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ORDERINGS WITH αTH JUMP DEGREE $\mathbf{0}^{(\alpha)}$

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ABSTRACT. This paper completes an investigation of "jumps" of orderings. The last few cases are given in the proof that for each recursive ordinal $\alpha \geq 1$ and for each Turing degree $\mathbf{d} \geq \mathbf{0}^{(\alpha)}$, there is a linear ordering $\mathbf{A}$ such that $\mathbf{d}$ is least among the αth jumps of degrees of (open diagrams of) isomorphic copies of $\mathbf{A}$, and for $\beta < \alpha$, the set of βth jumps of degrees of copies of $\mathbf{A}$ has no least element.

0. INTRODUCTION

The present paper continues the work on "jumps" of orderings in [K, AJK, AK, JS]. All structures considered here have universe ω and all languages are recursive. For any structure $\mathbf{A}$, the open diagram $D(\mathbf{A})$ can be thought of as a subset of ω. For simplicity, we may say that $\mathbf{A}$ is *recursive* (or *recursive in* X) if $D(\mathbf{A})$ is recursive (recursive in X). We write $\deg(X)$ for the Turing degree of a set $X \subseteq \omega$, and $\deg(\mathbf{A})$ for $\deg(D(\mathbf{A}))$. Then $\deg(\mathbf{A})$ provides some measure of the complexity of $\mathbf{A}$. This measure is not isomorphism invariant.

Jockusch suggested the following measures, which clearly are isomorphism invariant. For any structure $\mathbf{A}$ and any recursive ordinal α, $\mathbf{A}$ is said to have αth *jump degree* $\mathbf{d}$ if $\mathbf{d}$ is least among $\{\deg(\mathbf{B})^{(\alpha)} : \mathbf{B} \cong \mathbf{A}\}$. For some $\mathbf{A}$ and α, $\mathbf{A}$ fails to have αth jump degree. In fact, there are structures (such as the ordering ω_1^{CK}) that do not have αth jump degree for any recursive α (see [AJK]). It is easy to see that if $\beta < \alpha$, and $\mathbf{A}$ has βth jump degree, then $\mathbf{A}$ has αth jump degree. For each recursive ordinal α, there exist structures $\mathbf{A}$ such that $\mathbf{A}$ has αth jump degree and $\mathbf{A}$ does not have βth jump degree for any $\beta < \alpha$. When this occurs, $\mathbf{A}$ is said to have αth jump degree *sharply*.

Richter [Ri] showed that $\mathbf{0}$ is the only possible 0th jump degree for linear orderings, and in [K] it was shown that $\mathbf{0}'$ is the only possible 1st jump degree. In [AJK] it was shown that for each recursive ordinal $\alpha \geq 2$, and for each $\mathbf{d} > \mathbf{0}^{(\alpha)}$, there is an ordering that has αth jump degree $\mathbf{d}$ sharply. To complete the picture, we should say for which recursive ordinals α there are orderings that have αth jump degree $\mathbf{0}^{(\alpha)}$ sharply.

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In [JS] it was shown that there is an ordering having 1st jump degree $\mathbf{0}'$ sharply. Constructions in [AJK] yield orderings having αth jump degree $\mathbf{0}^{(\alpha)}$ sharply for odd finite $\alpha \geq 3$, and also for even infinite α. (This fact is not explicitly stated in the paper.) New constructions in [AK] yield orderings having αth jump degree $\mathbf{0}^{(\alpha)}$ sharply for even finite $\alpha \geq 4$, and for odd infinite α not the successor of a limit ordinal (as well as for odd finite $\alpha \geq 3$ and for even infinite nonlimit α). The remaining cases are

(1) $\alpha = 2$,
(2) α is the successor of a limit ordinal.

These two cases will be done in this paper.

All together, the constructions taken from [AJK] and [AK], and the new ones suffice to prove the following.

