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THE MINIMAL SUPPORT FOR A CONTINUOUS FUNCTIONAL ON A FUNCTION SPACE

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Dedicated to Professor Ryosuke Nakagawa on his sixtieth birthday

ABSTRACT. Let $C_p(X)$ be the function space with the pointwise convergent topology over a Tychonoff space X and ξ a continuous real-valued function on $C_p(X)$. A closed subset S of X is called a *support* for ξ if $\xi(f) = \xi(g)$ holds for any pair (f, g) of elements of $C_p(X)$ such that $f_{|S} = g_{|S}$. It is proven that the minimal support for any real-valued continuous function on the space $C_p(X)$ exists.

0. INTRODUCTION

In this paper we assume that all spaces are Tychonoff. Let $C_p(X)$ be the space of all real-valued continuous functions on X with the topology of pointwise convergence. We call a real-valued function on $C_p(X)$ a *functional*. For a family $\mathscr{A}$ of sets, we write $\bigcap\mathscr{A} = \bigcap\{A : A \in \mathscr{A}\}$. For a function f on X and a subset M of X, the restriction of f to M is denoted by $f_{|M}$. $\mathbb{R}$ and $\mathbb{N}$ denote the real line and the set of natural numbers respectively. Other undefined terms can be found in [E].

Let ξ be a continuous functional on $C_p(X)$. A subset S of X is called a *quasi-support* for ξ if $\xi(f) = \xi(g)$ holds for any pair (f, g) of elements of $C_p(X)$ such that $f_{|S} = g_{|S}$. And if a quasi-support S is closed, then S is called a *support*. We say that a support S is *minimal* if for every support T for ξ that is contained in S we have $S = T$. It is known [A] that a linear continuous functional ξ on $C_p(X)$ is expressed as a linear combination of a finite subset of X. That is $\xi = \sum_{i=1}^{n} \alpha_i x_i$ for some finite subset $\{x_1, \ldots, x_n\}$ of X and numbers $\{\alpha_1, \ldots, \alpha_n\}$. In this case the set $\{x_1, \ldots, x_n\}$ is clearly the minimal support for ξ. Our purpose of this paper is to give a general result for any continuous functional on $C_p(X)$:

Theorem 0. *There exists the minimal support S for any continuous functional on $C_p(X)$ and S is a separable subspace of X.*

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1. Proof of Theorem 0

We write

$$\langle f, \Gamma, \varepsilon\rangle = \{g \in C_p(X) \colon |f(x) - g(x)| < \varepsilon \text{ for any } x \in \Gamma\}$$

for a continuous function f on X, a finite subset Γ of X, and a positive number ε. $\pi_M \colon C_p(X) \to C_p(M)$ denotes the restriction map from X to a subspace M. It is known [A] that if M is a closed subset of X then π_M is an open map onto $\pi_M(C_p(X))$. Let $\operatorname{Supp}\xi$ be the set of all supports for a continuous functional ξ on $C_p(X)$. To prove the theorem, we show some results. We may assume that ξ is not constant.

Lemma 1. *For any pair* (S, T) *of elements of* $\operatorname{Supp}\xi$, $S \cap T$ *belongs to* $\operatorname{Supp}\xi$.

Proof of Lemma 1. Put $A = S \cap T$. Assume that there exist continuous functions f and g on X such that $\xi(f) \neq \xi(g)$ and $f_{|A} = g_{|A}$. It is easily checked the following equation

$$\pi_S^{-1}(\pi_S(\xi^{-1}(\xi(g)))) = \xi^{-1}(\xi(g))$$

for the restriction map π_S. From this, we have that $f_{|S}$ does not belong to $\pi_S(\xi^{-1}(\xi(g)))$. Since S is closed, π_S is open. We have that $\pi_S(\xi^{-1}(\xi(g)))$ is closed in $\pi_S(C_p(X))$. There exist a finite subset Γ of S and a positive number ε such that

$$\langle f, \Gamma, \varepsilon\rangle \cap \xi^{-1}(\xi(g)) = \varnothing.$$

We consider the following three cases.

Case I. If $\Gamma \cap T = \varnothing$ holds, then there exists a continuous function h on X such that $h_{|\Gamma} = f_{|\Gamma}$ and $h_{|T} = g_{|T}$. We have that $h \in \langle f, \Gamma, \varepsilon\rangle$ and $\xi(h) = \xi(g)$ because T is a support for ξ. This is a contradiction.

In the following two cases we assume that $\Gamma \cap T \neq \varnothing$ and put $\Gamma_1 = \Gamma \cap T$ and $\Gamma_2 = \Gamma \backslash \Gamma_1$.

Case II. If $\langle f, \Gamma_1, \varepsilon\rangle \cap \xi^{-1}(\xi(g)) \neq \varnothing$ holds, then we can take a function $h \in \langle f, \Gamma_1, \varepsilon\rangle \cap \xi^{-1}(\xi(g))$. There exists a continuous function $\tilde{h}$ on X such that $\tilde{h}_{|T} = h_{|T}$ and $\tilde{h}_{|\Gamma_2} = f_{|\Gamma_2}$. Here we have that $\tilde{h} \in \langle f, \Gamma, \varepsilon\rangle$ and $\xi(\tilde{h}) = \xi(h) = \xi(g)$. This is a contradiction.

