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ON GROUPS WITH A CENTRAL AUTOMORPHISM OF INFINITE ORDER

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ABSTRACT. It is shown that a group G, whose center has finite exponent, has a central automorphism of infinite order if and only if G has an infinite abelian direct factor. It is also shown that the group of central automorphisms of a nilpotent p-group of infinite exponent contains an uncountable torsionfree abelian subgroup

1. INTRODUCTION

In their paper [5] Menegazzo and Stonehewer show that, apart from a few obvious exceptions, a nilpotent p-group always has an outer automorphism of order p. They also observe that it is often easier, in the case of nilpotent p-groups, to construct automorphisms of finite order and, therefore, pose the question of which nilpotent p-groups have such an automorphism. A partial answer to this question is stated in [5] and an example of a nilpotent p-group that has no automorphism of infinite order is given.

The purpose of the current paper is to characterize those nilpotent p-groups G that have a central automorphism of infinite order. (Here an automorphism of G is called *central* if it acts trivially on the group modulo its center. We denote the group of central automorphisms of G by $\mathrm{Aut_c}G$.) There are two cases to consider. We show in §2 that if G is a nilpotent p-group of infinite exponent, then $\mathrm{Aut_c}G$ contains an uncountable torsionfree abelian subgroup. The arguments used here follow those of Buckley and Wiegold [1, Theorems 2.2, 2.6], for the most part. However, some additional results are needed, because the automorphisms constructed in [1] do not always have infinite order. In §3 we obtain our main result concerning central automorphisms of nilpotent p-groups. We show that if G is such a group of finite exponent, then G has a central automorphism of infinite order if and only if G has an infinite abelian direct factor. However, our result is a corollary to the following much more general theorem, which we prove in §3.

Theorem 3.1. *Suppose G is a group and $Z(G)$ has finite exponent. Then the following are equivalent.*

(i) *G has a central automorphism of infinite order.*

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(ii) *G has an infinite abelian direct factor.*
(iii) $\mathrm{Aut}_c G$ *contains an uncountable torsionfree abelian subgroup.*

Our notation is standard for the most part. For any group G, the commutator subgroup is G', the center is $Z(G)$, which we abbreviate to Z, and we write G_{ab} for G/G'. We use additive notation for abelian groups and, in particular, for subgroups of Z. For a prime p and natural number n, $Z[p^n]$ denotes the subgroup of elements of Z of order dividing p^n, and p^nZ denotes the subgroup of Z consisting of $\{p^n z | z \in Z\}$. We also recall that if $N \leq Z$ then $\mathrm{Hom}(G/N, N)$ can be identified with the subgroup of elements of $\mathrm{Aut}_c G$ that act trivially on both N and G/N.

2. Nilpotent p-groups of infinite exponent

Throughout this section we let G be a nilpotent p-group of infinite exponent. We refer the reader to [1] and [8] for standard facts concerning basic subgroups of nilpotent p-groups; recall that a subgroup B of the group G is called *basic* if $G' \leq B$ and B/G' is a basic subgroup of the abelian group G_{ab}. We require two preliminary results before proving the main result of this section.

Lemma 2.1. *Suppose B is a basic subgroup of G and $N \leq Z(G)$. Suppose $G = BN$ and $B \cap N$ has finite exponent. Then* $\mathrm{Aut}_c G$ *contains an uncountable torsionfree abelian subgroup.*

Proof. Let $Q = N/B \cap N$, so Q is a periodic radicable abelian group. Because $B \cap N$ has finite exponent, it follows that $\mathrm{Ext}(Q, B \cap N)$ has finite exponent m, say and clearly m is a power of p. If π is a p-adic integer of the form $1 + pm + s_2p^2 + s_3p^3 + \dots$, where each s_i is either 0 or m and infinitely many of them are nonzero, then multiplication by π induces an automorphism of Q and by [4, Lemma 52.1], π^* is multiplication by π on $\mathrm{Ext}(Q, B \cap N)$. Thus if $\Delta \in \mathrm{Ext}(Q, B \cap N)$,

$$\pi^*\Delta = \Delta \quad \text{because } m\Delta = 0.$$

On the other hand, because Q is divisible and $B \cap N$ has finite exponent, the theory of automorphisms of group extensions (see [7, p. 70]) shows that

$$C_{\mathrm{Aut}\, Q}(\Delta) \leq \mathrm{Aut}\, N.$$

Hence $\mathrm{Aut}\, N$ contains an uncountable, torsionfree abelian subgroup A, fixing $B \cap N$. Finally if we define, for each $\alpha \in A$,

$$(bn)\overline{\alpha} = b(n\alpha) \quad \text{for } b \in B,\ n \in N,$$

then $\overline{\alpha} \in \mathrm{Aut}\, G$ and it follows that $\mathrm{Aut}\, G$ contains a subgroup isomorphic to A. □