Theorem. *For each recursive ordinal $\alpha \geq 2$, and for each $\mathbf{d} \geq \mathbf{0}^{(\alpha)}$, there is an ordering having αth jump degree $\mathbf{d}$ sharply.*

The case $\alpha = 2$ is discussed in §1. Section 2 reviews the results of [AJK] and [AK]—handling the cases where α is finite and at least 3, or infinite and not the successor of a limit ordinal. Section 3 handles the case where α is the successor of a limit ordinal. Section 4 raises some questions.

1. The case $\alpha = 2$

Jockusch and Soare [JS] showed, by a clever permitting argument, that for any r.e. nonrecursive set C, there is an ordering $\mathbf{A}$ such that $\mathbf{A} \leq_T C$ and $\mathbf{A}$ has no recursive copy. If C is taken to be low, then $\mathbf{A}$ has 1st jump degree $\mathbf{0}'$ sharply. Our aim in this section is to show that there is an ordering $\mathbf{A}_2$ having 2nd jump degree $\mathbf{0}''$ sharply. We use the following relativized version of the result of Jockusch and Soare.

Lemma 1.1. *If C is r.e. in X but not recursive in X, then there is an ordering $\mathbf{A}$ such that $\mathbf{A}$ is recursive in C and no copy of $\mathbf{A}$ is recursive in X.*

Watnik [W] showed that for any ordering $\mathbf{A}$ and any set X, there is a copy of $Z \cdot \mathbf{A}$ recursive in X iff there is a copy of $\mathbf{A}$ recursive in X''. Ash, Jockusch, and Knight, unaware of Watnik's work, rediscovered this result and included it as Lemma 2.2 of [AJK]. The lemma below is similar except that Z is replaced by the ordering $\varphi = \eta + 2 + \eta$, and only one jump is needed to decode $\mathbf{A}$.

Lemma 1.2. *For any ordering $\mathbf{A}$ and any set X, $\varphi \cdot \mathbf{A}$ has a copy recursive in X iff $\mathbf{A}$ has a copy recursive in X'.*

Proof. First, suppose $\mathbf{B} \cong \varphi \cdot \mathbf{A}$, where $\mathbf{B} \leq_T X$. With an oracle for X', we can enumerate the pairs (b, c) in $\mathbf{B}$ such that c is the immediate successor of b. We enumerate the diagram of the desired $\mathbf{C} \cong \mathbf{A}$ by assigning elements of the universe of $\mathbf{C}$ to these pairs (as we come to them).

Now, suppose $\mathbf{C} \cong \mathbf{A}$, where $\mathbf{C} \leq_T X'$. We must produce $\mathbf{B} \leq_T X$ such that $\mathbf{B} \cong \varphi \cdot \mathbf{A}$. Recursive in X, we guess the ordering of finite sets in $\mathbf{C}$. At each stage, we will have determined, once and for all, the ordering of finitely many elements of $\mathbf{B}$. In addition, we will have tentatively located these in copies of φ (with the middle pair of elements designated) corresponding to elements of $\mathbf{C}$. At stage s, we first check our previous work, making sure that the guess on which it was based still looks correct. If it does, then we add a new copy of φ (and we designate a pair of new elements for the middle), corresponding to the first element of the universe of $\mathbf{C}$ that presently lacks one. If we find an error, we back up to the last stage $t < s$ for which the arrangement of φ's (corresponding to elements of $\mathbf{C}$) still seems correct. We remove from our picture the copies of φ added since stage t, incorporating their elements into the dense parts of adjacent copies of φ. To complete the work at stage s, we add new elements to each dense part of each existing copy of φ—one between each pair and one at each end. This construction clearly yields the desired $\mathbf{B} \cong \varphi \cdot \mathbf{A}$.

We can now produce the ordering $\mathbf{A}_2$ that was the aim of the section.

