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NOTE ON COUNTEREXAMPLES IN STRONG UNIQUE CONTINUATION PROBLEMS

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ABSTRACT. There are smooth functions on $\mathbb{R}^d$ vanishing to infinite order at a point and satisfying the differential inequality $|\Delta u| \le V|u|$ with $V \in$ weak $L^{d/2}$, and with $V \in L^1$ if $d = 2$.

INTRODUCTION

A differential equation or inequality has the strong unique continuation property (SUCP) if every C^∞ solution vanishing to infinite order at a point vanishes identically. Jerison and Kenig [5] proved that in $\mathbb{R}^d$ with $d \ge 3$, $V \in L^{d/2}_{\text{loc}}$ implies

$$(1) \qquad |\Delta u| \le V|u|$$

has the strong unique continuation property. The exponent $d/2$ is sharp ($u = \exp(-|x|^{-\varepsilon})$), but the $L^{d/2}$ condition can be replaced with certain other conditions having the same scale [2, 8]. We're concerned here with a result of Stein [8] that (1) has the SUCP if V is weak $L^{d/2}_{\text{loc}}$ with small norm, i.e. if

$$\sup_{a \in \mathbb{R}^d} \limsup_{\varepsilon \to 0} \sup_{\lambda} \lambda^{d/2} |\{x \colon |x - a| < \varepsilon \text{ and } V(x) > \lambda\}|$$

is sufficiently small. We will show that in this result of Stein, the "sufficiently small" condition is needed.

Theorem 1. *For $d \ge 3$ there is $u \colon \mathbb{R}^d \to \mathbb{R}$, smooth and not identically zero, and vanishing to infinite order at the origin, such that $|\Delta u| \le V|u|$ with $V \in$ weak $L^{d/2}$, i.e.*

$$\sup_{\lambda > 0} \lambda^{d/2} |\{x \in \mathbb{R}^d \colon V(x) > \lambda\}| < \infty.$$

It turns out that the $\mathbb{R}^2$ version of the same construction disproves the $\mathbb{R}^2$ analogue of the Jerison and Kenig result.

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Theorem 2. *There is $u\colon \mathbb{R}^2 \to \mathbb{R}$ smooth and not identically zero, vanishing to infinite order at the origin and such that $|\Delta u| \le V|u|$ with $V \in L^1$.*

Remarks. (1) The examples have to be a bit more complicated than $e^{-|x|^{-\varepsilon}}$; e.g., it can be shown that $|\Delta u| \le V|u|$ has the SUCP if V is weak $L^{d/2}$ and radial.

(2) There are some other open questions about the Laplacian and the SUCP, e.g., whether the SUCP holds for $|\Delta u| \le V|u|$ with "Kato-class" potentials V (see e.g. [2]) and whether it holds for $|\Delta u| \le V|\nabla u|$ with $V \in L^d$ (see [1, 6]). Our construction does not answer those questions. See the remark at the end of the paper.

Proof of the theorems

Let $\mathbf{e} \in \mathbb{R}^d$ be a fixed unit vector and let $\Gamma(x) = |x-\mathbf{e}|^{-(d-2)}$ if $d \ge 3$, $\Gamma(x) = \log\frac{1}{|x-\mathbf{e}|}$ if $d = 2$. Let p_{n-1} be the degree $n-1$ Taylor polynomial of Γ at the origin and let $f_n = \Gamma - p_{n-1}$.

Note that the functions f_n already "explain" the difference between small and large weak $L^{d/2}$ norms. Let $\overline{f}_n$ be obtained from f_n by smoothing off on a disc D of radius $\varepsilon \ll \frac{1}{n}$ centered at $\mathbf{e}$. Then $\overline{f}_n$ has the following properties:

(i) $\overline{f}_n = \mathcal{O}(|x|^{n-1})$ at ∞;
(ii) $\overline{f}_n = \mathcal{O}(|x|^n)$ at 0;
(iii) $\Delta\overline{f}_n \in C_0^\infty(\mathbb{R}^d/\{0\})$.

