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# BOUNDEDNESS OF SOLUTIONS TO FUNCTIONAL INTEGRO-DIFFERENTIAL EQUATIONS

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ABSTRACT. In this paper we discuss the boundedness of the solutions of a class of functional integro-differential equations. We give an example to answer a question posed by S. M. Kuen and K. P. Rybakowski, and we establish a sufficient condition, which is more general than the condition they obtained.

## 1. INTRODUCTION

The scalar equation

$$x'(t) = -\int_{-1}^{0} b(\theta) g(x(t+\theta))\, d\theta + f(t) \tag{1.1}$$

occurs frequently as a model in mathematical biology and the physical sciences. For example, it is a good model in one-dimensional viscoelasticity in which $x$ is the strain and $b$ is the relaxation function. It is also a model for a circulating fuel nuclear reactor in which $x$ is the neutron density (see [1]).

Equation (1.1) has been studied extensively by a number of authors, especially by Levin and Nohel (see, e.g. [3, 4, 6, 7]). In these papers authors studied the asymptotic behavior, boundedness, and oscillation of the solution to (1.1). Smets [8, 9] also studied the boundedness of the solutions of (1.1). In these earlier results, to guarantee the boundedness of solutions, all the authors supposed that $g(x)$ satisfies the following hypothesis:

$$xg(x) > 0 \qquad (x \neq 0)\ g(x) \in c(-\infty, +\infty), \tag{1.2}$$

$$g(x) \geq -\lambda \quad (\text{or } g(x) \leq \lambda), \qquad 0 < \lambda < \infty, \ -\infty < x < +\infty. \tag{1.3}$$

In other words, $g(x)$ must have a one-sided bound except the sign condition (1.2). S. M. Kuen and K. P. Rybakowski [2] discussed the boundedness of the solutions to the pure delay equation

$$x'(t) = -\int_{-1}^{0} b(\theta) g(x(t+\theta))\, d\theta, \tag{1.4}$$

where $b\colon [-1, 0] \to R$, $g\colon R^n \to R^n$, $x \in R^n$. They obtained the following result.

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**Theorem A.** *Suppose that* $g\colon R^n \to R^n$ *is locally Lipschitzian, and* $b\colon [-1, 0] \to R$ *is continuous, nonnegative, and not identically zero. Let* $i \in \{1, 2, \dots, n\}$. *Assume that* $M = \int_{-1}^{0} b(\theta)\, d\theta = 1$, *and that there are constants* $a > 0$, $c \geq 0$, $\beta$, $c_1$, $c_2 \in R$, $c_2 \leq c_1$ *such that*

(a) $g_i(x) \geq 0$ *whenever* $x \in R^n$ *and* $c_1 \leq x_i$,
$g_i(x) \leq 0$ *whenever* $x \in R^n$ *and* $x_i \leq c_2$;

(b) $g_i(x) \leq a(x_i - c_1 - \beta)$ *whenever* $x \in R^n$ *and* $c_1 + c \leq x_i$,
$g_i(x) \geq (1/a)(x_i - c_2 - a\beta)$ *whenever* $x \in R^n$ *and* $x_i \leq c_2 - c$.

*If* $x = x(\phi)(t)\colon 0 \leq t < \omega_\phi$ *is a solution of* (1.4) *with* $x_j$ *bounded for* $j \neq i$, *then* $x = x(\phi)(t)\colon 0 \leq t < \omega_\phi$ *is bounded, where* $g_i(x)$ *and* $x_i$ *are the ith component of* $g(x)$ *and* $x$, *respectively.*

In Theorem A condition (a) is equivalent to (1.2), and condition (b) shows that $|g_i(x)|$ is allowed to be unbounded as long as it satisfies a special linear growth condition.

In their paper the authors wrote "The question arises whether the sign condition (assumption (a) of Theorem A) above suffices to guarantee the boundedness of $x_i$ (if $x_j$ is bounded for $j \neq i$)?", and they also wrote "This seems to be an open question. In our proofs we either assume that $g_i$ is bounded or else that $g_i$ satisfies a rather special linear growth condition. It is not clear how to relax these hypothesis."

In this paper we give an example to illustrate that the answer to the question in [2] is not, (Levin once gave a similar example in [3]), and condition (b) is relaxed. In other words, $|g_i(x)|$ is not required to satisfy a special linear growth condition.

## 2. Counterexample and lemmas

Consider the scalar equation

$$x(t) = -\int_{-1}^{0} ax(t+\theta)\, d\theta, \tag{2.1$_a$}$$

where $a > 1$ is some constant.