Proposition 1.3. *There is an ordering having* 2nd *jump degree* $\mathbf{0}''$ *sharply.*

Proof. Let C be r.e. in $0'$ such that $0' <_T C$ and $C' \leq_T 0''$. By Lemma 1.1 there is an ordering $\mathbf{A}$ such that $\mathbf{A}$ is recursive in C, and no copy of $\mathbf{A}$ is recursive in $0'$. Let $\mathbf{B} \cong \varphi \cdot \mathbf{A}$. Take X such that $X' \equiv_T C$. By Lemma 1.2 there is a copy of $\mathbf{B}$ recursive in X. Therefore, $\mathbf{B}$ has 2nd jump degree $\mathbf{0}''$. By remarks in the previous section, in order to show that for all $k < 2$, $\mathbf{B}$ does not have kth jump degree, it is enough to show that $\mathbf{B}$ does not have 1st jump degree $\mathbf{0}'$. If $\mathbf{B}$ has 1st jump degree $\mathbf{0}'$, there would be a copy of $\mathbf{B}$ of low degree. Then by Lemma 1.2, $\mathbf{A}$ would have a copy recursive in $\mathbf{0}'$, a contradiction. This completes the proof.

2. Cases obtained from known results

Our aim in this section is to indicate how it follows from results in [AJK] and [AK] that if α is a recursive ordinal, either finite and at least 3, or infinite and not the successor of a limit ordinal, then for each $\mathbf{d} \geq \mathbf{0}^{(\alpha)}$, there is an ordering $\mathbf{A}_\alpha(\mathbf{d})$ having αth jump degree $\mathbf{d}$ sharply. We begin with a definition. A *recursive ordering scheme* is an index e such that for any set $S \subseteq \omega$, W_e^S is the diagram of a linear ordering $\mathbf{A}_e(S)$. The following lemma describes some recursive ordering schemes.

Lemma 2.1. *For each recursive ordinal β such that either β is finite and at least 2, or β is infinite and not a limit ordinal, there is a recursive ordering scheme $e(\beta)$ such that for all sets S and X, $\mathbf{A}_{e(\beta)}(S)$ has a copy recursive in X iff S is r.e. in $X^{(\beta)}$.*

Lemma 2.3 of [AJK] yields the result for even finite β, and Lemmas 4.2 and 2.1 of [AJK] yield the result for odd infinite β. Lemmas 6.2 and 6.3 of [AK] extend the result to odd finite β and even nonlimit β. Or, we could combine Lemma 1.2 with the results from [AJK] to handle odd finite β and even nonlimit β.

The next lemma gives the existence of sets with some special recursion-theoretic properties.

Lemma 2.2. *For each recursive ordinal β, and each $\mathbf{d} \geq \mathbf{0}^{(\beta+1)}$, there exists S such that $S^{(\beta+1)} \equiv_T S \oplus 0^{(\beta+1)}$, $S^{(\beta+1)}$ has degree $\mathbf{d}$, and $\{X^{(\beta)}: S$ is r.e. in $X^{(\beta)}\}$ has no element of least degree.*

This is a combination of Lemmas 1.1 and 1.2 of [AJK].

From Lemmas 2.1 and 2.2, we obtain, in a uniform way, the desired orderings $\mathbf{A}_\alpha(\mathbf{d})$ for finite $\alpha \geq 3$, and for infinite α such that α is neither a limit ordinal nor the successor of one.

Proposition 2.3. *Let α be a recursive ordinal, either finite and at least 3, or infinite and neither a limit ordinal nor the successor of a limit ordinal. Then for each $\mathbf{d} \geq \mathbf{0}^{(\alpha)}$, there is an ordering $\mathbf{A}_\alpha(\mathbf{d})$ having αth jump degree $\mathbf{d}$ sharply.*

