PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 114, Number 2, February 1992

A REMARK ON BOURGAIN ALGEBRAS ON THE DISK

PRATIBHA G. GHATAGE, SHUNHUA SUN, AND DECHAO ZHENG

(Communicated by Paul S. Muhly)

ABSTRACT. It is shown that the Bourgain algebra X_b of the space $X = H^\infty$ considered as a subalgebra of $\mathscr{U} = \operatorname{alg}\{H^\infty, \overline{H}^\infty\}$ is $H^\infty(\mathbb{D}) + UC(\mathbb{D})$ where $UC(\mathbb{D})$ is the algebra of uniformly continuous functions on the open unit disk $\mathbb{D}$. This uses and extends a recent result of Cima–Janson–Yale on the Bourgain algebra of H^∞ on $\partial\mathbb{D}$. Further, $(X_b)_b = X_b$.

In [3] Cima and Timoney introduce the concept of Bourgain algebra X_b of a linear subspace X of a Banach algebra A and show that if X itself is an algebra, then $X \subseteq X_b$. In [2] Cima, Janson, and Yale describe the Bourgain algebra of $H^\infty(\partial\mathbb{D})$. Since then there has been further study of Bourgain algebras on the bitorus and the polydisk [8].

We look at $X = H^\infty$ as a subalgebra of $\mathscr{U}$ where $\mathscr{U} = \operatorname{alg}\{H^\infty, \overline{H}^\infty\} \subseteq \mathscr{L}^\infty(\mathbb{D})$. We recall that $\mathscr{U}$ may be identified with $C(\mathscr{M})$ where $\mathscr{M}$ is the maximal ideal space of H^∞. $UC(\mathbb{D})$ denotes the algebra of uniformly continuous functions on $\mathbb{D}$. It is known that $H^\infty + UC(\mathbb{D}) = H^\infty[\overline{z}]$, the subalgebra of $\mathscr{L}^\infty(\mathbb{D})$ generated by H^∞ and $\overline{z}$ (see [1, p. 721]). The only norms we use are the supnorms and essential supnorms; $\|\varphi\|_{\partial\mathbb{D}} = \operatorname{ess\,sup}\{|\varphi(z)|,\ z \in \partial\mathbb{D}\}$ whereas $\|\varphi\|_{\mathbb{D}} = \sup\{|\varphi(z)|,\ z \in \mathbb{D}\}$. An infinite sequence $\{w_n\}_1^\infty \subseteq D$ is called thin if

$$\prod_{\substack{m=1\\ m\neq n}}^{\infty} \left| \frac{w_n - w_m}{1 - \overline{w}_n w_m} \right| \to 1 \quad \text{as } n \to \infty.$$

A thin Blaschke product b is a function on $\mathbb{D}$ of the form

$$b(z) = \prod_{n=1}^{\infty} \frac{|w_n|}{w_n} \frac{w_n - z}{1 - \overline{w}_n z},$$

where $\{w_n\}_1$ is a thin sequence. For a Blaschke product b with zeros $\{z_n\}$ the zero set $Z(b)$ in $\mathscr{M} = \operatorname{cl}\{z_n,\ n \geq 1\}\backslash\{z_n,\ n \geq 1\}$ = zero set of b in $\mathscr{M}\backslash\mathbb{D}$.

Received by the editors July 7, 1990 and, in revised form, August 10, 1990.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 30D55, 46J15.
Key words and phrases. Bourgain algebra, zero set, thin Blaschke products.
Most of this research was done while the second author was visiting Cleveland State University. He would like to thank the Mathematics Department for its hospitality. He would also like to thank the NNSFC for partial support.

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Our main result shows that analogous to (1) we have: $(H^\infty(\mathbb{D}))_b = H^\infty + UC(\mathbb{D})$. This is not particularly surprising, but the investigation does raise some interesting questions about subalgebras of $\mathscr{L}^\infty(\mathbb{D})$.

Lemma 1. *If $X = H^\infty(\mathbb{D})$ and $A = \mathscr{U}$, then we have*:

(i) $H^\infty + UC(\mathbb{D}) \subseteq X_b$;

(ii) *if b is an infinite Blaschke product, then $\overline{b} \notin X_b$.*

Proof. The proof of (i) is an easy consequence of the description of $(H^\infty(\partial\mathbb{D}))_b$ and (ii) uses the fact that if $\hat{\varphi}$ denotes the harmonic extension of $\varphi \in \mathscr{L}^\infty(\partial\mathbb{D})$ to $\mathbb{D}$ via the Poisson kernel, then the map $\varphi \mapsto \hat{\varphi}$ is asymptotically multiplicative on $H^\infty + C(\partial\mathbb{D})$. (See [4, Lemma 6.44].) We sketch the details for completeness.