The Cartesian product of a family $\{G_i : i \in I\}$ is denoted by $\mathrm{Cr}_{i\in I} G_i$ and the direct sum by $\bigoplus_{i\in I} G_i$. For a natural number n, we let C_n be the cyclic group of order n, and for the prime p, we let C_{p^∞} be the quasicyclic p-group.

Lemma 2.2. *Suppose I is an index set and $B = \bigoplus_{i\in I} C_{p^{n(i)}}$. Suppose that C is a subgroup of B of infinite exponent. Then there is a direct sum decomposition $B = X \oplus Y$ such that both $X \cap C$ and $Y \cap C$ have infinite exponent.*

Proof. We may assume $D = \bigoplus_{i=1}^{\infty} C_{p^{n(i)}}$, where $n(i) \leq n(j)$ if $i \leq j$, is a subgroup of B such that $E = C \cap D$ has infinite exponent. Let $(c_{11}, \dots, c_{1,k_1},$

$0, \ldots)$, $(c_{21}, \ldots, c_{2,k_2}, 0, \ldots), \ldots$ be a sequence of elements of E with orders p^{m_1}, $p^{m_2}, \ldots$ where

$$p^{m_{i+1}} > p^{n(k_i)+m_i} \qquad i = 1, 2, \ldots.$$

Then $k_i < k_{i+1}$ for all i. Consider

$$p^{n(k_i)}(c_{i+1,1}, \ldots, c_{i+1,k_i}, \ldots, c_{i+1,k_{i+1}}, 0, \ldots).$$

This is an element of E of order $p^{m_{i+1}-n(k_i)} > p^{m_i}$. Set $I_1 = \{1, \ldots, k_1\}$ and $I_{j+1} = \{k_j + 1, \ldots, k_{j+1}\}$ for $j \geq 1$. We may find sets $J, K \subseteq I$ such that $J \cup K = I$, $J \cap K = \varnothing$, $\bigcup_{j=1}^{\infty} I_{2j} \subseteq J$, and $\bigcup_{j=1}^{\infty} I_{2j-1} \subseteq K$. Then set $X = \bigoplus_{i\in J} C_{p^{n(i)}}$ and $Y = \bigoplus_{i\in K} C_{p^{n(i)}}$. The subgroups X and Y have the desired properties. □

Theorem 2.3. *If G is a nilpotent p-group of infinite exponent then* $\mathrm{Aut_c}G$ *contains an uncountable torsionfree abelian subgroup.*

Proof. There are a number of cases to consider.

Case 1. Suppose G is nonreduced, and let B be a basic subgroup of G. Let N be a central subgroup of G such that $N \cong C_{p^\infty}$. If G/BN is nontrivial, then $\mathrm{Hom}(G/BN, N)$ is an uncountable torsionfree abelian subgroup of $\mathrm{Aut_c}G$ as required. If $G = BN$ then either $N \leq B$ or $B \cap N$ is finite. In the former case, $G_{\mathrm{ab}} = \bigoplus_{i\in I} C_{p^{n(i)}}$, for some index set I and G_{ab} has infinite exponent, since G does. Since G_{ab} is reduced, $N \leq G'$ and hence $\mathrm{Hom}(G_{\mathrm{ab}}, N) \leq \mathrm{Aut_c}G$. Thus $\mathrm{Cr}_{i\in I}N[p^{n(i)}]$ is a subgroup of $\mathrm{Aut_c}G$ and the result now follows from [1, Lemma 2.5]. If $B \cap N$ is finite then Lemma 2.1 applies, again giving the result.

