PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 114, Number 2, February 1992

INVARIANT SUBSPACES AND PERTURBATIONS

LYLE NOAKES

(Communicated by Palle E. T. Jorgensen)

ABSTRACT. We study the stability of proper closed invariant subspaces with respect to perturbations in norm of continuous operators on a Hilbert space, using a nonlinear C^∞ map of Banach spaces.

1. INTRODUCTION

When J and K are Banach spaces over $\mathbb{C}$ or $\mathbb{R}$, let $L(J, K)$ be the Banach space of continuous linear transformations from J to K, with the operator norm. Let $(H, <, >)$ be a Hilbert space; a subspace W of H is *proper* when $W \neq \{\mathbf{0}\}, H$. Let $P(H)$ be the closed subset of $L(H, H)$ consisting of the orthogonal projections p_W onto closed subspaces W of H, and let $W^\perp$ denote the orthogonal complement of W in H. If W is a closed subspace of H, identify $L(W, K)$ with $\{A \in L(H, K): A|W^\perp = \mathbf{0}\}$.

Let $A_0 \in L(H, H)$ and W_0 be a closed subspace of H: W_0 is *invariant* for A_0 when $A_0(W_0) \subseteq W_0$. Let W_0 be proper. In §2 $U = B(\mathbf{0}, 1) \cap L(W_0, W_0^\perp)$, and we define a C^∞ function $\Psi_{W_0}: L(H, H) \times U \to L(W_0, W_0^\perp)$ with the property that $\Psi_{W_0}(A, T) = \mathbf{0} \Leftrightarrow (\mathbf{1} + T)W_0$ is an invariant subspace for A. Then we compute the derivative of Ψ_{W_0}.

In §3 W_0 is a proper closed invariant subspace for A_0, and we define *stability* of (W_0, A_0) in $\mathscr{I}$ relative to a smaller closed invariant subspace $V_0 \subseteq W_0$ of A_0. Here $\mathscr{I}$ is closed ideal in $L(H, H)$. *Unique stability* is a stronger condition.

Let W_1 be the orthogonal complement of V_0 in W_0. In the definition of stability, A_0 is perturbed by elements of $\mathscr{I}$ which vanish on V_0, and we are concerned with invariant subspaces of the form $(\mathbf{1} + T)W_0$ where $T \in \mathscr{I} \cap L(W_1, W_0^\perp)$. So no nontrivial perturbation is allowed on V_0, but in some applications $V_0 = \{\mathbf{0}\}$. The case $\mathscr{I} = L(H, H)$ also occurs in applications. If (say) $\mathscr{I}$ is the ideal of compact operators then, roughly speaking, the perturbations of A_0 and W_0 are compact.

In §4 the calculation of $(d\Psi_{A_0})_{(A_0, \mathbf{0})}$ leads to a necessary condition for (W_0, A_0) to be stable. For example, take $V_0 = \{\mathbf{0}\}$, $\mathscr{I} = L(H, H)$, and let H

Received by the editors January 21, 1990.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 47A15.

Key words and phrases. Hilbert space, Banach space, invariant subspace, implicit function theorem.

be separable. If A_0 is compact and both W_0 and $W_0^\perp$ are infinite-dimensional, then (W_0, A_0) is not stable.

In §5 we give a sufficient condition for (W_0, A_0) to be uniquely stable, and this is sometimes satisfied when the spectrum of A_0 is connected: then a perturbation of A_0 also has a proper closed invariant subspace. When the spectrum of A_0 is disconnected, our stability result does not seem to follow easily from the Riesz decomposition.

2. Projections and derivatives

Let $A_0 \in L(H, H)$, and let W_0 be a proper closed subspace of H. Define $W \subseteq H$ and $\Psi_{A_0, W_0} : U \equiv B(\mathbf{0}, 1) \cap L(W_0, W_0^\perp) \to L(W_0, W_0^\perp)$ by $W = (\mathbf{1} + T)W_0$ and $\Psi_{A_0, W_0}(T) = (\mathbf{1} - p_{W_0}) \circ (\mathbf{1} - p_W) \circ A_0 \circ (\mathbf{1} + T)|W_0$. Because $T \in U$, $\mathbf{1} + T$ is invertible.

