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MAKING A HOLE IN THE SPACE

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ABSTRACT. The paper provides a simple recipe for a construction of homeomorphisms "removing" convex bodies from nonreflexive Banach spaces.

Let X be a Banach space. A set $K \subset X$ is said to be removable (or deleting) from X if there exists a homeomorphism T mapping X onto $X \backslash K$. We say also that T removes K from X.

The study of removable sets was initiated by Klee [1] with the observation that if X is of infinite dimension then whole space is homeomorphic to the punctured space $X \backslash \{0\}$. Detailed discussion of the theory can be found in Bessaga and Pełczyński's book [2].

For the newcomer to the theory some facts as mentioned above may seem surprising, especially that the methods used in proofs are often "nonconstructive."

Below we present a hopefully unknown recipe for a construction of homeomorphisms removing from the spaces a closed and bounded convex body, a ball or a point. Our construction works, however, only in nonreflexive Banach spaces.

Let X be nonreflexive and let B and S stand for the closed unit ball and the unit sphere, respectively.

Since X is nonreflexive, there exists a linear functional $f \in X^*$ such that $\|f\| = 1$ which does not achieve its maximum on B, which means that $|f(x)| < \|x\|$ for all $x \neq 0$.

Consider the function ϕ defined on X by

$$\begin{aligned}\phi(x) &= 1 + f(x) + \max[0, 2(\|x\| - 1)] \\ &= \begin{cases} 1 + f(x), & \text{for } x \in B, \\ 2 \cdot \|x\| - 1 + f(x), & \text{for } x \notin B, \end{cases}\end{aligned}$$

and observe that ϕ is continuous (lipschitzian) convex and even affine when restricted to B. Moreover $\lim_{\|x\| \to +\infty} \phi(x) = +\infty$ while

$$\inf[\phi(x) \colon x \in X] = \inf[\phi(x) \colon x \in B] = 0.$$

Observe also that this infimum is not taken and $\phi(B) = \phi(S) = (0, 2)$.

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Select a sequence of points $\{x_n\}$, $n = 0, 1, 2, \dots$ on the unit sphere S such that $\phi(x_n) = 2^{-n}$.

Form a curve γ by joining consecutive points by segments and complete it by joining x_0 with infinity by a radial ray.

Observe that since ϕ is affine on B this curve can be parametrized, $\gamma = \gamma(t)$, by $t \in (0, \infty)$ in such a way that $\phi(\gamma(t)) = t$.

To end the preparation, for any $a \geq 0$ denote:

$$K_a = [x\colon \phi(x) \leq a]$$

and notice that for $a > 0$ each K_a is closed convex subset of X having nonempty interior, while

$$K_0 = \bigcap_{a>0} K_a = \varnothing .$$

Now we can formulate our recipe.

Fix a number $a \geq 0$. Take any $x \in X$. Calculate $\phi(x)$. Find the point y on γ such that $\phi(y) = \frac{1}{2}\phi(x)$ (simply take $y = \gamma(\frac{1}{2}\phi(x))$). Obviously $y \neq x$. Follow the ray beginning at x and passing through y as far as you reach point z such that $\phi(z) = \phi(x) + a$ (there is only one such point).

Define $T_a(x) = z$.

Observe that T_a maps X onto $X\backslash K_a$ and is invertible. It is only a technicality to prove that T_a and T_a^{-1} are continuous (even locally lipschitzian).

Thus T_a removes K_a from X and for $a > 0$ the hole mentioned in the title has been made. To make the hole ball shaped observe that $B \subset K_2$ and define

$$T_B(x) = p(T_2(x))T_2(x)/\|T_2(x)\| ,$$

where $p(\ )$ denotes the Minkowski functional for K_2.

T_B removes the unit ball from X, finally

$$T(x) = \left(1 - \frac{T_2(x)}{\|T_2(x)\|}\right) T_B(x)$$

removes just one point, the origin.

Finally let us consider the case $a = 0$. T_0 does not remove anything from X but it shows another "exotic" behavior. T_0 is self-invertible which means $T_0^2 = I$ and at the same time T_0 does not have a fixed point $(x \neq T_0(x))$. T_0 is only locally lipschitzian. The problem of whether there exists a uniformly continuous mapping T having the above property was raised by Koter [3] and is still unsolved.

References

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