Evidently, equation $(2.1)_a$ satisfies the conditions of Theorem A except condition (b) for any $a > 1$. Now we show that there at least exists one constant $a > 1$ such that $(2.1)_a$ has an unbounded solution. For this we consider the characteristic equation of equation $(2.1)_a$

$$\lambda - a(e^{-\lambda} - 1)/\lambda = 0$$

or equivalently

$$\lambda^2 - a(e^{-\lambda} - 1) = 0. \tag{2.2$_a$}$$

It is easy to prove that there is at least one constant $a \in (1, \frac{9}{4}\pi^2)$ such that $\lambda(a) = (\frac{9}{4}\pi^2 - a)^{1/2} + \frac{3}{2}\pi i$ is a root of $(2.2)_a$ with positive real part, we omit the detailed argument. This implies that there at least exists one unbounded solution of equation $(2.1)_a$. Hence the sign condition (a) of Theorem A is not enough to guarantee the boundedness of the solutions of equation (1.4).

Now we given the following assumptions:

$(\mathrm{H})_1$ $b\colon [-1, 0] \to R$ is continuous, nonnegative, not identically zero.

$(\mathrm{H})_2$ $g\colon R^n \to R^n$ is locally Lipschitzian. Let $i \in \{1, 2, \dots, n\}$. There are constants $c_1, c_2 \in R$, $c_1 \geq c_2$ such that $g_i(x) \geq 0$ whenever $x \in R^n$ and $x_i \geq c_1$ and $g_i(x) \leq 0$ whenever $x \in R^n$ and $x_i \leq c_2$, where $g_i$ and $x_i$ are the $i$th component of $g(x)$ and $x$, respectively.

**Lemma 1.** *Suppose that $x\colon [a, b) \to R$ is a continuous function. If*

$$\sup_{a \leq t < b} x(t) = +\infty \qquad \left(\text{or } \inf_{a \leq t < b} x(t) = -\infty\right),$$

*then there must be a sequence $\{t_n\} \subset [a, b)$ such that*

(I) $t_n > t_{n-1}$ *for each* $n = 1, 2, \dots$ *and* $t_n \to b^-$ *as* $n \to +\infty$;
(II) $x(t) < x(t_n)$ *(or* $x(t) > x(t_n)$*) for all* $t \in [a, t_n)$;
(III) $x(t_n) \to +\infty$ *(or* $x(t_n) \to -\infty$*) as* $n \to \infty$.

We omit the proof of Lemma 1 because it is not difficult.

**Lemma 2.** *Let $b$ and $g$ satisfy conditions* $(\mathrm{H})_1$ *and* $(\mathrm{H})_2$. *Suppose that $x(\phi)(t)\colon 0 \leq t \leq \omega_\phi$ is a solution of equation* (1.4) *and that $s, t$ are arbitrary two points in the interval $[0, \omega_\phi)$ with $s > t$. If the inequality $c_1 \leq x_i(u) < x_i(s)$ (or $x_i(s) < x_i(u) \leq c_2$) holds for all $u \in [t, s)$, then we have $s - t \leq 1$.*

*Proof.* Suppose that Lemma 2 is not true; that is, that there are two points $s, t \in [0, \omega_\phi)$ with $s > t$ such that $c_1 \leq x_i(u) < x_i(s)$ (or $c_2 \geq x_i(u) > x_i(s)$) for all $u \in [t, s)$ and $s - t > 1$. From the continuity of $x_i(t)$, we obtain $c_1 \leq x_i(u) < x_i(s)$ (or $x_i(s) < x_i(u) \leq c_2$) for some $\varepsilon > 0$ and all $u \in [s - 1 - \varepsilon, s)$, therefore, we have

$$x_i'(u) = -\int_{-1}^{0} b(\theta) g_i(x(u+\theta))\, d\theta \leq 0$$

$$\left(\text{or } x_i'(u) = -\int_{-1}^{0} b(\theta) g_i(x(u+\theta))\, d\theta \geq 0\right)$$

for all $u \in [s-\varepsilon, s)$. Here we used assumption $(\mathrm{H})_2$ and the fact that $x_i(u) \geq c_1$ on $[s - \varepsilon - 1, s)$. Hence $x_i(s - \varepsilon) \geq x_i(s)$ (or $x_i(s - \varepsilon) \leq x_i(s)$), which is a contradiction. This completes the proof of Lemma 2.