If $w \in W_0 \cap W^\perp$ then $\mathbf{0} = \langle w, (\mathbf{1} + T)w \rangle = \langle w, w \rangle$, and therefore, $W_0 \cap W^\perp = \{\mathbf{0}\}$. If $(\mathbf{1} - p_{W_0})(\mathbf{1} - p_W)(u) = \mathbf{0}$ then $(\mathbf{1} - p_W)(u) \in W_0 \Rightarrow (\mathbf{1} - p_W)(u) \in W_0 \cap W^\perp = \{\mathbf{0}\}$. Therefore $\operatorname{Ker}(\mathbf{1} - p_{W_0}) \cap \operatorname{Im}(\mathbf{1} - p_W) = \{\mathbf{0}\}$. So W is an invariant subspace for $A_0 \Leftrightarrow \Psi_{A_0, W_0}(T) = 0$.

Lemma 1. *The assignment $T \mapsto p_W$ is C^∞ in the sense of* [1], *and $(dp_W)_T$ is given by $(dp_W)_T(D)|W = (\mathbf{1} - p_W) \circ D \circ (\mathbf{1} + T)^{-1}$ and $(dp_W)_T(D)|W^\perp = (\mathbf{1} + T^*)^{-1} \circ D^* \circ (\mathbf{1} - p_W)^*$ where $D \in L(W_0, W_0^\perp)$.*

Proof. Define a C^∞ function $\mathscr{F} : U \times L(H, H) \to L(W_0, H) \oplus L(W_0^\perp, H)$ by $\mathscr{F}(T, p) = ((\mathbf{1} - p) \circ (\mathbf{1} + T)|W_0,\ p \circ (\mathbf{1} + T^*)^{-1}|W_0^\perp)$. Then $\mathscr{F}(T, p) = \mathbf{0} \Leftrightarrow p = p_W$ where $W = (\mathbf{1} + T)W_0$. If $\mathscr{F}(T, p) = \mathbf{0}$, then $(d\mathscr{F})_{T,p}(0, q) = (-q \circ (\mathbf{1} + T)|W_0,\ q \circ (\mathbf{1} + T^*)^{-1}|W_0^\perp)$, and $T \mapsto p_W$ is C^∞ by the implicit function theorem [1].

Because $(\mathbf{1}, -p_W) \circ (\mathbf{1} + T)|W_0 = 0$, $(dp_W)_T(D) \circ (\mathbf{1} + T)|W_0 = (\mathbf{1} - p_W) \circ D|W_0$, or rather $(dp_W)_T(D)|W = (\mathbf{1} - p_W) \circ D \circ (\mathbf{1} + T)^{-1}$. Now $p_W \circ p_W = p_W$ and therefore, $(dp_W)_T(D) \circ p_W + p_W \circ (dp_W)_T(D) = (dp_W)_T(D)$ so that, if $v \in W^\perp$, $p_W \circ (dp_W)_T(D)(v) = \mathbf{0}$; namely, $(dp_W)'_T(D)(W^\perp) \subseteq W$.

Now $\langle p_W u_1, u_2 \rangle = \langle u_1, p_W u_2 \rangle$ for all $u_1, u_2 \in H$, so $\langle (dp_W)_T(D)u_1, u_2 \rangle = \langle u_1, (dp_W)_T(D)u_2 \rangle$. In particular, when $u_1 = v \in W^\perp$ and $u_2 = w \in W$, $\langle (dp_W)_T(D)v, w \rangle = \langle v, (dp_W)_T(D)w \rangle = \langle v, (\mathbf{1} - p_W) \circ D \circ (\mathbf{1} + T)^{-1} w \rangle = \langle (\mathbf{1} + T^*)^{-1} \circ D^* \circ (\mathbf{1} - p_W)^* v, w \rangle$. This proves Lemma 1.

Define a C^∞ function $\Psi_{W_0} : L(H, H) \times U \to L(W_0, W_0^\perp)$, by $\Psi_{W_0}(A) = \Psi_{A, W_0}$. Then $\Psi_{W_0}(A, T) = (\mathbf{1} - p_{W_0}) \circ (\mathbf{1} - p_W) \circ A \circ (\mathbf{1} + T)|W_0$ is the composite of

(i) inclusion of W_0 in H (independent of T),
(ii) $(\mathbf{1} + T) : H \to H$,
(iii) $A|H : H \to H$ (independent of T),
(iv) $\mathbf{1} - p_W : H \to H$.
(v) $\mathbf{1} - p_{W_0} : H \to H$ (independent of T).