To prove (i) we only need to prove that $\overline{z} \in X_b$. So assume that $f_n \in H^\infty$, $f_n \xrightarrow{\text{wk}} 0$ and let $\varphi_n(z) = (f_n(z) - f(0))/z$. Then $\varphi_n \in H^\infty$ and $\varphi_n \xrightarrow{\text{wk}} 0$. Hence $f_n \to 0$, $\varphi_n \to 0$, both uniformly on compacta, and $\|f_n\| \le c$, $\|\varphi_n\| \le c$ for some $c > 0$. $(\overline{z}f_n(z) - \varphi_n(z))|_{\partial\mathbb{D}} = e^{i\theta}f_n(0) \in C(\partial\mathbb{D})$ and hence $\|\overline{z}f_n(z) - \varphi_n(z)\|_{\partial\mathbb{D}} \to 0$, which implies that $\|f_n(z) - z\varphi_n(z)\|_{\partial\mathbb{D}} \to 0$. However $f_n(z) - z\varphi_n(z) \in H^\infty$ and hence $\|f_n(z) - z\varphi_n(z)\|_{\mathbb{D}} \to 0$. Clearly, $\|\overline{z}f_n(z) - z\overline{z}\varphi_n(z)\|_{\mathbb{D}} \to 0$. It follows easily that for $\varepsilon > 0$ $\exists \rho < 1$ such that $|\overline{z}f_n(z) - \varphi_n(z)| \le |\overline{z}f_n(z) - z\overline{z}\varphi_n(z)| + (1-|z|^2)\|\varphi_n\|_{\mathbb{D}} < \varepsilon$ for $\rho < |z| < 1$. Since f_n and $\varphi_n \to 0$ uniformly on $\{z, |z| \le \rho\}$, $|\overline{z}f_n(z) - \varphi_n(z)| \to 0$ uniformly on $\{z, |z| \le \rho\}$. This shows that $\|\overline{z}f_n(z) - \varphi_n(z)\|_{\mathbb{D}} \to 0$ and hence $\overline{z} \in X_b$.

(ii) If b is an infinite Blaschke product such that $\overline{b} \in X_b$ then clearly $\overline{b}|_{\partial\mathbb{D}} \in (H^\infty(\partial\mathbb{D}))_b$. Hence by [2, Theorem 1] $\overline{b}|_{\partial\mathbb{D}} = \varphi + \psi$ where $\varphi \in H^\infty$ and $\psi \in C(\partial\mathbb{D})$. In particular $1 = b\varphi + b\psi$ a.e. on $\partial\mathbb{D}$. Using [4, Lemma 6.44] we have $1 \approx b\varphi + b\hat{\psi}$ in an annulus $\{z, \rho < |z| < 1\}$. This is clearly impossible since $\{z, \rho < |z| < 1\} \cap \{\text{zeros of } b\} \ne \varphi$.

Remark 1. In the context of Hankel operators and their compactness a natural analogue of $H^\infty + C(\partial\mathbb{D})$ on the disk is $H^\infty + COP$, see [1, §6], and on first thought $H^\infty + COP$ may be considered a natural candidate for $(H^\infty(\mathbb{D}))_b$. However it follows as an immediate consequence of Lemma 1 that $COP \not\subseteq (H^\infty(\mathbb{D}))_b$, because COP contains $\overline{b}$ whenever b is a Blaschke product in the little Bloch space $\mathscr{B}_0$ and $\mathscr{B}_0$ is known to contain infinite Blaschke products [5, p. 442]. K. Stroethoff and K. Yale have studied $H^\infty(\mathbb{D})_b$ as a subalgebra of $\mathscr{L}^\infty(\mathbb{D})$ and found a function $f \in \mathscr{L}^\infty(\mathbb{D})$ such that the Hankel operator H_f is compact but $f \notin H^\infty(\mathbb{D})_b$. They also show that in the larger context of $\mathscr{L}^\infty(\mathbb{D})$, $H^\infty(\mathbb{D})_b \supsetneqq H^\infty + UC(\mathbb{D})$ (private communication).

As the next proposition shows the proper analogue of $H^\infty + C(\partial\mathbb{D})$ in the context of Bourgain algebras within $\mathscr{U}$ is $H^\infty + UC(\mathbb{D})$.