**Lemma 3.** *Suppose that conditions* $(\mathrm{H})_1$ *and* $(\mathrm{H})_2$ *are satisfied. Let $x(t) = x(\phi)(t)\colon 0 \leq t < \omega_\phi$ be a solution of equation* (1.4) *with bounded components $x_j(t)\colon 0 \leq t < \omega_\phi$ for $j \neq i$. If $x_i(t)\colon 0 \leq t < \omega_\phi$ is unbounded, then $x_i(t)\colon 0 \leq t < \omega_\phi$ must oscillate between $-\infty$ and $+\infty$, in other words*

$$\sup_{0 \leq t < \omega_\phi} x_i(t) = +\infty \quad \textit{and} \quad \inf_{0 \leq t < \omega_\phi} x_i(t) = -\infty.$$

*Proof.* Suppose that Lemma 3 is not true, i.e. that $\sup_{0 \leq t < \omega_\phi} x_i(t) = +\infty$ but $\inf_{0 \leq t < \omega_\phi} x_i(t) > -\infty$. By the continuity of $g_i(x)$ and the boundedness of $x_j(t)$ for $j \neq i$, there is a constant $M > 0$ such that $g_i(x(t)) \geq -M$ for all $t \in [-1, \omega_\phi)$. Thus we have

$$x_i'(t) = -\int_{-1}^{0} b(\theta) g_i(x(t+\theta))\, d\theta \leq M \int_{-1}^{0} b(\theta)\, d\theta = M^* \quad \text{for all } t \in [0, \omega_\phi).$$

This implies

$$x_i(s) - x_i(t) = \int_t^s x_i'(u)\, du \leq M^*(s - t) \quad \text{for any } s, t \in [0, \omega_\phi). \tag{2.3}$$

Assume first that $\omega_\phi < +\infty$. By (2.3) we have $\sup_{0\le t<\omega_\phi} x_i(t) \le M^*\omega_\phi < +\infty$, which is a contradiction to our assumption. Hence $\omega_\phi = +\infty$. By our assumption and Lemma 1, there exists a sequence $\{t_n\} \subset [0, +\infty)$ such that

$$\begin{aligned} &t_n < t_{n+1} \quad \text{for } n = 1, 2\ldots \text{ and } t_n \to +\infty \text{ as } n \to +\infty; \\ &x_i(t) < x_i(t_n) \quad \text{for all } t \in [-1, t_n), \\ &x_i(t_n) \to +\infty \quad \text{as } n \to +\infty. \end{aligned}$$

Therefore, we may pick a positive sufficiently large integer $n$ such that

$$x_i(t_n) > c_1 + 2M^* \quad \text{and} \quad t_n - 1 > 1. \tag{2.4}$$

Let $s = t_n$ in (2.3), we obtain

$$x_i(t_n) - x_i(t) \le M^*(t_n - t) \quad \text{for all } t \in [t_n - 1, t_n]. \tag{2.5}$$

From (2.4) and (2.5), we have

$$x_i(t) \ge x_i(t_n) - M^*(t_n - t) \ge c_1 + 2M^* - M^* > c_1 \quad \text{for all } t \in [t_n - 1, t_n].$$

By the continuity of $x_i(t)$ on the interval $[-1, +\infty)$ for some $\varepsilon > 0$, we obtain

$$x_i(t) \ge c_1 \quad \text{for all } t \in [t_n - 1 - \varepsilon, t_n].$$

On the other hand, we have

$$x_i(t) \le x_i(t_n) \quad \text{for all } t \in [t_n - 1 - \varepsilon, t_n]$$

by the properties of $\{t_n\}$. Hence, we obtain

$$c_1 \le x_i(t) \le x_i(t_n) \quad \text{for all } t \in [t_n - 1 - \varepsilon, t_n].$$

This contradicts Lemma 2. The proof of the lemma is completed.

## 3. Main results

We give the following hypotheses:

$(\mathrm{H})_3$ There is a constant $K > 0$ such that

$$g_i(x) \le K \qquad (\text{or } g_i(x) \ge -K) \quad \text{whenever } x \in R^n;$$

$(\mathrm{H})_4$ Let $(\mathrm{H})_2$ be satisfied. Assume that there is a constant $c > 0$, two strictly increasing functions $\alpha, \beta\colon R^+ \to R^+ = [0, +\infty)$ such that

(a) there is a large constant $G > 0$ such that

$$\alpha(\beta(r)) \le r \quad \text{or} \quad \beta(\alpha(r)) \le r \quad \text{for all } r > G;$$

(b) $g_i(x) \le \alpha(x_i - c_1)$ whenever $x \in R^n$ and $x_i \ge c_1 + c$;

(c) $g_i(x) \ge -\beta(c_2 - x_i)$ whenever $x \in R^n$ and $x_i \le c_2 - c$.