Therefore $(d\Psi_{W_0})_{(A, T)}(\mathbf{0}, D) \in L(W_0, W_0^\perp)$ is $(\mathbf{1} - p_{W_0}) \circ \{(\mathbf{1} - p_W) \circ A \circ D - (dp_W)_T(D) \circ A \circ (\mathbf{1} + T)|W_0)\}$, which we rewrite as $(\mathbf{1} - p_{W_0}) p_W \circ \{(\mathbf{1} - p_W) \circ A \circ D - (dp_W)_T(D) \circ A \circ (\mathbf{1} + T)|W_0 - (dp_W)_T(D) \circ (\mathbf{1} - p_W) \circ A \circ (\mathbf{1} + T)|W_0\}$. From Lemma 1 we now obtain

Lemma 2.

$$(d\Psi_{W_0})_{(A,T)}(\mathbf{0}, D) = (\mathbf{1}-p_{W_0})\circ\{(\mathbf{1}-p_W)\circ A\circ D - \{(\mathbf{1}-p_W)\circ D\circ(\mathbf{1}+T)^{-1}\circ p_W + (\mathbf{1}+T^*)^{-1}\circ D^*\circ(\mathbf{1}-p_W)^*\circ(\mathbf{1}-p_W)\}\circ A\circ(\mathbf{1}+T)|W_0\}.$$

In particular,

$$(d\Psi_{W_0})_{(A,\mathbf{0})}(\mathbf{0}, D) = (\mathbf{1}-p_{W_0})\circ\{A\circ D - D\circ p_{W_0}\circ A|W_0 - D^*\circ(\mathbf{1}-p_{W_0})^*\circ(\mathbf{1}-p_{W_0})\circ A|W_0\}.$$

Suppose now that W_0 is an invariant subspace for $A_0\in L(H,H)$. Then $(\mathbf{1}-p_{W_0})\circ A_0|W_0=\mathbf{0}$, and we have

Lemma 3.

$$(d_{\Psi_{W_0}})_{(A_0,\mathbf{0})}(\mathbf{0}, D), D) = \Lambda(W_0, A_0)(D) \equiv (\mathbf{1}-p_{W_0})\{A_0\circ D - D\circ p_{W_0}\circ A_0|W_0\}.$$

3. Stability in an ideal relative to a closed invariant subspace

Let W_0 be a closed subspace of H, and let $\mathscr{I}$ be a closed two-sided ideal of the algebra $L(H,H)$. (Note that H is not necessarily separable, so that $\mathscr{I}$ is not necessarily the ideal of compact operators.)

Lemma 4. *For some $\varepsilon\in(0,1]$, there is a C^∞ function $\delta: U_\varepsilon\equiv B(\mathbf{0},\varepsilon)\cap\mathscr{I}\to\mathscr{I}$, such that $\delta(T)=p_W-p_{W_0}$ for all $T\in U_\varepsilon$. Here $W=(\mathbf{1}+T)W_0$.*

Proof. Define a C^∞ function $\mathscr{G}: U_1\times\mathscr{I}\to\mathscr{I}\cap L(W_0,H)\oplus\mathscr{I}\cap L(W_0^\perp,H)$ by

$$\mathscr{G}(T,d) = ((\mathbf{1}-p_{W_0}-d)\circ(\mathbf{1}+T)|W_0,\ (p_{W_0}+d)\circ(\mathbf{1}+T^*)^{-1}|W_0^\perp).$$

Note that $\mathscr{G}$ maps into $\mathscr{I}\times\mathscr{I}$ because $\mathscr{I}$ is a two-sided ideal, and that $\mathscr{G}(T,d)=\mathbf{0}\Leftrightarrow p_{W_0}+d=p_W$. If $\mathscr{G}(T,d)=\mathbf{0}$ then $(d\mathscr{G})_{T,d}(\mathbf{0},e) = (-e\circ(\mathbf{1}+T)|W_0,\ e\circ(\mathbf{1}+T^*)^{-1}|W_0^\perp)$. Therefore, by the implicit function theorem, for some ε, there is a C^∞ function δ such that $\mathscr{G}(T,\delta(T))=\mathbf{0}$ for all $T\in U_\varepsilon$. This proves Lemma 4.

