BOUNDENESS OF SOLUTIONS TO FUNCTIONAL INTEGRO-DIFFERENTIAL EQUATIONS

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Abstract. In this paper we discuss the boundedness of the solutions of a class of functional integro-differential equations. We give an example to answer a question posed by S. M. Kuen and K. P. Rybakowski, and we establish a sufficient condition, which is more general than the condition they obtained.

1. Introduction

The scalar equation

\[ x'(t) = - \int_{-1}^{0} b(\theta) g(x(t + \theta)) \, d\theta + f(t) \]

occurs frequently as a model in mathematical biology and the physical sciences. For example, it is a good model in one-dimensional viscoelasticity in which \( x \) is the strain and \( b \) is the relaxation function. It is also a model for a circulating fuel nuclear reactor in which \( x \) is the neutron density (see [1]).

Equation (1.1) has been studied extensively by a number of authors, especially by Levin and Nohel (see, e.g. [3, 4, 6, 7]). In these papers authors studied the asymptotic behavior, boundedness, and oscillation of the solution to (1.1). Smets [8, 9] also studied the boundedness of the solutions of (1.1). In these earlier results, to guarantee the boundedness of solutions, all the authors supposed that \( g(x) \) satisfies the following hypothesis:

\[
\begin{align*}
(1.2) \quad x g(x) > 0 & \quad (x \neq 0) \quad g(x) \in \mathcal{C}(-\infty, +\infty), \\
(1.3) \quad g(x) \geq -\lambda & \quad \text{or} \quad g(x) \leq \lambda, \quad 0 < \lambda < \infty, \quad -\infty < x < +\infty.
\end{align*}
\]

In other words, \( g(x) \) must have a one-sided bound except the sign condition (1.2). S. M. Kuen and K. P. Rybakowski [2] discussed the boundedness of the solutions to the pure delay equation

\[ x'(t) = - \int_{-1}^{0} b(\theta) g(x(t + \theta)) \, d\theta, \]

where \( b: [-1, 0] \to \mathbb{R} \), \( g: \mathbb{R}^{n} \to \mathbb{R}^{n} \), \( x \in \mathbb{R}^{n} \). They obtained the following result.

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Theorem A. Suppose that \( g : \mathbb{R}^n \to \mathbb{R}^n \) is locally Lipschitzian, and \( b : [-1, 0] \to \mathbb{R} \) is continuous, nonnegative, and not identically zero. Let \( i \in \{1, 2, \ldots, n\} \). Assume that \( M = \int_{-1}^0 b(\theta) d\theta = 1 \), and that there are constants \( a > 0, c \geq 0, \beta, c_1, c_2 \in \mathbb{R}, c_2 \leq c_1 \) such that

(a) \( g_i(x) \geq 0 \) whenever \( x \in \mathbb{R}^n \) and \( c_1 \leq x_i \),
(b) \( g_i(x) \leq a(x_i - c_1 - \beta) \) whenever \( x \in \mathbb{R}^n \) and \( c_1 + c \leq x_i \),
(c) \( g_i(x) \geq (1/a)(x_i - c_2 - a\beta) \) whenever \( x \in \mathbb{R}^n \) and \( x_i \leq c_2 - c \).

If \( x = x(\phi)(t) : 0 < t < \omega_\phi \) is a solution of (1.4) with \( x_j \) bounded for \( j \neq i \), then \( x = x(\phi)(t) : 0 \leq t < \omega_\phi \) is bounded, where \( g_i(x) \) and \( x_i \) are the \( i \)th component of \( g(x) \) and \( x \), respectively.

In Theorem A condition (a) is equivalent to (1.2), and condition (b) shows that \( |g_i(x)| \) is allowed to be unbounded as long as it satisfies a special linear growth condition.

In their paper the authors wrote “The question arises whether the sign condition (assumption (a) of Theorem A) above suffices to guarantee the boundedness of \( x_i \) (if \( x_j \) is bounded for \( j \neq i \))”, and they also wrote “This seems to be an open question. In our proofs we either assume that \( g_i \) is bounded or else that \( g_i \) satisfies a rather special linear growth condition. It is not clear how to relax these hypothesis.”