**Theorem 1.** *Let the hypotheses* $(\mathrm{H})_1$, $(\mathrm{H})_2$, *and* $(\mathrm{H})_3$ *be satisfied. If* $x(t) = x(\phi)(t)\colon 0 \le t < \omega_\phi$ *is a solution of equation* (1.4) *with bounded components* $x_j(t)\colon 0 \le t < \omega_\phi$ *for* $j \ne i$, *then* $x_i(t)\colon 0 \le t < \omega_\phi$ *is bounded.*

*Proof.* From Lemma 3 we only prove that $x_i(t)\colon 0 \le t < \omega_\phi$ is bounded from above. Suppose that $x_i(t)\colon 0 \le t < \omega_\phi$ is not bounded from above, i.e. that $\sup_{0\le t<\omega} x(t) = +\infty$. By the hypothesis $(\mathrm{H})_3$, we have

$$x_i'(t) = -\int_{-1}^{0} b(\theta) g_i(x(t+\theta))\, d\theta \le K^* \quad \text{for all } t \in [0, \omega_\phi),$$

$$K^* = K \int_{-1}^{0} b(\theta)\, d\theta.$$

This implies

$$x_i(s) - x_i(t) = \int_t^s x_i'(u)\,du \le k^*(s-t) \quad \text{for any } s\,,\, t \in [0\,,\, \omega_\phi)\,.$$

The remainder of the proof is the same as the proof of Lemma 3, hence we omit it.

**Theorem 2.** *Let the hypotheses* $(\mathrm{H})_1$, $(\mathrm{H})_2$, *and* $(\mathrm{H})_3$ *be satisfied. Assume that* $\int_{-1}^0 b(\theta)\,d\theta = 1$. *If* $x(t) = x(\phi)(t)\colon 0 \le t < \omega_\phi$ *is a solution of equation* (1.4) *with bounded components* $x_j(t)\colon 0 \le t < \omega_\phi$ *for* $j \ne i$, *then* $x_i(t)\colon 0 \le t < \omega_\phi$ *is bounded.*

*Proof.* Suppose that Theorem 2 is not true, i.e. that $x_i(t)\colon 0 \le t < \omega_\phi$ is unbounded. From Lemma 3 we have

$$\sup_{0\le t<\omega_\phi} x_i(t) = -\inf_{0\le t<\omega_\phi} x_i(t) = +\infty\,. \tag{3.1}$$

By Lemma 1, there are two sequences $\{s_n\}$, $\{t_n\} \subset [0\,,\, \omega_\phi)$ such that:

$$\begin{aligned} &s_n > s_{n-1}\,,\ t_n > t_{n-1}\,, \quad \text{for } n = 1\,,2\,,\dots \quad \text{and} \\ &s_n \to \omega_\phi\,,\ t_n \to \omega_\phi\,, \quad \text{as } n \to +\infty\,; \end{aligned} \tag{3.2}$$

$$\begin{aligned} &x_i(t) < x_i(s_n) \quad \text{for all } t \in [0\,,\, s_n) \quad \text{and} \\ &x_i(t) > x_i(t_n) \quad \text{for all } t \in [0\,,\, t_n)\,,\ n = 1\,,2\,,\dots\,; \end{aligned} \tag{3.3}$$

$$x_i(t_n) \to -\infty\,, \qquad x_i(s_n) \to +\infty \quad \text{as } n \to +\infty\,. \tag{3.4}$$

We can choose $\{s_n\}$ and $\{t_n\}$. If it is necessary, we can take their subsequences such that

$$s_n < t_n < s_{n+1}\,, \qquad n = 1\,,2\,,\dots\,.$$

According to the continuity of $x_i(t)$ and (3.1) we see that there are also two sequences $\{u_n\}$, $\{v_n\} \subset [0\,,\, \omega_\phi)$ such that

$$x_i(u_n) = c_1\,, \qquad x_i(v_n) = c_2 \quad \text{and} \quad u_n < s_n < v_n < t_n < u_{n+1}\,, \qquad n = 1\,,2\,,\dots\,. \tag{3.5}$$