Now suppose that W_0 is a proper closed invariant subspace for $A_0\in L(H,H)$, and let V_0 be a closed subspace of W_0 (not necessarily proper), which is also invariant for A_0. Write $W_0=V_0\oplus W_1$. Let $\mathscr{I}_{W_0,V_0,A_0}$ be the closed affine subspace $\{A_0+S: S\in\mathscr{I}, S|V_0=\mathbf{0}\}$ of $L(H,H)$.

We say that (W_0,A_0) is *stable in $\mathscr{I}$ relative to* V_0, when there is an open neighborhood N_0 of A_0 in $\mathscr{I}_{W_0,V_0,A_0}$ and a C^1 function $f: N_0\to\mathscr{I}\cap L(W_1,W_0^\perp)$, such that $f(A_0)=\mathbf{0}$, and $(\mathbf{1}+f(A))W_0$ is an invariant subspace for A, for each $A\in N_0$. When N_0 can be chosen so that f is unique we call (W_0,A_0) *uniquely stable in $\mathscr{I}$ relative to* V_0. Note that if $\mu\in\mathbb{C}$ then (W_0,A_0) is (uniquely) stable in $\mathscr{I}$ relative to $V_0\Leftrightarrow(W_0,A_0-\mu\mathbf{1})$ is (uniquely) stable in $\mathscr{I}$ relative to V_0.

If $A\in\mathscr{I}_{W_0,V_0,A_0}$, $T\in\mathscr{I}\cap L(W_1,W_0^\perp)$, write $W=(\mathbf{1}+T)W_0$. Then

$$\begin{aligned}\Psi_{A,W_0}(T) &\equiv (\mathbf{1}-p_{W_0})\circ(\mathbf{1}-p_W)\circ A_0\circ(\mathbf{1}+T)|W_0\\ &= (\mathbf{1}-p_{W_0})\circ A_0\circ(\mathbf{1}+T)|W_0 + (\mathbf{1}-p_{W_0})\circ S\circ(\mathbf{1}+T)|W_0\\ &\quad - (\mathbf{1}-p_{W_0})\circ p_W\circ(A_0+S)\circ(\mathbf{1}+T)|W_0\\ &= (\mathbf{1}-p_{W_0})\circ A_0\circ T + (\mathbf{1}-p_{W_0})\circ S\circ(\mathbf{1}+T)|W_0\\ &\quad - (\mathbf{1}-p_{W_0})\circ p_W\circ(A_0+S)\circ(\mathbf{1}+T)|W_0\end{aligned}$$

because W_0 is an invariant subspace for A_0.

Choose ε as in Lemma 4, and suppose that $T \in U_\varepsilon \cap L(W_1, W_0^\perp)$. Then $(\mathbf{1} - p_{W_0}) \circ p_W \in \mathcal{I}$, and therefore $\Psi_{A, W_0}(T) \in \mathcal{I} \cap L(W_0, W_0^\perp)$. If $v_0 \in V_0$ then

$$\begin{aligned}\Psi_{A, W_0}(T)(v_0) &= (\mathbf{1} - p_{W_0}) \circ (\mathbf{1} - p_W) \circ A(v_0) = (\mathbf{1} - p_{W_0}) \circ (\mathbf{1} - p_W) \circ A_0(v_0)\\ &= (\mathbf{1} - p_{W_0}) \circ (\mathbf{1} - p_W) \circ (\mathbf{1} + T) \circ A_0(v_0) = \mathbf{0}\end{aligned}$$

because $T(v_0) = \mathbf{0}, S(v_0) = \mathbf{0}$, $A_0(v_0) \in W_0$, and $T \circ A_0(v_0) = \mathbf{0}$. This, together with Lemma 3, proves

Lemma 5. *Ψ_{W_0} restricts to a C^∞ map*

$$\Psi'_{W_0} : \mathcal{I}_{W_0, V_0, A_0} \times \{U_\varepsilon \cap L(W_1, W_0^\perp)\} \to \mathcal{I} \cap L(W_1, W_0^\perp)$$

and

$$(d\Psi'_{W_0})_{(A_0, \mathbf{0})}(\mathbf{0}, D) = \Lambda(W_0, A_0)(D) \in \mathcal{I} \cap L(W_1, W_0^\perp),$$

where $D \in \mathcal{I} \cap L(W_1, W_0^\perp)$.