Case 2. Suppose G is reduced. According to [8, XVI] every basic subgroup B is infinite. If B has finite exponent then [8, XV] shows that $G = BZ$. Hence by Lemma 2.1, $\mathrm{Aut_c}G$ satisfies the desired conclusion. If B has infinite exponent but $G' \cap Z$ has finite exponent, then G/Z also has finite exponent (the correct version of [8, IX]). Hence $B \cap Z$ has infinite exponent. Then $(B \cap Z)G'/G'$ is a subgroup of infinite exponent in the group B/G', a direct sum of cyclic groups. By Lemma 2.2, we can find X/G', Y/G' so that $B/G' = X/G' \oplus Y/G'$ and $X/G' \cap (B \cap Z)G'/G'$, $Y/G' \cap (B \cap Z)G'/G'$ both have infinite exponent.

Hence $X \cap Z$ and $Y \cap Z$ both have infinite exponent. However, B/X is a basic subgroup of G/X and the epimorphism of Szele [4, 36.1] yields

$$\mathrm{Hom}(Y/G', X \cap Z) \leq \mathrm{Hom}(G/X \cap Z, X \cap Z) \leq \mathrm{Aut_c}G$$

and $\mathrm{Aut_c}G$ contains an uncountable torsionfree abelian subgroup in this case.

Finally, if both B and $G' \cap Z$ have infinite exponent then the proof of [1, Theorem 2.6] gives the result in this case. This completes the proof. □

3. The finite exponent case

We first give a proof of Theorem 3.1.

Let π denote the set of primes dividing the exponent of $Z(G)$.

(i) *implies* (ii). Let α be a central automorphism of G that has infinite order and note that α induces an automorphism of infinite order on $Z = Z(G)$. (We say that α acts *infinitely* on Z.) Since π is finite and, for each prime p, the Sylow p-subgroups of Z are characteristic in G, there is some prime $p \in \pi$

such that α acts infinitely on the Sylow p-subgroup of Z, which we denote by K. Let L be the p'-part of Z so that $Z = K \oplus L$.

Let $k \in \mathbb{N}$ be minimal such that some nontrivial power of α acts trivially on $p^k K$. Replacing α with a nontrivial power of itself if necessary we may assume that α acts trivially on $p^k K$. It then follows that α acts infinitely on $p^{k-1}K/p^k K$, otherwise some nontrivial power of α acts trivially on both $p^{k-1}K/p^k K$ and $p^k K$. However, this implies some nontrivial power of α acts trivially on $p^{k-1}K$, contrary to the minimality of k.

We set $M = p^{k-1}K(\alpha - 1) = \{z^\alpha - z | z \in p^{k-1}K\}$ and note that M is an α-invariant subgroup of $p^{k-1}K$. To complete the proof we now establish a series of claims.

(a) *M has exponent p.*

For if $m \in M$, then $m = z^\alpha - z$ for some $z \in p^{k-1}K$. Then $pz \in p^k K$ so

$$pz = (pz)\alpha = p(z\alpha) = pz + pm.$$

Hence $pm = 0$ and (a) follows.

(b) *α acts infinitely on M.*

For if α acts finitely on M then some nontrivial power of α acts trivially on M. Clearly α acts trivially on $p^{k-1}K/M$ so some power of α acts trivially on $p^{k-1}K$, which contradicts the choice of k.

(c) *α acts infinitely on $P \equiv MG'(p^k K)/G'(p^k K)$.*

Otherwise α acts finitely on P, so some nontrivial power of α acts trivially on P. Since α is central, it acts trivially on G'. Hence some nontrivial power of α acts trivially on $MG'(p^k K)$, contrary to (b). Hence (c) follows.

We let $C = C_P(\alpha)$. It follows from (a) that $P = C \oplus D$ for some subgroup D (which is not necessarily α-invariant.) Furthermore D is infinite by (c). Let I be an index set and, for $i \in I$, choose $r_i \in M$ so that $\{r_i G'(p^k K) : i \in I\}$ is a basis of D. Define $N = \langle r_i : i \in I \rangle$ and note that N is an infinite elementary abelian subgroup of $M \leq p^{k-1}K$. It is clear that:

(d) $n^\alpha \not\equiv n \bmod G'(p^k K)$ *for all nontrivial* $n \in N$.

It follows from [4, p. 119, Example 5] that K (and hence Z) has a direct summand A such that $A[p] = N$. Set $Z = A \oplus F$. Since A is a direct summand of Z, each $0 \neq n \in N$ has the same p-height in A as in Z. Furthermore $N \leq M \leq p^{k-1}K$. Hence:

(e) *The p-height of $0 \neq n \in N$ in A is at least $k-1$.* (In fact one can easily see that it is exactly $k-1$.)