Proposition 1. $(H^\infty(\mathbb{D}))_b = H^\infty(\mathbb{D}) + UC(\mathbb{D})$.

Proof. In view of Lemma 1 we only need to prove that $(H^\infty(\mathbb{D}))_b \subseteq H^\infty(\mathbb{D}) + UC(\mathbb{D})$. Suppose $f \in \mathscr{U}$ and $f \in (H^\infty(\mathbb{D}))_b$. Then f has nontangential limits a.e. and we denote this function by $f|_{\partial\mathbb{D}}$. Clearly $\psi = f - (f|_{\partial\mathbb{D}})\hat{} \in C(\mathbb{D})$ and $\psi|_{\partial\mathbb{D}} = 0$ a.e.

Claim 1. $\psi \in (H^\infty(\mathbb{D}))_b$.

To prove the claim note that by hypothesis, $f \in (H^\infty(\mathbb{D}))_b$ and hence $f|_{\partial\mathbb{D}} \in H^\infty + C(\partial\mathbb{D})$ [2, Theorem 1]. It follows that $(f|_{\partial\mathbb{D}})^\frown \in H^\infty + UC(\mathbb{D})$. By Lemma 1, $(f|_{\partial\mathbb{D}})^\frown \in (H^\infty(\mathbb{D}))_b$ and this completes proof of the claim.

Claim 2. Given $\varepsilon > 0$ $\exists r(0 < r < 1)$ such that $|\psi(z)| < \varepsilon$ for $z \in \{r < |z| < 1\}$.

If not $\exists\{r_n\}$, $0 < r_n < 1$, $r_n \to 1$ and $\{z_n\} \subseteq \mathbb{D}$ such that $|z_n| > r_n$ and $|\psi(z_n)| \geq \delta$. Choosing a subsequence if necessary, we may assume that $\{z_n\}$ is an interpolating sequence. As in [2, p. 123] we have a sequence $\{f_n\} \subseteq H^\infty$, such that $f_n(z_m) = \delta_{nm}$ and $f_n \xrightarrow{\text{wk}} 0$. By Claim 1, $\exists\{\varphi_n\} \subseteq H^\infty(\mathbb{D})$ such that $\|\psi f_n - \varphi_n\|_{\mathbb{D}} < \varepsilon_n$ where $\varepsilon_n \to 0$. In particular, $\|\varphi_n\|_{\partial\mathbb{D}} < \varepsilon_n$ and φ_n being in H^∞ this implies $\|\varphi_n\|_{\mathbb{D}} < \varepsilon_n$. Thus $\|\psi f_n\| < 2\varepsilon_n \to 0$, which is clearly a contradiction since $|(\psi f_n)(z_n)| = |\psi(z_n)| \geq \delta$ for all n. This proves Claim 2 and we conclude that $\psi \in C_0(\overline{\mathbb{D}}) \subseteq UC(\mathbb{D})$ and hence $f = (f|_{\partial\mathbb{D}})^\frown + \psi \in H^\infty + UC(\mathbb{D})$.

Proposition 2. $(H^\infty + UC(\mathbb{D}))_b = H^\infty + UC(\mathbb{D})$.

Proof. It is sufficient to show that $(H^\infty + UC(\mathbb{D}))_b \subseteq H^\infty + UC(\mathbb{D})$. Let $f \in (H^\infty + UC(\mathbb{D}))_b$ and write $g = f|_{\mathscr{M}(\mathscr{L}^\infty(\partial\mathbb{D}))}$. This is well defined since by assumption $f \in \mathscr{U} = C(\mathscr{M})$. Clearly $g \in (H^\infty + C(\partial\mathbb{D}))_b$ and hence $g \in H^\infty + C(\partial\mathbb{D})$. See [6]. In particular, $\hat{g} \equiv$ the harmonic extension of g to $\mathbb{D}$ belongs to $H^\infty + UC(\mathbb{D}) \subseteq (H^\infty + UC(\mathbb{D}))_b$ and hence $\psi = f - \hat{g} \in (H^\infty + UC(\mathbb{D}))_b$. We claim that $\psi \in C_0(\overline{\mathbb{D}})$. If not, (dropping to a subsequence) we may choose $\{z_n\} \subseteq \mathbb{D}$ such that $\{z_n\}$ is a thin sequence satisfying $|\psi(z_n)| \geq \delta$ for some $\delta > 0$. Let b be the (thin) Blaschke product with zeros $\{z_n\}$. Following a construction of Izuchi's (the ideas behind which are outlined in [7]) we can find $\{y_n\} \subseteq Z(b)$ and $\{f_n\} \subseteq H^\infty$ such that

(1) $f_n \xrightarrow{\text{wk}} 0$ and

(2) $|f_n(y_n)| > 1 - \varepsilon_n$ where $\varepsilon_n \to 0$.