Proof. Let $\alpha = \beta + 1$. By Lemma 2.2, there exists S such that $S^{(\alpha)} \equiv_T S \oplus 0^{(\alpha)}$, $S^{(\alpha)}$ has degree $\mathbf{d}$, and the set $C = \{X^{(\beta)} : S \text{ is r.e. in } X^{(\beta)}\}$ has no element of least degree. Let $\mathbf{A}_\alpha(\mathbf{d}) = \mathbf{A}_{e(\beta)}(S)$, where $e(\beta)$ is as in Lemma 2.1. We now proceed exactly as in [AJK]. We may assume that $\mathbf{A}_\alpha(\mathbf{d})$ is recursive in S. If $\mathbf{B} \cong \mathbf{A}_\alpha(\mathbf{d})$, then we have $\mathbf{A}_\alpha(\mathbf{d})^{(\alpha)} \leq_T S^{(\alpha)} \leq_T S \oplus 0^{(\alpha)} \leq_T \mathbf{B}^{(\alpha)}$. Therefore, $\mathbf{A}_\alpha(\mathbf{d})$ has αth jump degree $\mathbf{d}$. By Lemma 2.1, $\mathbf{A}_\alpha(\mathbf{d})$ has a copy recursive in X iff S is r.e. in $X^{(\beta)}$. Then the fact that C has no element of least degree means that $\mathbf{A}_\alpha(\mathbf{d})$ cannot have βth jump degree.

In the case α is a limit ordinal (and also in the case α is the successor of a limit ordinal), we use a recursive ordering scheme coding a family of sets. If $S \subseteq \omega \times \omega$, we write S_m for the set $\{k : (m, k) \in S\}$. We say that S is an *enumeration* for the family consisting of the sets S_m. The lemma below describes a recursive ordering scheme in terms of a "fundamental sequence" for a limit ordinal. Each recursive ordinal has one or more notations in Kleene's system O. If α is a limit ordinal, then each notation picks out an increasing sequence of ordinals $(\alpha_n)_{n\in\omega}$—the *fundamental sequence*—that converges to α. (For more about O, see [Ro, pp. 208–210].) The lemma has four parts. Only part (a) is needed in this section, to produce $\mathbf{A}_\alpha(\mathbf{d})$ for recursive limit ordinals α. The other parts of the lemma will be used in §3.

Lemma 2.4. *Let α be a recursive limit ordinal, with a notation picking out the fundamental sequence $(\alpha_n)_{n\in\omega}$. There is a new fundamental sequence $(\beta_n)_{n\in\omega}$ picked out by a new notation for α (effectively determined from the original), and there is a recursive ordering scheme e which determines an ordering $\mathbf{A}_e(S)$ corresponding to each $S \subseteq \omega \times \omega$, such that*

(a) *for any S and X, $\mathbf{A}_e(S)$ has a copy recursive in X iff $S_n \leq_T X^{(\beta_n)}$ uniformly in n;*

(b) *for all S, $\mathbf{A}_e(S)$ has no dense subinterval;*

(c) *for each $\beta < \alpha$, there is some N such that if $S' \subseteq \omega \times \omega$ and for all $n < N$, $S_n = S'_n$, then $\mathbf{A}_e(S)$ and $\mathbf{A}_e(S')$ satisfy the same Σ_β sentences of $L_{\omega_1\omega}$;*

(d) *there exist recursive functions σ and τ such that for all S and X,*

  (i) *if a is an index for the diagram of a copy of $\mathbf{A}_e(S)$ relative to X, then for all n, $\varphi_{\sigma(a)}(n)$ is an index for S_n relative to $X^{(\beta_n)}$, and*

  (ii) *if b is a number such that for all n, $\varphi_b(n)$ is an index for S_n relative to $X^{(\beta_n)}$, then $\tau(b)$ is an index for the diagram of a copy of $\mathbf{A}_e(S)$ relative to X.*

Part (a) of Lemma 2.4 is Lemma 4.5 (3) of [AJK]. Parts (b) and (d) are both mentioned in [AJK]. Part (c) is not stated, although it could be arrived at by examining the orderings used in the proof. In order to make (c) plausible, and (b) and (d) as well, we display some specific ordering $\mathbf{A}_e(S)$. (These are not precisely the same ones as in [AJK].)