In this paper we give an example to illustrate that the answer to the question in [2] is not, (Levin once gave a similar example in [3]), and condition (b) is relaxed. In other words, \( |g_i(x)| \) is not required to satisfy a special linear growth condition.

2. Counterexample and lemmas

Consider the scalar equation

\[
(2.1)_a \quad x(t) = -\int_{-1}^0 ax(t + \theta) d\theta,
\]

where \( a > 1 \) is some constant.

Evidently, equation (2.1)_a satisfies the conditions of Theorem A except condition (b) for any \( a > 1 \). Now we show that there at least exists one constant \( a > 1 \) such that (2.1)_a has an unbounded solution. For this we consider the characteristic equation of equation (2.1)_a

\[
\lambda - a(e^{-\lambda} - 1)/\lambda = 0
\]

or equivalently

\[
(2.2)_a \quad \lambda^2 - a(e^{-\lambda} - 1) = 0.
\]

It is easy to prove that there is at least one constant \( a \in (1, \frac{9}{4}\pi^2) \) such that \( \lambda(a) = (\frac{9}{4}\pi^2 - a)^{1/2} + \frac{3}{2}\pi i \) is a root of (2.2)_a with positive real part, we omit the detailed argument. This implies that there at least exists one unbounded solution of equation (2.1)_a. Hence the sign condition (a) of Theorem A is not enough to guarantee the boundedness of the solutions of equation (1.4).

Now we given the following assumptions:

(H) \( b : [-1, 0] \to \mathbb{R} \) is continuous, nonnegative, not identically zero.
boundedness of solutions

\( (H)_2 \) \( g : \mathbb{R}^n \rightarrow \mathbb{R}^n \) is locally Lipschitzian. Let \( i \in \{1, 2, \ldots, n\} \). There are constants \( c_1, c_2 \in \mathbb{R} \), \( c_1 \geq c_2 \) such that \( g_i(x) \geq 0 \) whenever \( x \in \mathbb{R}^n \) and \( x_i \geq c_1 \) and \( g_i(x) \leq 0 \) whenever \( x \in \mathbb{R}^n \) and \( x_i \leq c_2 \), where \( g_i \) and \( x_i \) are the \( i \)th component of \( g(x) \) and \( x \), respectively.

**Lemma 1.** Suppose that \( x : [a, b) \rightarrow \mathbb{R} \) is a continuous function. If

\[
\sup_{a \leq t < b} x(t) = +\infty \quad \left( \text{or} \quad \inf_{a \leq t < b} x(t) = -\infty \right),
\]

then there must be a sequence \( \{t_n\} \subset [a, b) \) such that

1. \( t_n > t_{n-1} \) for each \( n = 1, 2, \ldots \) and \( t_n \rightarrow b^- \) as \( n \rightarrow +\infty \);
2. \( x(t) < x(t_n) \) (or \( x(t) > x(t_n) \)) for all \( t \in [a, t_n) \);
3. \( x(t_n) \rightarrow +\infty \) (or \( x(t_n) \rightarrow -\infty \)) as \( n \rightarrow +\infty \).

We omit the proof of Lemma 1 because it is not difficult.

**Lemma 2.** Let \( b \) and \( g \) satisfy conditions \( (H)_1 \) and \( (H)_2 \). Suppose that \( x(\phi)(t) : 0 \leq t \leq \omega_\phi \) is a solution of equation (1.4) and that \( s, t \) are arbitrary two points in the interval \( [0, \omega_\phi) \) with \( s > t \). If the inequality \( c_1 \leq x_i(u) < x_i(s) \) (or \( x_i(s) < x_i(u) \leq c_2 \)) holds for all \( u \in [t, s) \), then we have \( s - t \leq 1 \).

