^{N}\frac{1}{a(k)}\phi_i\circ T^k\Big) = O(\log a(N)).$$

Choose a sequence N_k such that $\sum_{k\geq 1}(\log a(N_k))^{-1} < \infty$ and

$$\frac{\log a(N_k)}{\log a(N_{k+1})} \to 1,$$

e.g., choose $N_k = \inf\{m : a(m) > \exp(k^2)\}$. Then, by the Borel-Cantelli Lemma, for any $\eta > 0$,

$$\sum_{k=1}^{\infty} P_i(\{x : T_{N_k} - E(T_{N_k}) \geq \eta\log a(N_k)\}) = O\left(\eta^{-2}\sum_{k=1}^{\infty}\frac{1}{\log a(N_k)}\right) < \infty.$$

Hence $\lim_{k\to\infty} T_{N_k}/\log a(N_k) = 1$ a.s. and therefore we obtain

$$\lim_{N\to\infty}\frac{T_N}{\log a(N)} = 1 \qquad \text{a.s.}$$

as in the proof of Theorem 2.

Remark 3. If $a(n)$ is regularly varying with index > 0, then we also have that

$$\lim_{N\to\infty}\frac{1}{\log N}\sum_{n=1}^{N}\frac{1}{na(n)}S_n f = \int_{\mathbf{R}} f\,d\lambda \qquad \text{a.s.}$$

by Proposition 1.

We show finally that there exist odd inner functions with $a(n)$ any regularly varying sequence of index in $(0, 1/2)$. This result extends Lemma 3.7 in [1].

Proposition 4. *Suppose* $0 < \gamma < 1/2$ *and that* $b(n)$ *is regularly varying with index* γ *as* $n \to \infty$. *Then there is an odd inner function*

$$T(x) = x + \int_{\mathbf{R}}\frac{1+tx}{t-x}\,d\mu(t)$$

such that $a(n) \sim b(n)$ *as* $n \to \infty$.

Proof. Let

$$T(x) = x + \int_{\mathbb{R}} \frac{1+tx}{t-x}\, d\mu(t)$$

be an odd inner function where the symmetric measure μ has a tail distribution

$$c_\mu(b) = \mu(\{t : |t| \geq b\}),$$

which is regularly varying at infinity with index $-\alpha$, where $\alpha \in (1, 2)$. Let $T^n(i) = b_n i$. By (15), the definition of T, we obtain that

$$b_{n+1} = b_n(1 + F(b_n)) \qquad (n \geq 1),$$

where

$$F(b) = \int_{-\infty}^{\infty} \frac{t^2+1}{t^2+b^2}\, \mu(dt).$$

It can be shown as in the proof of Lemma 3.7 in [1] that

$$F(b) = b^{-2} + 2(1 - b^{-2}) \int_0^\infty c_\mu(bz) \frac{z}{(z^2+1)^2}\, dz \sim d_\alpha c_\mu(b),$$

where

$$0 < d_\alpha = 2\int_0^\infty \frac{z^{1-\alpha}}{(z^2+1)^2}\, dz < \infty.$$

Hence $F(b)$ is also regularly varying at infinity with index $-\alpha$.

Consequently $C(b) = (F(b))^{-1}$ is regularly varying with index α and and satisfies

$$C(b_{n+1}) - C(b_n) \to \alpha \quad \text{as } n \to \infty.$$

Therefore $C(b_n) \sim n\alpha$ and

$$b_n \sim C^{-1}(n\alpha) \sim \alpha^{1/\alpha} C^{-1}(n)$$

as $n \to \infty$, and is α^{-1}-regularly varying.

Since $1 < \alpha < 2$, $\sum_{n \geq 1} b_n^{-1} = \infty$, whence T is conservative and the return sequence satisfies (by (17), (18), and (24))

$$a(n) \sim \frac{1}{\pi} \sum_{k=1}^{n} \frac{1}{b_k} \sim \frac{n}{b_n \pi(1-\alpha^{-1})} \sim \frac{1}{\pi(1-\alpha^{-1})\alpha^{1/\alpha}} \frac{n}{C^{-1}(n)}.$$

In order to obtain the proposition from the last statement, let $0 < \gamma < 1/2$ and a γ-regularly varying sequence $b(n)$ be given. Define $\alpha = \frac{1}{1-\gamma} \in (1, 2)$, and set

$$\psi(x) = \frac{1}{\pi(1-\alpha^{-1})\alpha^{1/\alpha}} \frac{x}{b(x)}.$$

Let μ be a symmetric measure on $\mathbb{R}$, singular with respect to Lebesgue measure, such that

$$c_\mu(b) \sim \frac{1}{d_\alpha \psi^{-1}(b)}$$

as $b \to \infty$. From the above, it follows that if

$$T(x) = x + \int_{\mathbb{R}} \frac{1+tx}{t-x}\, d\mu(t)$$

then the associated sequence $a(n)$ defined by (17) satisfies $a(n) \sim b(n)$.

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