For F a countable family of orderings, the *shuffle* of F, denoted by $\sigma(F)$, consists of densely many copies of each of the orderings in F, with no first or last copy of anything.

Let $\mathbf{A}_e(S) = \sum_n((n+1) + Z \cdot \sigma(F_n))$, where F_n consists of all orderings of the following forms:

(1) $\eta + 1 + \omega^{\alpha_n} + (k+1) + \eta$, for $k \in S_n \oplus (\omega - S_n)$, and
(2) $\eta + 1 + \omega^{\alpha_n + 1} + (k+1) + \eta$, for $k \in \omega$.

Claim. For any sets S and X, $\mathbf{A}_e(S)$ has a copy recursive in X iff S_n is $\Delta^0_{2\alpha_n+4}$ relative to X uniformly in n.

By Lemmas 6.2 and 6.3 of [AK], $\sigma(F_n)$ has a copy recursive in X iff S_n is $\Delta^0_{2\alpha_n+2}$ relative to X, and by the result of Watnik mentioned in §1, $Z \cdot \sigma(F_n)$ has a copy recursive in X iff S_n is $\Delta^0_{2\alpha_n+4}$ relative to X. If $\mathbf{B} \cong \mathbf{A}_e(S)$, where $\mathbf{B} \leq_T X$, then there is a sequence of pairs (a_n, b_n), Δ^0_3 relative to $\mathbf{B}$, marking off the intervals of type $Z \cdot \sigma(F_n)$. It follows that S_n is $\Delta^0_{2\alpha_n+4}$ relative to X, uniformly in n. If S_n is $\Delta^0_{2\alpha_n+4}$ relative to X, uniformly in n, then there is a sequence $\mathbf{C}_n \cong \sigma(F_n)$, uniformly recursive in X. Combining these $\mathbf{C}_n$, we get a copy of $\mathbf{A}_e(S)$ recursive in X. This proves the claim.

The relation we have in mind here between the two fundamental sequences is that $1+\beta_n = 2\alpha_n+4$, and then the claim yields (a) for this recursive ordering scheme. By inspection, (b) is clear. By Propositions 5.2 and 5.3 of [AK], we get the fact that if $2\gamma \leq \alpha_N$ and for all $n < N$, $S_n = S'_n$, then $\mathbf{A}_e(S)$ and $\mathbf{A}_e(S')$ satisfy the same $\Sigma_{2\gamma}$ sentences of $L_{\omega_1\omega}$. This proves (c). We get (d) from the fact that everything involved in the proof of the claim is true uniformly.

The next lemma asserts the existence of an enumeration of a family of sets with special recursion-theoretic properties.

Lemma 2.5. *Let α be a recursive limit ordinal, with a fundamental sequence $(\beta_n)_{n\in\omega}$ picked out by a notation for α. Then for any $\mathbf{d} \geq \mathbf{0}^{(\alpha)}$, there exists $S \subseteq \omega \times \omega$ such that $S^{(\alpha)} = S \oplus 0^{(\alpha)}$, $S^{(\alpha)}$ has degree $\mathbf{d}$, and for all $\beta < \alpha$, $\{X^{(\beta)}: S_n \leq_T X^{(\beta_n)}$ uniformly in $n\}$.*

This follows from Lemma 1.4 of [AJK].

We can now produce $\mathbf{A}_\alpha(\mathbf{d})$ for any recursive limit ordinal α.

Proposition 2.6. *If α is a recursive limit ordinal and $\mathbf{d} \geq \mathbf{0}^{(\alpha)}$, then there is an ordering $\mathbf{A}_\alpha(\mathbf{d})$ having αth jump degree $\mathbf{d}$ sharply.*

Proof. For the given α, picking out the fundamental sequence $(\alpha_n)_{n\in\omega}$, let $(\beta_n)_{n\in\omega}$ and e be as in Lemma 2.4. Now, for α with this fundamental sequence $(\beta_n)_{n\in\omega}$, and for the given $\mathbf{d}$, let $S \subseteq \omega \times \omega$ be as in Lemma 2.5. It is easy to see that $\mathbf{A}_e(S)$ is the desired ordering.