**Proof.** Suppose that Lemma 2 is not true; that is, there are two points \( s, t \in [0, \omega_\phi) \) with \( s > t \) such that \( c_1 \leq x_i(u) < x_i(s) \) (or \( c_2 > x_i(u) > x_i(s) \)) for all \( u \in [t, s) \) and \( s - t > 1 \). From the continuity of \( x_i(t) \), we obtain \( c_1 \leq x_i(u) < x_i(s) \) (or \( x_i(s) < x_i(u) \leq c_2 \)) for some \( \varepsilon > 0 \) and all \( u \in [s - 1 - \varepsilon, s) \), therefore, we have

\[
x'_i(u) = - \int_{-1}^{0} b(\theta)g_i(x(u + \theta)) \, d\theta \leq 0
\]

(\text{or} \( x'_i(u) = - \int_{-1}^{0} b(\theta)g_i(x(u + \theta)) \, d\theta \geq 0 \))

for all \( u \in [s - \varepsilon, s) \). Here we used assumption \( (H)_2 \) and the fact that \( x_i(u) \geq c_1 \) on \( [s - \varepsilon - 1, s) \). Hence \( x_i(s - \varepsilon) \geq x_i(s) \) (or \( x_i(s - \varepsilon) \leq x_i(s) \)), which is a contradiction. This completes the proof of Lemma 2.

**Lemma 3.** Suppose that conditions \( (H)_1 \) and \( (H)_2 \) are satisfied. Let \( x(t) = x(\phi)(t) : 0 \leq t < \omega_\phi \) be a solution of equation (1.4) with bounded components \( x_j(t) : 0 \leq t < \omega_\phi \) for \( j \neq i \). If \( x_i(t) : 0 \leq t < \omega_\phi \) is unbounded, then \( x_i(t) : 0 \leq t < \omega_\phi \) must oscillate between \( -\infty \) and \( +\infty \), in other words

\[
\sup_{0 \leq t < \omega_\phi} x_i(t) = +\infty \quad \text{and} \quad \inf_{0 \leq t < \omega_\phi} x_i(t) = -\infty.
\]

**Proof.** Suppose that Lemma 3 is not true, i.e. that \( \sup_{0 \leq t < \omega_\phi} x_i(t) = +\infty \) but \( \inf_{0 \leq t < \omega_\phi} x_i(t) > -\infty \). By the continuity of \( g_i(x) \) and the boundedness of \( x_j(t) \) for \( j \neq i \), there is a constant \( M > 0 \) such that \( g_i(x(t)) \geq -M \) for all \( t \in [-1, \omega_\phi) \). Thus we have

\[
x'_i(t) = - \int_{-1}^{0} b(\theta)g_i(x(t + \theta)) \, d\theta \leq M \int_{-1}^{0} b(\theta) \, d\theta = M^* \quad \text{for all} \ t \in [0, \omega_\phi).
\]

This implies

\[
(2.3) \quad x_i(s) - x_i(t) = \int_{t}^{s} x'_i(u) \, du \leq M^* (s - t) \quad \text{for any} \ s, t \in [0, \omega_\phi).
\]
Assume first that \( \omega_\phi < +\infty \). By (2.3) we have \( \sup_{0 \leq t < \omega_\phi} x_{i}(t) \leq M^* \omega_\phi < +\infty \), which is a contradiction to our assumption. Hence \( \omega_\phi = +\infty \). By our assumption and Lemma 1, there exists a sequence \( \{t_n\} \subset [0, +\infty) \) such that

\[
t_n < t_{n+1} \quad \text{for } n = 1, 2 \ldots \text{ and } t_n \to +\infty \text{ as } n \to +\infty;
\]

\[
x_{i}(t) < x_{i}(t_n) \quad \text{for all } t \in [-1, t_n),
\]

\[
x_{i}(t_n) \to +\infty \quad \text{as } n \to +\infty.
\]

Therefore, we may pick a positive sufficiently large integer \( n \) such that

\[
(2.4) \quad x_{i}(t_n) > c_1 + 2M^* \quad \text{and} \quad t_n - 1 > 1.
\]

Let \( s = t_n \) in (2.3), we obtain

\[
(2.5) \quad x_{i}(t_n) - x_{i}(t) \leq M^*(t_n - t) \quad \text{for all } t \in [t_n - 1, t_n].
\]

From (2.4) and (2.5), we have

\[
x_{i}(t) \geq x_{i}(t_n) - M^*(t_n - t) \geq c_1 + 2M^* - M^* > c_1 \quad \text{for all } t \in [t_n - 1, t_n].
\]