4. A necessary condition for stability

In the setting of §3 we have

Theorem 1. *If (W_0, A_0) is stable in $\mathcal{I}$ relative to V_0, then $\Lambda(W_0, A_0)|\mathcal{I} \cap L(W_1, W_0^\perp)$ maps onto the whole of $\mathcal{I} \cap L(W_1, W_0^\perp)$.*

Proof. Suppose that (W_0, A_0) is stable in $\mathcal{I}$ relative to V_0. For $A \in \mathrm{f}^{-1}\{U_\varepsilon \cap L(W_1, W_0^\perp)\}$, $\Psi'_{W_0}(A, f(A)) = \mathbf{0}$. For $B \in \mathcal{I} \cap L(W_1, W_0^\perp)$,

$$(d\Psi'_{W_0})_{(A_0, \mathbf{0})} \circ (\mathbf{0}, (df)_{A_0}(B)) + (d\Psi'_{W_0})_{(A_0, \mathbf{0})}(B, \mathbf{0}) = \mathbf{0}.$$

Now $(d\Psi'_{W_0})_{(A_0, \mathbf{0})}(\mathbf{0}, D) = \Lambda(W_0, A_0)(D)$, and $\Lambda(W_0, A_0) \circ (df)_{A_0}(B) = -(d\Psi'_{W_0})_{(A_0, \mathbf{0})}(B, \mathbf{0}) = -\Psi'_{W_0}(B, \mathbf{0}) = -\Psi_{W_0}(B, \mathbf{0})$, since $\Psi'_{W_0}(A, T)$ is linear in the variable A. Let $\rho : \mathcal{I} \cap L(W_1, W_0^\perp) \to L(H, H)$ be the inclusion. Then $\Psi_{W_0}(\rho(T), \mathbf{0}) = T$ for all $T \in \mathcal{I} \cap L(W_1, W_0^\perp)$. Therefore $\Lambda(W_0, A_0) \circ \chi = \mathbf{1}_{\mathcal{I} \cap L(W_1, W_0^\perp)}$ where $\chi = -(d\mathrm{f})_{A_0} \circ \rho$.

Remark. $\Lambda(W_0, A_0)|\mathcal{I} \cap L(W_0, W_0^\perp)$ has nontrivial kernel $\Leftrightarrow A_0$ commutes with some $D \in \mathcal{I}$ for which W_0 is not an invariant subspace.

Example 1. Let $A_0 = \mathbf{0}$. Then $\Lambda(W_0, A_0)|\mathcal{I} \cap L(W_1, W_0^\perp)$ is trivial, and therefore $(W_0, \mathbf{0})$ is not stable in $\mathcal{I}$ relative to V_0 unless $\mathcal{I} \cap L(W_1, W_0^\perp)$ is also trivial. Let $V_0 = \{\mathbf{0}\}$.

Then there is no C^1 assignment of proper closed invariant subspaces to operators in a neighborhood of $\mathbf{0}$ in $\mathcal{I}$. The C^1 assignment would be by continuous isomorphisms, which are perturbations of $\mathbf{1}$ by elements of $\mathcal{I}$.

We have in mind the case where H is infinite-dimensional, and $\mathcal{I}$ is $L(H, H)$, or perhaps the ideal of compact operators in $L(H, H)$, but when $H = \mathbb{R}^2$, with the Euclidean inner product, note the following alternative proof that $(\mathbf{0}, W_0)$ is not stable in $L(H, H)$ relative to $\{\mathbf{0}\}$.

Without loss of generality, $W_0 = \mathbb{R} \times \{\mathbf{0}\}$. Suppose that, for some $\beta > 0$, and some continuous $\mathrm{f}: B(\mathbf{0}, \beta) \to L(\mathbb{R}^2, \mathbb{R}^2)$, we have $f(\mathbf{0}) = \mathbf{0}$, and $(\mathbf{1} + f(A))W_0$ is an invariant subspace of A whenever $A \in B(\mathbf{0}, \beta)$.