3. Successors of limit ordinals

Our aim in this section is to show that for any recursive limit ordinal α and any $\mathbf{d} \geq \mathbf{0}^{(\alpha+1)}$, there is an ordering $\mathbf{A}_{\alpha+1}(\mathbf{d})$ having $(\alpha+1)$st jump degree $\mathbf{d}$ sharply. Here we need a new construction. For a given set $S \subseteq \omega$, we form a family of sets as follows. Let $(S^*)_n = (S \cap n) \oplus (\omega - (S \cap n))$, and let $S^* = \{(n, j): j \in (S^*)_n\}$. We can recover S from S^*, since $i \in S$ iff for all (or for some) $n > i$, $2i \in (S^*)_n$.

Let α, $(\beta_n)_{n\in\omega}$, and e be as in Lemma 2.4. We may assume that $\beta_0 > 2$. Let $P \subseteq \omega \times \omega$. For each $m \in \omega$, let $\rho(m) = \eta + 2 + \mathbf{A}_e((P_m)^*) + 2 + \eta$, and let $\pi(P) = \sigma\{\rho(m) \colon m \in \omega\}$. It follows from Lemma 2.4 that for each $\beta < \alpha$, there exists $N \in \omega$ such that if $Q \subseteq \omega \times \omega$ and the intersections with N of the sets enumerated by Q are the same as the intersections with N of the sets enumerated by P, then $\pi(P)$ and $\pi(Q)$ satisfy the same Σ_β sentences of $L_{\omega_1\omega}$.

For a countable family of sets $\mathscr{S} \subseteq P(\omega)$, let $\mathscr{E}(\mathscr{S})$ be the set of all enumerations. Let $C(\mathscr{S}) = \{X \colon (\exists R \in \mathscr{E}(\mathscr{S}))(R_k)^*_n \leq_T X^{(\beta_n)}$ uniformly in k and $n\}$.

Remark. If $X \in C(\mathscr{S})$, then all of the sets in $\mathscr{S}$ are recursive in $X^{(\alpha)}$.

Lemma 3.1. *Let $\mathscr{S}$ be a countable family of subsets of ω, and let $P \in \mathscr{E}(\mathscr{S})$. Then for $X \subseteq \omega$, $\pi(P)$ has a copy recursive in X iff $X \in C(\mathscr{S})$.*

Proof. Let $\mathbf{A} \cong \pi(P)$, where $\mathbf{A} \leq_T X$. Using an oracle for X'', we can form a list of the 6-tuples $\mathbf{a}_k$ from $\mathbf{A}$ of the form $(a_1, a_2, a_3, a_4, a_5, a_6)$, where $a_1 < a_2 < a_3 < a_4 < a_5 < a_6$, the intervals (a_1, a_2) and (a_5, a_6) have type η, and the intervals (a_2, a_3) and (a_4, a_5) are empty. For such tuple, the interval (a_3, a_4) must be a copy of $\mathbf{A}_e((P_m)^*)$ for some $m = m(k)$. Let R_k be the set $P_{m(k)}$ recovered (as described above) from $(P_{m(k)})^*$, and let $R = \{(k, i) \colon i \in P_{m(k)}\}$. Then $R \in \mathscr{E}(\mathscr{S})$. The fact that Lemma 2.4 holds uniformly implies that $(R_k)^*_n = (P_{m(k)})^*_n \leq_T X^{(\alpha_n)}$ uniformly in k and n. Therefore, $X \in C(\mathscr{S})$. This proves one direction. For the other direction, suppose we have $R \in \mathscr{E}(\mathscr{S})$, where $(R_k)^*_n \leq_T X^{(\alpha_n)}$ uniformly in k and n. Then, again using the fact that Lemma 2.4 holds uniformly, we have a sequence of structures $\mathbf{C}_k \cong \mathbf{A}_e((R_k)^*)$, uniformly recursive in X. From this, we get a copy of $\pi(P)$ recursive in X.