Moreover, $\Delta\overline{f}_n/\overline{f}_n$ is essentially ε^{-2} on D and zero otherwise, and therefore belongs to $L^{d/2}$ with norm independent of n; on the other hand, the Carleman inequality of Stein [8] implies a lower bound on the weak $L^{d/2}$ norm of any function satisfying (i)–(iii).

Going back to the proof, we have the following "gluing lemma."

Lemma. *Suppose $n \in \mathbb{Z}^+$, $\varepsilon > 0$. If r is large enough and $s > 0$ is small enough, then there is a smooth function $g = g^{rs}_{n\varepsilon}\colon \mathbb{R}^d/\{r\mathbf{e}, s\mathbf{e}\} \to \mathbb{R}$ such that*

$$\begin{aligned} g(x) &= r^n f_n(r^{-1}x) && \text{when } |x| > 2, \\ g(x) &= -s^n f_{n+1}(s^{-1}x) && \text{when } |x| < \tfrac{1}{2}, \\ |\Delta g| &< \varepsilon|g| && \text{when } \tfrac{1}{2} \le |x| \le 2. \end{aligned}$$

Moreover g satisfies the following estimates when $|\alpha| < n$, $(1-\frac{1}{n+2})s \le |x| \le (1-\frac{1}{n+1})r$:

$$|D^\alpha g| \le \begin{cases} C_{n\alpha}|x|^{n-|\alpha|}, & \text{if } |\frac{x}{s}-\mathbf{e}| > \frac{1}{n+2} \\ C_\alpha s^{n-|\alpha|}|\frac{x}{s}-\mathbf{e}|^{-(d-2+|\alpha|)}, & \text{if } |\frac{x}{s}-\mathbf{e}| < \frac{1}{n+2}. \end{cases}$$

Here we replace $|\frac{x}{s}-\mathbf{e}|^{-(d-2+|\alpha|)}$ by $\log(1/|\frac{x}{s}-\mathbf{e}|)$ if $d-2 = |\alpha| = 0$.

Proof. Let Z_k be the term of homogeneity k in the expansion of Γ at zero. A standard estimate (e.g. [9], p. 167, Equation 7.33.6) is then that $|Z_k(x)| \le Ck^{d-3}|x|^k$, and considering Z_k as a harmonic function on a ball of radius $\frac{|x|}{k}$, it follows that $|D^\alpha Z_k(x)| \le C_\alpha k^{d-3+|\alpha|}|x|^{k-|\alpha|}$. The expansion of $r^n f_n(r^{-1}x)$ at zero is

$$\sum_{k\ge n} r^{n-k} Z_k(x). \tag{2}$$

Choosing r large, we can make $r^n f_n(r^{-1}x) - Z_n(x)$ and its first four derivatives arbitrarily small on $\frac{1}{4} < |x| < 4$. Also

$$(3) \qquad -s^n f_{n+1}(s^{-1}x) = -s^n|s^{-1}x - \mathbf{e}|^{-(d-2)} + \sum_{k=0}^{n} s^{n-k} Z_k(x).$$

If s is small, $-s^n f_{n+1}(s^{-1}x) - Z_n(x)$, and its first four derivatives will be small when $\frac{1}{4} < |x| < 4$. Now let Ψ be C^∞, with $\Psi = 1$ when $|x| < \frac{1}{2}$, $\Psi = 0$ when $|x| > 2$, and

$$\tilde{g}(x) = r^n f_n(r^{-1}x) + \Psi(x)(-s^n f_{n+1}(s^{-1}x) - r^n f_n(r^{-1}x)).$$