Let S^1 be the unit circle in $\mathbb{R}^2 \cong \mathbb{C}$. For $z \in S^1$, let $R_z \in L(\mathbb{R}^2, \mathbb{R}^2)$ be a complex multiplication by z. Define $g(z) \in B(\mathbf{0}, \beta)$ to be

$$R_z \circ \begin{bmatrix} 0 & \beta/2 \\ 0 & 0 \end{bmatrix} \circ R_z^{-1}.$$

The only one-dimensional invariant subspace for $g(z)$ is $R_z W_0$ and therefore, $(\mathbf{1} + f \circ g(z))W_0 = R_z W_0$.

Define $f: B(\mathbf{0}, \beta) \to \mathbb{R}P^1$ by $f(A) = (\mathbf{1} + f(A))W_0$, and $R: S^1 \to \mathbb{R}P^1$ by $R(z) = R_z W_0$. Then $R = f \circ g: S^1 \to B(\mathbf{1}, \beta) \to \mathbb{R}P^1$, which is a contradiction, since R is the double cover, and $B(\mathbf{0}, \beta)$ is contractible.

Example 2. Suppose $A_0 \in \mathscr{J}$ where $\mathscr{J}$ is a two-sided idea of $L(H, H)$ and $\mathscr{J} \cap L(W_1, W_0^{\perp}) \neq \mathscr{J} \cap L(W_1, W_0^{\perp})$. Then $\Lambda(W_0, A_0)\{\mathscr{J} \cap L(W_1, W_0^{\perp})\} \neq \mathscr{J} \cap L(W_1, W_0^{\perp})$, and therefore (W_0, A_0) is not stable in $\mathscr{J}$ relative to V_0. For instance, let H be separable, and let $\mathscr{J}$ be the ideal of compact transformations in $L(H, H)$. Suppose neither W_0 nor $W_0^{\perp}$ is finite dimensional: then $W_0 \cong W_0^{\perp} \cong L^2[0, 1]$, and therefore, $\mathscr{J} \cap L(W_0, W_0^{\perp}) \neq L(W_0, W_0^{\perp})$. So (W_0, A_0) is not stable in $L(H, H)$ relative to $\{\mathbf{0}\}$.

5. A SUFFICIENT CONDITION FOR UNIQUE STABILITY

Still in the setting of §3, we have

Lemma 6. *If* $\Lambda(W_0, A_0)|\mathscr{J} \cap L(W_1, W_0^{\perp}) : \mathscr{J} \cap L(W_1, W_0^{\perp}) \to \mathscr{J} \cap L(W_1, W_0^{\perp})$ *has a continuous inverse then* (W_0, A_0) *is uniquely stable in* $\mathscr{J}$ *relative to* V_0.

Proof. If $\Lambda(W_0, A_0)|\mathscr{J} \cap L(W_1, W_0^{\perp})$ has a continuous inverse then , by Lemma 5 and the implicit function theorem, there is a unique C^∞ function f defined over some neighborhood N_0 of A_0 in $\mathscr{J}_{W_0, V_0, A_0}$ with $f(A_0) = \mathbf{0}$ and $\Psi'_{W_0}(A, f(A)) = \mathbf{0}$ for all $A \in f^{-1}\{U_\varepsilon \cap L(W_1, W_0^{\perp})\}$. Then $(\mathbf{1} + f(A))W_0$ is an invariant subspace for A

If $A \in L(H, H)$ write

$$A_1 = p_{W_0} \circ A|W_0 : W_0 \to W_0; \quad A_2 = p_{W_0} \circ A|W_0^{\perp} : W_0^{\perp} \to W_0;$$
$$A_3 = (\mathbf{1} - p_{W_0} \circ A|W_0 : W_0 \to W_0^{\perp}; \quad A_4 = (\mathbf{1} - p_{W_0}) \circ A|W_0^{\perp} : W_0^{\perp} \to W_0^{\perp}.$$

The spectrum of a linear operator B is denoted by $\sigma(B)$.

Theorem 2. *If* $\sigma(p_{W_1} \circ A_{0,1}|W_1) \cap \sigma(A_{0,4})$ *is empty then* (W_0, A_0) *is uniquely stable in* $\mathscr{J}$ *relative to* V_0.