By the continuity of \( x_{i}(t) \) on the interval \([-1, +\infty)\) for some \( \varepsilon > 0 \), we obtain

\[
x_{i}(t) \geq c_1 \quad \text{for all } t \in [t_n - 1 - \varepsilon, t_n].
\]

On the other hand, we have

\[
x_{i}(t) \leq x_{i}(t_n) \quad \text{for all } t \in [t_n - 1 - \varepsilon, t_n]
\]

by the properties of \( \{t_n\} \). Hence, we obtain

\[
c_1 \leq x_{i}(t) \leq x_{i}(t_n) \quad \text{for all } t \in [t_n - 1 - \varepsilon, t_n].
\]

This contradicts Lemma 2. The proof of the lemma is completed.

### 3. Main results

We give the following hypotheses:

\[(H)_3 \quad \text{There is a constant } K > 0 \text{ such that}
\]

\[
g_{i}(x) \leq K \quad \text{(or } g_{i}(x) \geq -K) \quad \text{whenever } x \in \mathbb{R}^n;
\]

\[(H)_4 \quad \text{Let } (H)_2 \text{ be satisfied. Assume that there is a constant } c > 0, \text{ two}
\]

\[
\text{strictly increasing functions } \alpha, \beta: \mathbb{R}^+ \to \mathbb{R}^+ = [0, +\infty) \text{ such that}
\]

\[(a) \quad \text{there is a large constant } G > 0 \text{ such that}
\]

\[
\alpha(\beta(r)) \leq r \quad \text{or} \quad \beta(\alpha(r)) \leq r \quad \text{for all } r > G;
\]

\[(b) \quad g_{i}(x) \leq \alpha(x_i - c_1) \quad \text{whenever } x \in \mathbb{R}^n \text{ and } x_i \geq c_1 + c;
\]

\[(c) \quad g_{i}(x) \geq -\beta(c_2 - x_i) \quad \text{whenever } x \in \mathbb{R}^n \text{ and } x_i \leq c_2 - c.
\]

**Theorem 1.** Let the hypotheses \((H)_1\), \((H)_2\), and \((H)_3\) be satisfied. If \( x(t) = x(\phi)(t) : 0 \leq t < \omega_{\phi} \) is a solution of equation (1.4) with bounded components \( x_{j}(t) : 0 \leq t < \omega_{\phi} \) for \( j \neq i \), then \( x_{i}(t) : 0 \leq t < \omega_{\phi} \) is bounded.

**Proof.** From Lemma 3 we only prove that \( x_{i}(t) : 0 \leq t < \omega_{\phi} \) is bounded from above. Suppose that \( x_{i}(t) : 0 \leq t < \omega_{\phi} \) is not bounded from above, i.e. that \( \sup_{0 \leq t < \omega_{\phi}} x(t) = +\infty \). By the hypothesis \((H)_3\), we have

\[
x_{i}'(t) = - \int_{-1}^{0} b(\theta) g_{i}(x(t + \theta)) d\theta \leq K^* \quad \text{for all } t \in [0, \omega_{\phi}],
\]

\[
K^* = K \int_{-1}^{0} b(\theta) d\theta.
\]
This implies
\[ x_i(s) - x_i(t) = \int_t^s x'_i(u) \, du \leq k^*(s - t) \quad \text{for any } s, t \in [0, \omega_\phi). \]

The remainder of the proof is the same as the proof of Lemma 3, hence we omit it.

**Theorem 2.** Let the hypotheses \((H)_1, (H)_2, (H)_3\) be satisfied. Assume that \(\int_0^\infty b(\theta) \, d\theta = 1\). If \(x(t) = x(\phi)(t); 0 \leq t < \omega_\phi\) is a solution of equation (1.4) with bounded components \(x_j(t); 0 \leq t < \omega_\phi\) for \(j \neq i\), then \(x_i(t); 0 \leq t < \omega_\phi\) is bounded.