Then $\tilde{g}$ satisfies the first two requirements of the lemma and is C^4 close to Z_n on $\frac{1}{4} < |x| < 4$. Since $Z_n(\frac{x}{|x|})$ is a solution of a second order ODE, we know that ∇Z_n vanishes only at zero. So we have a lower bound on $|\nabla \tilde{g}|$ for $\frac{1}{4} < |x| < 4$, provided that r, s^{-1} are large. Let M be the zero set of $\tilde{g}$ and $\delta(x) = \operatorname{dist}(x, M)$. For suitable ρ_0, δ is smooth on $\{x\colon \delta(x) < \rho_0\}$ (tubular neighborhood theorem) with bounds on its derivatives, and this is uniform in r, s as r, $s^{-1} \to \infty$. Let $\chi\colon R \to \mathbb{R}$ be smooth with $\chi(t) = 1$ when $|t| < \frac{1}{2}\rho_0$, $\chi(t) = 0$ when $|t| < \rho_0$, and let

$$g = \tilde{g} - \tfrac{1}{2}\delta^2 \chi \circ \delta \Delta \tilde{g}.$$

If we differentiate g by the product rule, we obtain

$$\Delta g = \left(\Delta \tilde{g} - \Delta\left(\frac{1}{2}\delta^2\right) \chi \circ \delta \Delta \tilde{g}\right) - 2\nabla\left(\tfrac{1}{2}\delta^2\right) \cdot \nabla(\chi \circ \delta \Delta \tilde{g}) - \tfrac{1}{2}\delta^2 \Delta(\chi \circ \delta \Delta \tilde{g}).$$

Since $\Delta(\frac{1}{2}\delta^2) = 1$ on M, it is clear that all three terms are

$$(4) \qquad \leq C\delta \|\Delta \tilde{g}\|_{C^2},$$

with C independent of r and s. On the other hand, since $|\nabla \tilde{g}|$ is bounded away from zero and $|\nabla\nabla \tilde{g}|$ is bounded, we know that $|\tilde{g}| \geq C^{-1}\delta$ when $\frac{1}{4} < |x| < 4$. So

$$(5) \qquad |g| > C^{-1}\delta - \frac{1}{2}\|\Delta \tilde{g}\|_\infty \chi \circ \delta \delta^2.$$

Since Z_n is harmonic, we make $\|\Delta \tilde{g}\|_{C^2}$ arbitrarily small by taking r, s^{-1} large. Then (5) implies $|g| > \frac{1}{2}C^{-1}\delta$ when $\frac{1}{4} < |x| < 4$, so (4) becomes

$$|\Delta g| \leq C\|\Delta \tilde{g}\|_{C^2}|g| \leq \varepsilon |g|.$$

When $\frac{1}{2} < |x| < 2$, the last statement of the lemma follows from the construction—all that is being claimed is that the derivative bounds are independent of r and s as r, $s^{-1} \to \infty$. When $|x| > 2$ or $|x| < \frac{1}{2}$, the last statement follows from (2) or (3) respectively, using $|D^\alpha Z_k| \leq C_\alpha k^{d-3+|\alpha|}|x|^{k-|\alpha|}$. $\square$

We now smooth off the singularity of $g^{rs}_{n\varepsilon}$ at $s\mathbf{e}$. The estimate $|Z_k| < Ck^{d-3}|x|^k$ implies that $|f_n(x)| \geq \frac{1}{2}|x - \mathbf{e}|^{-(d-2)}(\frac{1}{2}\log\frac{1}{|x-\mathbf{e}|}$ if $d = 2$), provided that $(n+1)|x - \mathbf{e}|$ is less than a suitable constant $b < 1$. Moreover, for $\rho < \frac{b}{n+1}$ the variation of f_n on the set $\{x\colon \frac{1}{2}\rho < |x - \mathbf{e}| < \rho\}$ is $\leq C\rho^{2-d}$ for all d (i.e. $\leq C$ when $d = 2$; this follows since $|\nabla Z_k| \leq C|x|^k$). Let Ψ be smooth with $0 \le \alpha \le 1$ and $\psi(x) = 1$ when $|x - \mathbf{e}| < \frac{1}{2}$, $\Psi(x) = 0$ when $|x - \mathbf{e}| > 1$. For $\rho < \frac{b}{n+1}$ choose a point x_0 with $|\frac{x_0}{s} - \mathbf{e}| = \rho$ and define