We wish to show that for sufficiently generic P, the ordering $\pi(P)$ does not have αth jump degree. This would follow from the lemma below. Since the statement of the lemma is recursion-theoretic, we believe that it ought to have a recursion-theoretic proof. However, the only proof we have found is model-theoretic. We appeal to Lemma 3.1 and establish directly that $\pi(P)$ does not have αth jump degree. Logically, we have no need for the lemma. Our reason for stating it is to call attention to it, and to encourage the reader to try to find a more natural proof.

Lemma 3.2. *Let α be a recursive limit ordinal, let $P \subseteq \omega\times\omega$ be $(\alpha+1)$-generic, and let $\mathscr{S}$ be the family of sets enumerated by P. Then $\{X^{(\alpha)} \colon X \in C(\mathscr{S})\}$ has no element of least degree.*

Proof. Suppose that $Y \leq_T X^{(\alpha)}$, for all $X \in C(\mathscr{S})$. We shall show that for all $X \in C(\mathscr{S})$, $X^{(\alpha)} \not\leq_T Y$. By Lemma 3.1, $Y \leq_T \mathbf{A}^{(\alpha)}$ for all $\mathbf{A} \cong \pi(P)$. Then by Theorem 5 of [AKMS] (which generalizes Theorem 1.4′ from [K]), if $Y \leq_T \mathbf{A}^{(\alpha)}$ for all $\mathbf{A} \cong \pi(P)$, then there is some finite tuple $\mathbf{a}$ in $\pi(P)$ such that Y is recursive in every enumeration of the recursive infinitary Σ_α-type of $\mathbf{a}$. Our plan is to produce a structure $\mathbf{B}$ with a tuple $\mathbf{b}$ realizing the same recursive infinitary Σ_α-type as $\mathbf{a}$, and such that $\mathbf{B} \leq_T P_0 \oplus P_1 \oplus \cdots \oplus P_k$ for some k. Suppose for the moment that we have $\mathbf{B}$ and $\mathbf{b}$. Recursively in $\mathbf{B}^{(\alpha)}$, we can enumerate the recursive infinitary Σ_α-type. Therefore, $Y \leq_T$

$(P_0 \oplus P_1 \oplus \cdots \oplus P_k)^{(\alpha)}$. By genericity, $P_{k+1} \not\leq_T (P_0 \oplus P_1 \oplus \cdots \oplus P_k)^{(\alpha)}$, so $P_{k+1} \not\leq_T Y$. By the remark before Lemma 3.1, $P_{k+1} \leq_T X^{(\alpha)}$ for all $X \in C(\mathscr{S})$. Therefore, $X^{(\alpha)} \not\leq_T Y$.

It remains to describe $\mathbf{B}$ and $\mathbf{b}$. Recall that $\pi(P)$ is a shuffle of structures $\rho(m) = \eta + 2 + \mathbf{A}_e((P_m)^*) + 2 + \eta$, for $m \in \omega$. Let $\mathbf{a} = \mathbf{a}_1 {}^\frown \mathbf{a}_2^\frown \cdots {}^\frown \mathbf{a}_r$, where $\mathbf{a}_i$ is the portion of $\mathbf{a}$ lying in a single copy of $\rho(m)$ for some $m = m(i)$, and $\mathbf{a}_i$ comes before $\mathbf{a}_{i+1}$. Let Q be an enumeration of all finite sets, together with the finite collection of sets $P_{m(i)}$. Let $\mathbf{B} = \pi(Q)$, and let $\mathbf{b} = \mathbf{b}_1 {}^\frown \mathbf{b}_2^\frown \cdots {}^\frown \mathbf{b}_r$, where $\mathbf{b}_i$ sits just as $\mathbf{a}_i$ does in a copy of $\rho(m(i))$, and $\mathbf{b}_i$ comes before $\mathbf{b}_{i+1}$. We must show that $\mathbf{a}$ and $\mathbf{b}$ satisfy at least the same recursive infinitary Σ_α formulas.