First, recall Rosenblum's theorem [2, Theorem 0.12]: Let J, K be Hilbert spaces. Given $B \in L(J, J)$, $C \in L(K, K)$, define B^*, $C_*: L(J, K) \to L(J, K)$ by $B^*(D) = D \circ B$ and $C_*(D) = C \circ D$. Then $\sigma(C_* - B^*) \subseteq \sigma(C) - \sigma(B)$. It follows that if M is a closed subspace of $L(J, K)$, which is invariant for both B^* and C_*, then $\sigma(C_*|M - B^*|M) \subseteq \sigma(B)$.

To prove Theorem 2, take $J = W_1$, $K = W_0^{\perp}$, $B = p_{W_1} \circ A_{0,1}|W_1$, $C = A_{0,4}$, $M = \mathscr{J} \cap L(W_1, W_0^{\perp})$. By hypothesis, $0 \notin \sigma(C) - \sigma(B)$ and, therefore $\Lambda(W_0, A_0)|\mathscr{J} \cap L(W_1, W_0^{\perp}) = C_* - B^*$ has a continuous inverse. Theorem 2 now follows from Lemma 6.

Example 3. When H is a complex Hilbert space, a disconnection of $\sigma(A_0)$ leads to the Riesz decomposition of H, as a direct sum of invariant subspaces W_0, Y_0 for A_0 [2, Theorem 2.10]. If $\sigma(A_0)$ is disconnected and $\|A - A_0\|$ is small, then $\sigma(A)$ is also disconnected; therefore H is a direct sum of invariant subspaces W, Y for A. A stronger form of stability follows from Theorem 2; namely, (W_0, A_0) is uniquely stable in $L(H, H)$ relative to $\{\mathbf{0}\}$.

To apply Theorem 2 when $\sigma(A_0)$ is not necessarily disconnected, we take $V_0 \neq \{\mathbf{0}\}$.

Example 4. Let $A_0 \in L(H, H)$ be selfadjoint, where H is a complex Hilbert space. Let $\{E_\lambda : \lambda \in \mathbb{R}\} \subseteq P(H)$ be its resolution of the identity: then $A_0 = \int_{\mathbb{R}} \lambda \, dE_\lambda$. Let $\lambda_0 \in \sigma(A_0)$. Then $W_0 \equiv E_{\lambda_0}(H)$ is a closed invariant subspace for A_0.

Suppose that, for some $\alpha > 0$, $(\lambda_0 - \alpha, \lambda_0 + \alpha) \subseteq \sigma(A_0)$. Let V_0 be the orthogonal complement of $E_{\lambda_0 - \alpha}(H)$ in $W_0 \equiv E_{\lambda_0}(H)$. By Theorem 2, (W_0, A_0) is uniquely stable in $L(H, H)$ relative to V_0.

For instance, let $H = L^2[-1, 1]$ and define $A_0 \in L(H, H)$ by $(A_0(f))x = xf(x)$ a.e. Let $W_0 = L^2[-1, 1/2]$ and $V_0 = L^2[1/2 - \alpha, 1/2]$ where $\alpha \in (0, 1/2)$. Then (W_0, A_0) is uniquely stable in $L(H, H)$ relative to V_0.

In particular, if we perturb A_0 to a nearly $A \in L(L^2[-1, 1], L^2[-1, 1])$, which agrees with A_0 on $L^2[1/2 - \alpha, 1/2]$, then A has an invariant subspace of the form $B(L^2[-1, 1/2])$ where $B: L^2[-1, 1] \to L^2[-1, 1]$ is continuous linear and close in norm to 1.

Acknowledgment

I am especially grateful to Professor Radjavi, whose seminars sparked my interest in the stability problem, and for a number of stimulating conversations. It was also a pleasure to talk to Professors M. Lambrou, W. E. Longstaff, and K. J. Harrison. I want to thank the referee for his careful reading and helpful comments.

References

1. S. Lang, *Introduction to differentiable manifolds*, Interscience-Wiley, New York 1962.
2. H. Radjavi and P. Rosenthal, *Invarint subspaces*, Ergeb. Math. (3) Band 77, Springer-Verlag, London 1973.

Department of Mathematics, University of Western Australia, Nedlands, WA 6009, Australia