**Proof.** Suppose that Theorem 2 is not true, i.e. that \(x_i(t); 0 \leq t < \omega_\phi\) is unbounded. From Lemma 3 we have
\[ \sup_{0 \leq t < \omega_\phi} x_i(t) = -\inf_{0 \leq t < \omega_\phi} x_i(t) = +\infty. \]

By Lemma 1, there are two sequences \(\{s_n\}, \{t_n\} \subset [0, \omega_\phi)\) such that:
\[ s_n > s_{n-1}, t_n > t_{n-1}, \quad \text{for } n = 1, 2, \ldots \quad \text{and} \]
\[ s_n \to \omega_\phi, \quad t_n \to \omega_\phi, \quad \text{as } n \to +\infty; \]
\[ x_i(t) < x_i(s_n) \quad \text{for all } t \in [0, s_n) \quad \text{and} \]
\[ x_i(t) > x_i(t_n) \quad \text{for all } t \in [0, t_n), \quad n = 1, 2, \ldots; \]
\[ x_i(t_n) \to -\infty, \quad x_i(s_n) \to +\infty \quad \text{as } n \to +\infty. \]

We can choose \(\{s_n\}\) and \(\{t_n\}\). If it is necessary, we can take their subsequences such that
\[ s_n < t_n < s_{n+1}, \quad n = 1, 2, \ldots. \]

According to the continuity of \(x_i(t)\) and (3.1) we see that there are also two sequences \(\{u_n\}, \{v_n\} \subset [0, \omega_\phi)\) such that
\[ x_i(u_n) = c_1, \quad x_i(v_n) = c_2 \quad \text{and} \quad u_n < s_n < v_n < t_n < u_{n+1}, \]
\[ n = 1, 2, \ldots. \]

(See Figure 1). Evidently, we can choose the subsequences of these sequences, which still are denoted by \(\{s_n\}, \{t_n\}, \{u_n\}, \{v_n\}\), such that
\[ x_i(s_n) \geq x_i(t) \geq x_i(t_n) \quad \text{whenever } t \in [s_n, t_n], \]
\[ x_i(t_n) \leq x_i(t) \leq x_i(s_{n+1}) \quad \text{whenever } t \in [t_n, s_{n+1}], \]
\[ c_1 \leq x_i(t) < x_i(s_n) \quad \text{whenever } t \in [u_n, s_n], \]
\[ x_i(t_n) < x_i(t) \leq c_2 \quad \text{whenever } t \in [v_n, t_n]. \]

Inequalities (3.8), (3.9), and Lemma 2 imply that
\[ s_n - u_n \leq 1, \quad t_n - v_n \leq 1, \quad n = 1, 2, 3, \ldots. \]

By our assumption, there is a constant \(T > 0\) such that
\[ |x_j(t)| \leq T \quad \text{for all } -1 \leq t < \omega_\phi, \quad j \neq i. \]
Let $B = \{ x \in \mathbb{R}^n | |x_j| \leq T, j \neq i, c_2 - c \leq x_i \leq c_1 + c \} \subset \mathbb{R}^n$. Let $K = \max \{ \sup_{x \in B} g_i(x), \alpha(c), \beta(c) \}$ and $L = \max \{ K, G, \sup_{-\theta \leq \theta \leq 0} |\phi_i(\theta)| \}$, where $G$ is the constant in (H)$_4$, and $\phi_i$ is the $i$th component of $\phi$, which is the initial function.

We suppose that $\alpha(r) \to +\infty$ and $\beta(r) \to +\infty$, otherwise it is not w.l.o.g. (If not, $\alpha(r)$ or $\beta(r)$ is bounded, then $g_i(x)$ is bounded from above or below, therefore $x_i(t)$ is bounded from Theorem 1), and that the inequality $\beta(\alpha(r)) < r$ is satisfied. (If the inequality $\alpha(\beta(r)) \leq r$ is true, the proof is similar.)