$$g^{rs}_{n\varepsilon\rho}(x) = \dot{g}^{rs}_{r\varepsilon}(x) + \Psi\left(\rho^{-1}\left(\frac{x}{s} - \mathbf{e}\right)\right)\left(g^{rs}_{n\varepsilon}(x_0) - g^{rs}_{n\varepsilon}(x)\right);$$

if $d = 2$, use $\log\frac{1}{\rho}$ instead of $\rho^{-(d-2)}$. Then $C^{-1}s^n\rho^{-(d-2)} \le |g^{rs}_{n\varepsilon\rho}| \le Cs^n\rho^{-(d-2)}$ for $|\frac{x}{s} - \mathbf{e}| < \rho$ if $d \ge 3$, and $C^{-1}s^n\log\frac{1}{\rho} \le |g^{rs}_{n\varepsilon\rho}| \le Cs^n\log\frac{1}{\rho}$ for $|\frac{x}{s} - \mathbf{e}| < \rho$ if $d = 2$. Using the lemma, the remark about the variation of f_{n+1} and the product rule, if $|\alpha| \ge 1$ and $|\frac{x}{s} - \mathbf{e}| < \rho$, then

$$|D^\alpha g^{rs}_{n\varepsilon\rho}| \le C_\alpha s^{n-|\alpha|}\rho^{-(d-2+|\alpha|)}.$$

This implies in particular that for $|\frac{x}{s} - \mathbf{e}| < \rho$,

$$|\Delta g^{rs}_{n\varepsilon\rho}| \le C(\rho s)^{-2}|g^{rs}_{n\varepsilon\rho}| \quad \text{if } d \ge 3, \tag{6}$$

$$|\Delta g^{rs}_{n\varepsilon\rho}| \le C(\rho s)^{-2}\left(\log\frac{1}{\rho}\right)^{-1}|g^{rs}_{n\varepsilon\rho}| \quad \text{if } d = 2. \tag{7}$$

Moreover, the previous bound and the lemma give

$$|D^\alpha g^{rs}_{n\varepsilon\rho}| \le A_{n\alpha\rho}|x|^{n-|\alpha|} \tag{8}$$

for $|\alpha| \ge 0$, $(1 - \frac{1}{n+2})s \le |x| \le (1 - \frac{1}{n+1})r$.

Now suppose $\{\rho_n\}_1^\infty$ is any sequence with $\rho_j < \frac{b}{j+1}$. Let $r_1 = 1$ and define $u(x)$ for $|x| \ge \frac{1}{2}$ by $u(x) = -f_1(x)$, smoothed off slightly when $|x - \mathbf{e}| < \rho_1$, so as to be C^∞. For $j \ge 2$ we will define recursively numbers η_j and r_j with $r_j < \eta_j < r_{j-1}$ and the values of the function u on the set $E_j = \{x\colon (1-\frac{1}{j+1})r_j \le |x| \le (1-\frac{1}{j})r_{j-1}\}$. Namely, if this has been done for $j \le n$, then make η_{n+1}/r_n and r_{n+1}/η_{n+1} small enough that a function $g^{r_n/\eta_{n+1}\, r_{n+1}/\eta_{n+1}}_{n2^{-n}}$ as in the lemma exists, and so that (with $A_{n\alpha\rho}$ as in (8))

$$A_{n+1\alpha\rho_{n+2}}(r_1\cdots r_n)^{-1}r_{n+1}^{n-|\alpha|} < 2^{-n}$$

for $|\alpha| < \frac{n}{2}$. Then define u on E_{n+1} by

$$u(x) = C_{n+1}g^{r_n/\eta_{n+1}\, r_{n+1}/\eta_{n+1}}_{n2^{-n}\rho_{n+1}}(\eta_{n+1}^{-1}x), \tag{9}$$

where the constant C_{n+1} is chosen to make the definition consistent when $|x| = (1 - \frac{1}{n+1})r_n$. This works out to $C_2 = -\eta_2$ and $C_{n+1}(\frac{r_n}{\eta_{n+1}})^n = -C_n(\frac{r_n}{\eta_n})^{n-1}$; i.e., $C_{n+1} = (-1)^n(r_1\cdots r_n)^{-1}\eta_{n+1}^n$. There are two conditions to check.