(See Figure 1). Evidently, we can choose the subsequences of these sequences, which still are denoted by $\{s_n\}$, $\{t_n\}$, $\{u_n\}$, and $\{v_n\}$, such that

$$x_i(s_n) \ge x_i(t) \ge x_i(t_n) \quad \text{whenever } t \in [s_n\,,\, t_n]\,, \tag{3.6}$$

$$x_i(t_n) \le x_i(t) \le x_i(s_{n+1}) \quad \text{whenever } t \in [t_n\,,\, s_{n+1}]\,, \tag{3.7}$$

$$c_1 \le x_i(t) < x_i(s_n) \quad \text{whenever } t \in [u_n\,,\, s_n)\,, \tag{3.8}$$

$$x_i(t_n) < x_i(t) \le c_2 \quad \text{whenever } t \in [v_n\,,\, t_n)\,. \tag{3.9}$$

Inequalities (3.8), (3.9), and Lemma 2 imply that

$$s_n - u_n \le 1\,, \quad t_n - v_n \le 1\,, \qquad n = 1\,,2\,,3\,,\dots\,. \tag{3.10}$$

By our assumption, there is a constant $T > 0$ such that

$$|x_j(t)| \le T \quad \text{for all } -1 \le t < \omega_\phi\,,\ j \ne i\,.$$

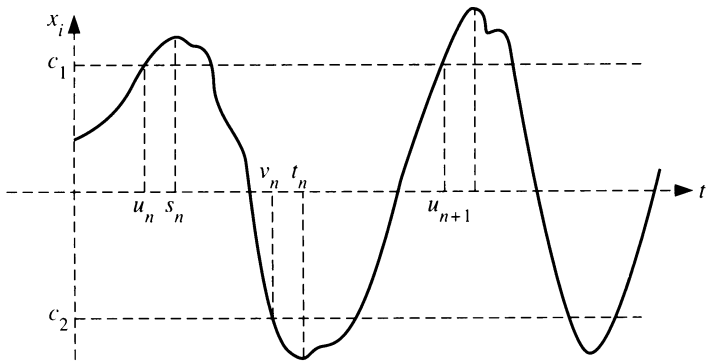


FIGURE 1

Let $B = \{x \in R^n |, \ |x_j| \le T, \ j \ne i, \ c_2 - c \le x_i \le c_1 + c\} \subset R^n$. Let $K = \max\{\sup_{x \in B} |g_i(x)|, \ \alpha(c), \beta(c)\}$ and $L = \max\{K, G, \sup_{-1 \le \theta \le 0} |\phi_i(\theta)|\}$, where $G$ is the constant in $(\mathrm{H})_4$, and $\phi_i$ is the $i$th component of $\phi$, which is the initial function.

We suppose that $\alpha(r) \to +\infty$ and $\beta(r) \to +\infty$, otherwise it is not w.l.o.g. (If not, $\alpha(r)$ or $\beta(r)$ is bounded, then $g_i(x)$ is bounded from above or below, therefore $x_i(t)$ is bounded from Theorem 1), and that the inequality $\beta(\alpha(r)) \le r$ is satisfied. (If the inequality $\alpha(\beta(r)) \le r$ is true, the proof is similar.)

Now, we choose a sufficiently large integer $n$ such that

$$\begin{gathered} \alpha(x_i(s_n) - c_1) > L, \qquad \beta(c_2 - x_i(t_{n-1})) > L; \\ x_i(s_n) > L, \qquad x_i(t_{n-1}) < -L, \qquad x_i(s_{n-1}) - c_1 > L. \end{gathered} \tag{3.11}$$

Here we use the assumption of $(\mathrm{H})_4$ and (3.4). We assert that for all $t \in [u_n, s_n]$, $\theta \in [-1, 0]$, the inequality

$$g_i(x(t+\theta)) \ge -\beta(c_2 - x_i(t_{n-1})) \tag{3.12}$$

holds. In fact, for all $t \in [u_n, s_n]$, $\theta \in [-1, 0]$, $x(t+\theta)$ must belong to one of the following cases: e (I) $x(t+\theta) \in B$,

(II) $x_i(t+\theta) \ge c_1 + c$,

(III) $x_i(t+\theta) \le c_2 - c$.