Case III. If $\langle f, \Gamma_1, \varepsilon\rangle \cap \xi^{-1}(\xi(g)) = \varnothing$ holds, then we have $\pi_A(\langle f, \Gamma_1, \varepsilon\rangle) \cap \pi_A(\xi^{-1}(\xi(g))) = \varnothing$ for the restriction map π_A because Γ_1 is contained in A. However, by our assumption, we have

$$\pi_A(f) = \pi_A(g) \in \pi_A(\langle f, \Gamma_1, \varepsilon\rangle) \cap \pi_A(\xi^{-1}(\xi(g))).$$

This is a contradiction. Lemma 1 is proved.

Lemma 2. $\bigcap \operatorname{Supp}\xi$ *is a support for* ξ.

Proof of Lemma 2. Put $S = \bigcap \operatorname{Supp}\xi$. Assume that there exist continuous functions f and g on X such that $\xi(f) \neq \xi(g)$ and $f_{|S} = g_{|S}$. Since f does not belong to a closed set $\xi^{-1}(\xi(g))$ in $C_p(X)$, there exist a finite subset Γ of X and a positive number ε such that $\langle f, \Gamma, \varepsilon\rangle \cap \xi^{-1}(\xi(g)) = \varnothing$. Using Lemma 1, by the definition of S, we can find an element T of $\operatorname{Supp}\xi$ such that $T \cap (\Gamma \backslash S) = \varnothing$. Consider three cases in the same way as we did in the proof of Lemma 1. We have a contradiction in any case. The proof of the lemma is completed.

Lemma 3. $\bigcap \operatorname{Supp} \xi$ *is a separable subspace of* X.

Proof of Lemma 3. Put $S = \bigcap \operatorname{Supp} \xi$. Since S is a support for ξ (Lemma 2), there exists a functional $\tilde{\xi}$ on $\pi_S(C_p(X))$ such that $\xi = \tilde{\xi} \circ \pi_S$. Since S is closed, π_S is open. We have that $\tilde{\xi}$ is continuous. And the cellurality of the space $\pi_S(C_p(X))$ is countable (See [A]). There exists a countable quasi-support A for $\tilde{\xi}$ in S. Obviously $\mathrm{cl}_X A$ is a support for ξ. So $S = \mathrm{cl}_X A$ holds by the minimality of the support S (Lemma 2). The lemma is proved.

2. Remarks and comments

I. For any countable subset A of X we can find a continuous functional ξ on $C_p(X)$ such that $\bigcap \operatorname{Supp} \xi = \mathrm{cl}\, A$. To do this, we assume that A is indexed as $A = \{x_n : n \in \mathbb{N}\}$. For every $f \in C_p(X)$, we put

$$\xi(f) = \sum \{2^{-n} r(f(x_n)) : n \in \mathbb{N}\},$$

where $r : \mathbb{R} \to [0, 1]$ is a continuous function defined by

$$r(\alpha) = \begin{cases} 1 & (1 \le \alpha), \\ \alpha & (0 \le \alpha \le 1), \\ 0 & (\alpha \le 0). \end{cases}$$

II. Using the same idea in the proof of our theorem, we can prove the following lemma.

Lemma 4. *Let* $\mathscr{F}$ *be a nonempty proper closed subset of* $C_p(X)$. *We put* $\operatorname{Supp} \mathscr{F} = \{S \subset X : S$ *is closed in* X, $\pi_S^{-1}(\pi_S(\mathscr{F})) = \mathscr{F}\}$. *Then the set* $\bigcap \operatorname{Supp} \mathscr{F}$ *belongs to* $\operatorname{Supp} \mathscr{F}$.

This lemma gives a result on the minimal support.

Theorem 5. *Let* ξ *be a nonconstant continuous functional on* $C_p(X)$. *For an* $\alpha \in \xi(C_p(X))$, *we put* $S_\alpha = \cap \operatorname{Supp} \xi^{-1}(\alpha)$. *Then we have* $\bigcap \operatorname{Supp} \xi = \mathrm{cl}(\bigcup \{S_\alpha : \alpha \in \xi(C_p(X))\})$.

Proof. We can easily show the following two claims.

(1) For any $\alpha \in \xi(C_p(X))$, $\bigcap \operatorname{Supp} \xi \in \operatorname{Supp} \xi^{-1}(\alpha)$ holds.

(2) $\bigcup \{S_\alpha : \alpha \in \xi(C_p(X))\}$ is a quasi-support for ξ.

We have $S_\alpha \subset \bigcap \operatorname{Supp} \xi$ for any $\alpha \in \xi(C_p(X))$ by (1), and $\bigcap \operatorname{Supp} \xi \subset \mathrm{cl}(\bigcup \{S_\alpha : \alpha \in \xi(C_p(X))\})$ by (2). The theorem is proved.

Added in proof. The author recently proved that the similar results hold for the compact-open topology but fail in general for the norm topology.

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