(f) *AG'/G' is a direct summand of G_{ab}.*

It suffices by [4, Corollary 27.5] to show that AG'/G' is pure in G_{ab}, and to do this it is enough to show that every element of order p in AG'/G' has the same p-height in G_{ab} as in AG'/G' (see [4, p. 114, (h)] and also note that the Sylow p-subgroup of G_{ab} is pure in G_{ab}).

Now $A \cap G' = 1$ since α fixes no nontrivial element of $A[p] = N$, by (d), whereas the central automorphism α acts trivially on G'. It follows that $(AG'/G')[p] = NG'/G'$. Suppose for a contradiction that $n + G' \in NG'/G'$ has larger p-height in G_{ab} than in AG'/G'. Then by (e), $n + G' = p^l g + G'$ for some $g \in G$ and some $l \geq k$. Since α is central, $g\alpha = g + z$ for some $z \in Z$ so that

$$(1) \qquad n\alpha + G' = (p^l g)\alpha + G' = p^l g + p^l z + G' = n + p^l z + G'.$$

Also $pn = 0$ so $p^{l+1}g \in G'$ and since α is central $p^{l+1}g = (p^{l+1}g)\alpha = p^{l+1}g + p^{l+1}z$. Hence $z \in K$, the Sylow p-subgroup of Z so (1) shows

$$n\alpha \equiv n \mod G'p^kK,$$

contrary to (d). This proves (f).

The result now follows immediately. For there is a subgroup H of G such that

$$G_{\mathrm{ab}} = AG'/G' \oplus H/G'.$$

Hence $G = AH$. But $A \cap H \le A \cap G' = 1$ and $A \le Z(G)$ so it follows that $G = A \times H$.

(ii) *implies* (iii). Clearly $\mathrm{Cr}_{m=1}^{\infty} C_{p^m} \le \mathrm{Aut_c}G$. Hence by [1, Lemma 2.5] the result follows.

(iii) *implies* (i). This is clear.

In particular, we have

Corollary 3.2. *Suppose G is a nilpotent p-group of finite exponent. Then G has a central automorphism of infinite order if and only if G has an infinite abelian direct factor.*

This result is analogous to [3, Lemma 2] although the proof is rather different. The reader is also referred to [2, 6] where further results have been obtained on central automorphisms of infinite groups.

An example. We note that Theorem 3.1 fails if $Z(G)$ is allowed to have infinite exponent. For each prime p, let G_p be a nilpotent p-group as in [5, 3.2(ii)] such that $\mathrm{Aut}\, G_p = \mathrm{Aut_c}G_p$ is an elementary abelian p-group. Set $G = \bigoplus G_p$, the direct sum being taken over all primes p. Then $\mathrm{Aut}\, G = \mathrm{Cr}(\mathrm{Aut}\, G_p)$ and $\mathrm{Aut_c}G$ clearly contains elements of infinite order. However G has no infinite abelian direct factor.

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References

1. J. Buckley and J. Wiegold, *On the number of outer automorphisms of an infinite nilpotent p-group*, Arch. Math. **31** (1978), 321–328.
2. M. Curzio, D. J. S. Robinson, H. Smith, and J. Wiegold, *Some remarks on central automorphisms of hypercentral groups*, Arch. Math. **53** (1989), 327–331.
3. S. Franciosi and F. de Giovanni, *On torsion groups with nilpotent automorphism groups*, Comm. Algebra **14** (1986), 1909–1935.
4. L. Fuchs, *Infinite abelian groups*, vol. 1, Academic Press, New York and London, 1970.
5. F. Menegazzo and S. E. Stonehewer, *On the automorphism group of a nilpotent p-group*, J. London Math. Soc. (2) **31** (1985), 272–276.
6. M. R. Pettet, *Central automorphisms of periodic groups*, Arch. Math. **51** (1988), 20–33.
7. D. J. S. Robinson, *Applications of cohomology to the theory of groups*, Groups-St. Andrews 1981, London Mathematical Society Lecture Note series, vol. 71, Cambridge Univ. Press, Cambridge, 1982.

8. A. E. Zaleskiĭ, *A nilpotent p-group has an outer automorphism*, Dokl. Akad. Nauk SSSR **196** (1971); Soviet Math. Dokl. **12** (1971), 227–230.

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