Since we have not seen it in print, we provide a very brief sketch for completeness. Suppose $x \in \mathscr{M}$ and μ_x is its unique representing measure supported on the Shilov boundary of $\mathscr{M}$. By a well-known but unpublished result of Hoffman's, whenever x_1 and x_2 are in the zero set $Z(b)$ of a thin Blaschke product b, μ_{x_1}, and μ_{x_2} have disjoint supports and hence starting with an arbitrary sequence $\{x_n\}$ in $Z(b)$ with a cluster point x_0 $(\in Z(b))$ we have $\mu_{x_n} \cap \operatorname{spt}\mu_{x_0} = \varphi$ for all n. The f_n's are now built inductively using the fact that $\operatorname{spt}\mu_{x_n}$ $(n \geq 0)$ is a weak peak set for H^∞ (unpublished manuscript).

Since $\psi \in (H^\infty + UC(\mathbb{D})_b$ $\exists g_n \in H^\infty$ and $\varphi_n \in UC(\mathbb{D})$ such that $\|\psi f_n - g_n - \varphi_n\| < \varepsilon_n \to 0$. As $\psi|_{\mathscr{M}(\mathscr{L}^\infty(\mathbb{D}))} = 0$, we have $\|g_n + \varphi_n\|_{\mathscr{M}(\mathscr{L}^\infty(\mathbb{D}))} \leq \varepsilon_n$ and replacing φ_n by the harmonic extension $\hat{\varphi}_n$ of $\varphi_n|_{\mathscr{M}(\mathscr{L}^\infty(\mathbb{D}))}$ we have

$$(*) \qquad \|g_n + \hat{\varphi}_n\| \leq \varepsilon_n .$$

Clearly $\varphi_n - \hat{\varphi}_n$ vanishes on $\mathscr{M}\backslash\mathbb{D}$ and for any $m \in \mathscr{M}\backslash\mathbb{D}$, $g_n(m) + \varphi_n(m) = g_n(m) + \hat{\varphi}_n(m)$ and hence $\|\psi f_n - g_n - \hat{\varphi}_n\|_{Z(b)} \leq \varepsilon_n$. So $\|g_n + \hat{\varphi}_n\|_{Z(b)} \geq \|\psi f_n\|_{Z(b)} - \|\psi f_n - g_n - \hat{\varphi}_n\|_{Z(b)} \geq \|\psi f_n\|_{Z(b)} - \varepsilon_n$.

However, $\psi \in C(\mathscr{M})$ and $|\psi(z_n)| \geq \delta$, so $|\psi(m)| \geq \delta$ for all $m \in Z(b)$ and hence $\|\psi f_n\|_{Z(b)} \geq \delta|f_n(y_n)| \geq (1-\varepsilon_n)\delta$. It follows that $\|g + \hat{\varphi}_n\|_{Z(b)} \geq \delta/2$

for n sufficiently large. This clearly contradicts $(*)$ and proves the claim that $\psi \in C_0(\overline{\mathbb{D}})$. In particular $f = \hat{g} + \psi \in H^\infty + UC(\mathbb{D})$.

Remark 2. In [6] Gorkin and Izuchi showed that the only Douglas algebra B for which $B_b = \mathscr{L}^\infty(\partial\mathbb{D})$ is $\mathscr{L}^\infty(\partial\mathbb{D})$ and that if A and B are Douglas algebras with $A \subseteq B$ then $A_b \subseteq B_b$. Our work indicates that both of these analogues cannot be valid on the disk. Dechao Zheng has some further results on subalgebras A of $\mathscr{U}$ for which $A_b = \mathscr{U}$. However, the proofs are fairly involved and will be published elsewhere.

Acknowledgment

P. Gorkin has communicated to us that Gorkin, Izuchi and Mortini have independently proved some of our results. We are grateful to P. Gorkin and K. Yale for their kind communications.

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Department of Mathematics, Cleveland State University, Cleveland, Ohio 44115

Department of Mathematics, Sichuan University, Chengdu, 610064, Peoples' Republic of China

Department of Mathematics, State University of New York at Stony Brook, Stony Brook, New York 11794