(i) u vanishes to infinite order at zero.

Substituting the value of C_n into (9) and using (8), on E_{n+1}

$$|D^\alpha u| \le A_{n\alpha\rho_{n+1}}(r_1\cdots r_n)^{-1}|x|^{n-\alpha} \le A_{n\alpha\rho_{n+1}}(r_1\cdots r_{n-1})^{-1}r_n^{n-|\alpha|-1} < 2^{-(n-1)}$$

for $|\alpha| < \frac{n}{2}$, by choice of r_n. Statement (i) follows.

(ii) If $\{\rho_n\}$ are chosen appropriately, then $\frac{\Delta u}{u} \in \text{weak}\, L^{d/2}$ if $d \ge 3$ and $\frac{\Delta u}{u} \in L^1$ if $d = 2$.

Δu vanishes except on $\bigcup_n(B_n \cup D_n)$ where $B_n = \{x\colon \frac{1}{2}\eta_n \le |x| \le 2\eta_n\}$

and $D_n = \{x : |\frac{x}{r_n} - \mathbf{e}| < \rho_n\}$. We have $|\Delta g^{rs}_{n2^{-n}}| \leq 2^{-n}|g^{rs}_{n2^{-n}}|$, and therefore $|\Delta u| \leq C\eta_n^{-2}2^{-n}|u|$ on B_n.
So

$$\int_{\cup B_n} \left(\frac{\Delta u}{u}\right)^{d/2} \leq C \sum 2^{-n \times d/2} < \infty .$$

If $d \geq 3$ we simply take $\rho_n = \frac{b}{n+1}$. Let $E(\lambda)$ be the distribution function of $\frac{\Delta u}{n} \restriction \bigcup_n D_n$. D_n is a disc of radius $\frac{b}{n+1} r_n$ and $|\Delta u| \leq C(\frac{b}{n+1} r_n)^{-2}|u|$ there by (6). So

$$E(\lambda) \leq C \sum_{C(\frac{b}{n+1} r_n)^{-2} > \lambda} \left(\frac{b}{n+1} r_n\right)^d \leq C\lambda^{-d/2},$$

since the r_n are decreasing at least geometrically.

If $d = 2$ we choose ρ_n so that $\sum (\log \frac{1}{\rho_n})^{-1} < \infty$. Then by (7),

$$\int_{\bigcup B_n} \left|\frac{\Delta u}{u}\right| \leq C \sum (\rho_n r_n)^2 (\rho_n r_n)^{-2} \log \frac{1}{\rho_n} < \infty. \quad \square$$

Remark. The reason it was possible to do such a simple construction is that the Carleman inequalities, which would prove the opposite of Theorems 1 and 2, fail for a very simple reason: the relevant Sobolev inequalities are already false. For example, in $\mathbb{R}^2$, convolution with Γ does not map L^1 to L^∞. This fact allowed us to do our construction very close to the diagonal (where $|x - r_n\mathbf{e}| < \frac{b}{n+1} r_n$) and avoid considering cancellations. In the problems mentioned in Remark (2) of the Introduction, the Carleman inequalities fail for a different reason, discovered by Jerison [4] (see also [1]), and concerning what happens in the region $|x - y| \simeq \frac{|y|}{\sqrt{n}}$ where the relevant kernels have oscillatory behavior. See e.g. [7] and [10] for various constructions of kernels.

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