If (I) is true, then $g_i(x(t+\theta)) \ge -K \ge -L \ge -\beta(c_2 - x_i(t_{n-1}))$. For case (II), we have $g_i(x(t+\theta)) \ge 0 \ge -\beta(c_2 - x_i(t_{n-1}))$ from assumption $(\mathrm{H})_2$. By hypothesis $(\mathrm{H})_4$, case (III) implies

$$g_i(x(t+\theta)) \ge -\beta(c_2 - x_i(t+\theta)). \tag{3.13}$$

From (3.3) and (3.7), we have

$$x_i(t+\theta) \ge x_i(t_{n-1}),$$

and hence

$$g_i(x(t+\theta)) \ge -\beta(c_2 - x_i(t_{n-1})).$$

Here we used the monotonicity of $\beta(r)$ and (3.13). Therefore, we have proved that for all $t \in [u_n, s_n]$ and $\theta \in [-1, 0]$, the inequality (3.12) holds. In other words, our assertion is true.

From (3.12) and the assumption on $b$, we have

$$
\begin{aligned}
x_i(s_n) - x_i(u_n) &= \int_{u_n}^{s_n} x_t'(d)\,dt = -\int_{u_n}^{s_n}\int_{-1}^{0} b(\theta) g_i(x(t+\theta))\,d\theta\,dt \\
&\le \int_{u_n}^{s_n}\int_{-1}^{0} b(\theta)\beta(c_2 - x_i(t_{n-1}))\,d\theta\,dt \\
&= \beta(c_2 - x_i(t_{n-1}))\int_{u_n}^{s_n}\int_{-1}^{0} b(\theta)\,d\theta\,dt \\
&= \beta(c_2 - x_i(t_{n-1}))(s_n - u_n) \le \beta(c_2 - x_i(t_{n-1}))\,.
\end{aligned}
\tag{3.14}
$$

By a similar argument, we are able to obtain that for all $t \in [v_{n-1}, t_{n-1}]$ and $\theta \in [-1, 0]$, the inequality

$$g_i(x(t+\theta)) \le \alpha(x_i(s_{n-1}) - c)$$

holds, and hence we have

$$
\begin{aligned}
x_i(t_{n-1}) - x_i(v_{n-1}) &= \int_{v_{n-1}}^{t_{n-1}} x_i'(t)\,dt = -\int_{v_{n-1}}^{t_{n-1}}\int_{-1}^{0} b(\theta) g_i(x(t+\theta))\,d\theta\,dt \\
&\ge -\alpha(x_i(s_{n-1}) - c_1)\,.
\end{aligned}
\tag{3.15}
$$

Here we also used the fact that $t_{n-1} - v_{n-1} \le 1$ and $\int_{-1}^{0} b(\theta)\,d\theta = 1$. From (3.5), (3.14), and (3.15), we obtain

$$x_i(s_n) - c_1 \le \beta(c_2 - x_i(t_{n-1})) \tag{3.16}$$

and

$$c_2 - x_i(t_{n-1}) \le \alpha(x_i(s_{n-1}) - c_1)\,. \tag{3.17}$$

Substituting (3.17) into (3.16), we have

$$x_i(s_n) - c_1 \le \beta(\alpha(x_i(s_{n-1}) - c_1)) \le x_i(s_{n-1}) - c_1\,. \tag{3.18}$$

Here we use the monotonicity of $\beta(r)$, (3.11), and the fact that $\beta(\alpha(r)) \le r$ for $r > G$. The inequality (3.18) is a contradiction to (3.2) and (3.3). The theorem is proved.

## References


1. W. K. Ergen, *Kinetics of the circulating-fuel nuclear reactor*, J. Appl. Phys. **25** (1954), 702–711.
2. S. M. Kuen and K. P. Rybakowski, *Boundedness of solutions of a system of integro-differential equations*, J. Math. Anal. Appl. **112** (1985), 378–390.
3. J. J. Levin, *Boundedness and oscillation of some Volterra and delay equations*, J. Differential Equations **5** (1969), 369–398.
4. ——, *The asymptotic behavior of the solution of a Volterra equation*, Proc. Amer. Math. Soc. **14** (1963), 534–541.
5. ——, *The qualitative behavior of a nonlinear Volterra equation*, Proc. Amer. Math. Soc. **16** (1965), 711–718.
6. J. J. Levin and J. A. Nohel, *On a nonlinear delay equation*, J. Math. Anal. Appl. **8** (1964), 31–44.

7. J. A. Nohel, *Some problems in nonlinear Volterra integral equations*, Bull. Amer. Math. Soc. **68** (1962), 323–329.

8. H. B. Smets, *A general property of boundedness for the power of some stable and unstable nuclear reactors*, Nukleonik **2** (1960), 44–45.

9. ——, *Transitoires bornés pour certains systèmes à boucle unique contenant un élément non-linéaire*, Automatisme **10** (1965), 459–464.

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