Consider in $\pi(P)$ the interval to the left of the copy of $\rho(m(1))$ that contains $\mathbf{a}_1$, or the interval between the copy of $\rho(m(i))$ containing $\mathbf{a}_i$ and the copy of $\rho(m(i+1))$ containing $\mathbf{a}_{i+1}$, or the interval to the right of the copy of $\rho(m(r))$ containing $\mathbf{a}_r$. In each case, this interval is isomorphic to $\pi(P)$, and the corresponding interval in $\mathbf{B}$ is isomorphic to $\pi(Q)$. Therefore, it is enough to show that $\pi(P)$ and $\pi(Q)$ satisfy the same Σ_α sentences of $L_{\omega_1\omega}$. (Here we are using the infinitary version of the Feferman-Vaught Theorem—known for sums of orderings before it was extended to more general kinds of product structures. See Proposition 5.1 of [AK].) Since α is a limit ordinal, this is the same as showing that the structures satisfy the same Σ_β sentences for all $\beta < \alpha$. For each $N \in \omega$, the intersections with N of sets enumerated by P include all subsets of N (by genericity), and the same is true for Q. This implies that for all $\beta < \alpha$, $\pi(P)$ and $\pi(Q)$ satisfy the same Σ_β sentences. Therefore, $\mathbf{a}$ and $\mathbf{b}$ satisfy the same Σ_α formulas.

We can now produce $\mathbf{A}_{\alpha+1}(\mathbf{d})$ for any recursive limit ordinal α.

Proposition 3.3. *If α is a recursive limit ordinal, and $\mathbf{d} \geq \mathbf{0}^{(\alpha+1)}$, then there is an ordering having $(\alpha+1)$st jump degree $\mathbf{d}$ sharply.*

Proof. Let $P \subseteq \omega \times \omega$ be $(\alpha+1)$-generic, where $P^{(\alpha+1)}$ has degree $\mathbf{d}$. Then $P^{(\alpha+1)} \equiv_T P \oplus 0^{(\alpha+1)}$, so $\pi(P)$ has $(\alpha+1)$st jump degree $\mathbf{d}$. By Lemmas 3.1 and 3.2 (or, by the proof of Lemma 3.2), it does not have αth jump degree.

4. Conclusion

In §2, in dealing with finite $\alpha \geq 3$ and infinite α not a limit ordinal or the successor of one, we used a uniform method consisting of the following two steps:

(1) Find a recursive ordering scheme e such that for all S and X, there is a copy of $\mathbf{A}_e(S)$ recursive in X iff S is r.e. in $X^{(\beta)}$, where $\beta + 1 = \alpha$.
(2) Choose S such that the structure $\mathbf{A}_e(S)$ will have the desired αth jump degree $\mathbf{d}$ sharply.

It is natural to ask whether the method could be used where we did not use it. We want to know for which recursive ordinals β there is a recursive ordering scheme e such that for all S and X, $\mathbf{A}_e(S)$ has a copy recursive in X iff S is r.e. in $X^{(\beta)}$. We used the fact that for all nonlimit $\beta \geq 2$, there are such schemes. There cannot be a scheme for $\beta = 0$, because if there were, then we could produce orderings with 1st jump degree $\mathbf{d}$ for all $\mathbf{d} \geq \mathbf{0}'$. The two questions below correspond to the remaining cases.

Question 1. Is there a recursive ordering scheme e such that for all S and X, $\mathbf{A}_e(S)$ has a copy recursive in X iff S is r.e. in X'?

Question 2. For recursive limit β, is there a recursive ordering scheme such that for all S and X, $\mathbf{A}_e(S)$ has a copy recursive in X iff S is r.e. in $X^{(\beta)}$?

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