Now, we choose a sufficiently large integer $n$ such that
\begin{align*}
\alpha(x_i(s_n) - c_1) > L, & \quad \beta(c_2 - x_i(t_{n-1})) > L; \\
x_i(s_n) > L, & \quad x_i(t_{n-1}) < -L, \quad x_i(s_{n-1}) - c_1 > L.
\end{align*}
Here we use the assumption of (H)$_4$ and (3.4). We assert that for all $t \in [u_n, s_n]$, $\theta \in [-1, 0]$, the inequality
\begin{equation}
(3.12) \quad g_i(x(t + \theta)) \geq -\beta(c_2 - x_i(t_{n-1}))
\end{equation}
holds. In fact, for all $t \in [u_n, s_n]$, $\theta \in [-1, 0]$, $x(t + \theta)$ must belong to one of the following cases: 

(I) $x(t + \theta) \in B$, 

(II) $x_i(t + \theta) \geq c_1 + c$, 

(III) $x_i(t + \theta) \leq c_2 - c$.

If (I) is true, then $g_i(x(t + \theta)) \geq -K \geq -L \geq -\beta(c_2 - x_i(t_{n-1}))$. For case (II), we have $g_i(x(t + \theta)) \geq 0 \geq -\beta(c_2 - x_i(t_{n-1}))$ from assumption (H)$_2$. By hypothesis (H)$_4$, case (III) implies
\begin{equation}
(3.13) \quad g_i(x(t + \theta)) \geq -\beta(c_2 - x_i(t + \theta)).
\end{equation}
From (3.3) and (3.7), we have
\begin{equation*}
x_i(t + \theta) \geq x_i(t_{n-1}),
\end{equation*}
and hence
\begin{equation*}
g_i(x(t + \theta)) \geq -\beta(c_2 - x_i(t_{n-1})).
\end{equation*}
Here we used the monotonicity of $\beta(r)$ and (3.13). Therefore, we have proved that for all $t \in [u_n, s_n]$ and $\theta \in [-1, 0]$, the inequality (3.12) holds. In other words, our assertion is true.
From (3.12) and the assumption on $b$, we have

$$X_i(S_n) - X_i(U_n) = \int_{U_n}^{S_n} x'_i(d) \, dt = -\int_{U_n}^{S_n} \int_{-1}^{0} b(\theta) g_i(x(t + \theta)) \, d\theta \, dt$$

(3.14)

$$\leq \int_{U_n}^{S_n} \int_{-1}^{0} b(\theta) \beta(c_2 - x_i(t_{n-1})) \, d\theta \, dt$$

$$= \beta(c_2 - x_i(t_{n-1})) \int_{U_n}^{S_n} \int_{-1}^{0} b(\theta) \, d\theta \, dt$$

$$= \beta(c_2 - x_i(t_{n-1}))(S_n - U_n) \leq \beta(c_2 - x_i(t_{n-1})).$$

By a similar argument, we are able to obtain that for all $t \in [v_{n-1}, t_{n-1}]$ and $\theta \in [-1, 0]$, the inequality

$$g_i(x(t + \theta)) \leq \alpha(x_i(S_{n-1}) - c)$$

holds, and hence we have

(3.15)

$$x_i(t_{n-1}) - x_i(v_{n-1}) = \int_{v_{n-1}}^{t_{n-1}} x'_i(t) \, dt = -\int_{v_{n-1}}^{t_{n-1}} \int_{-1}^{0} b(\theta) g_i(x(t + \theta)) \, d\theta \, dt$$

$$\geq -\alpha(x_i(S_{n-1}) - c_1).$$

Here we also used the fact that $t_{n-1} - v_{n-1} \leq 1$ and $\int_{-1}^{0} b(\theta) \, d\theta = 1$. From (3.5), (3.14), and (3.15), we obtain

(3.16) $$x_i(S_n) - c_1 \leq \beta(c_2 - x_i(t_{n-1}))$$

and

(3.17) $$c_2 - x_i(t_{n-1}) \leq \alpha(x_i(S_{n-1}) - c_1).$$

Substituting (3.17) into (3.16), we have

(3.18) $$x_i(S_n) - c_1 \leq \beta(\alpha(x_i(S_{n-1}) - c_1)) \leq x_i(S_{n-1}) - c_1.$$

Here we use the monotonicity of $\beta(r)$, (3.11), and the fact that $\beta(\alpha(r)) \leq r$ for $r > G$. The inequality (3.18) is a contradiction to (3.2) and (3.3). The theorem is proved.

References


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