IN Variant subspaces and Perturbations

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Abstract. We study the stability of proper closed invariant subspaces with respect to perturbations in norm of continuous operators on a Hilbert space, using a nonlinear $C^\infty$ map of Banach spaces.

1. Introduction

When $J$ and $K$ are Banach spaces over $\mathbb{C}$ or $\mathbb{R}$, let $L(J, K)$ be the Banach space of continuous linear transformations from $J$ to $K$, with the operator norm. Let $(H, \langle , \rangle)$ be a Hilbert space; a subspace $W$ of $H$ is proper when $W \neq \{0\}$, $H$. Let $P(H)$ be the closed subset of $L(H, H)$ consisting of the orthogonal projections $p_W$ onto closed subspaces $W$ of $H$, and let $W^\perp$ denote the orthogonal complement of $W$ in $H$. If $W$ is a closed subspace of $H$, identify $L(W, K)$ with $\{A \in L(H, K): A|W^\perp = 0\}$.

Let $A_0 \in L(H, H)$ and $W_0$ be a closed subspace of $H$: $W_0$ is invariant for $A_0$ when $A_0(W_0) \subseteq W_0$. Let $W_0$ be proper. In §2 $U = B(0, 1) \cap L(W_0, W_0^\perp)$, and we define a $C^\infty$ function $\Psi_{W_0}: L(H, H) \times U \to L(W_0, W_0^\perp)$ with the property that $\Psi_{W_0}(A, T) = 0 \iff (1 + T)W_0$ is an invariant subspace for $A$. Then we compute the derivative of $\Psi_{W_0}$.

In §3 $W_0$ is a proper closed invariant subspace for $A_0$, and we define stability of $(W_0, A_0)$ in $\mathcal{J}$ relative to a smaller closed invariant subspace $V_0 \subseteq W_0$ of $A_0$. Here $\mathcal{J}$ is closed ideal in $L(H, H)$. Unique stability is a stronger condition.

Let $W_1$ be the orthogonal complement of $V_0$ in $W_0$. In the definition of stability, $A_0$ is perturbed by elements of $\mathcal{J}$ which vanish on $V_0$, and we are concerned with invariant subspaces of the form $(1 + T)W_0$ where $T \in \mathcal{J} \cap L(W_1, W_0^\perp)$. So nontrivial perturbation is allowed on $V_0$, but in some applications $V_0 = \{0\}$. The case $\mathcal{J} = L(H, H)$ also occurs in applications. If (say) $\mathcal{J}$ is the ideal of compact operators then, roughly speaking, the perturbations of $A_0$ and $W_0$ are compact.

In §4 the calculation of $(d \Psi_{A_0})(A_0, 0)$ leads to a necessary condition for $(W_0, A_0)$ to be stable. For example, take $V_0 = \{0\}$, $\mathcal{J} = L(H, H)$, and let $H$
be separable. If \( A_0 \) is compact and both \( W_0 \) and \( W_0^\perp \) are infinite-dimensional, then \((W_0, A_0)\) is not stable.

In §5 we give a sufficient condition for \((W_0, A_0)\) to be uniquely stable, and this is sometimes satisfied when the spectrum of \( A_0 \) is connected: then a perturbation of \( A_0 \) also has a proper closed invariant subspace. When the spectrum of \( A_0 \) is disconnected, our stability result does not seem to follow easily from the Riesz decomposition.

### 2. Projections and derivatives

Let \( A_0 \in L(H, H) \), and let \( W_0 \) be a proper closed subspace of \( H \). Define \( W \subseteq H \) and \( \Psi_{A_0, w_0} : U \equiv B(0, 1) \cap L(W_0, W_0^\perp) \to L(W_0, W_0^\perp) \) by \( W = (1 + T)W_0 \) and \( \Psi_{A_0, w_0}(T) = (1 - p_{w_0}) \circ (1 - p_{w'}) \circ A_0 \circ (1 + T)|W_0. \) Because \( T \in U \), \( 1 + T \) is invertible.

If \( w \in W_0 \cap W^\perp \) then \( 0 = \langle w, (1 + T)w \rangle = \langle w, w \rangle \), and therefore, \( W_0 \cap W^\perp = \{0\} \). If \((1 - p_{w_0})(1 - p_{w'}) (u) = 0 \) then \((1 - p_{w'})(u) \in W_0 \Rightarrow (1 - p_{w'})(u) \in W_0 \cap W^\perp = \{0\} \). Therefore \( \text{Ker}(1 - p_{w_0}) \cap \text{Im}(1 - p_{w'}) = \{0\} \). So \( W \) is an invariant subspace for \( A_0 \Leftrightarrow \Psi_{A_0, w_0}(T) = 0 \).

**Lemma 1.** The assignment \( T \mapsto p_w \) is \( C^\infty \) in the sense of [1], and \((dp_w)_T\) is given by \((dp_w)_T(D)|W = (1 - p_w) \circ D \circ (1 + T)^{-1} \) and \((dp_w)_T(D)|W^\perp = (1 + T^*)^{-1} \circ D^* \circ (1 - p_w)^* \) where \( D \in L(W_0, W_0^\perp) \).

**Proof.** Define a \( C^\infty \) function \( \mathcal{F} : U \times L(H, H) \to L(W_0, H) \oplus L(W_0^\perp, H) \) by \( \mathcal{F}(T, p) = ((1 - p) \circ (1 + T)|W_0, p \circ (1 + T^*)^{-1}|W_0^\perp) \). Then \( \mathcal{F}(T, p) = 0 \Leftrightarrow p = p_w \) where \( W = (1 + T)W_0 \). If \( \mathcal{F}(T, p) = 0 \), then \((d\mathcal{F})_T, p(0, q) = (-q \circ (1 + T)|W_0, q \circ (1 + T^*)^{-1}|W_0^\perp), \) and \( T \mapsto p_w \) is \( C^\infty \) by the implicit function theorem [1].

Because \((1, -p_w) \circ (1 + T)|W_0 = 0, (dp_w)_T(D) \circ (1 + T)|W_0 = (1 - p_w) \circ D|W_0 \), or rather \((dp_w)_T(D)|W = (1 - p_w) \circ D \circ (1 + T)^{-1} \). Now \( p_w \circ p_w = p_w \) and therefore, \((dp_w)_T(D) \circ p_w + p_w \circ (dp_w)_T(D) = (dp_w)_T(D) \) so that, if \( v \in W^\perp, p_w \circ (dp_w)_T(D)(v) = 0 \); namely, \((dp_w)_T(D)(W^\perp) \subseteq W \).

Now \( \langle p_w u_1, u_2 \rangle = \langle u_1, p_w u_2 \rangle \) for all \( u_1, u_2 \in H \), so \( \langle (dp_w)_T(D)u_1, u_2 \rangle = \langle u_1, (dp_w)_T(D)u_2 \rangle \). In particular, when \( u_1 = v \in W^\perp \) and \( u_2 = w \in W \), \( \langle (dp_w)_T(D)w, w \rangle = \langle v, (dp_w)_T(D)w \rangle = \langle v, (1 - p_w) \circ D \circ (1 + T)^{-1} w \rangle = \langle (1 + T^*)^{-1} \circ D^* \circ (1 - p_w)^* v, w \rangle \). This proves Lemma 1.

Define a \( C^\infty \) function \( \Psi_{w_0} : L(H, H) \times U \to L(W_0, W_0^\perp) \), by \( \Psi_{w_0}(A) = \Psi_{A, w_0} \). Then \( \Psi_{w_0}(A, T) = (1 - p_{w_0}) \circ (1 - p_w) \circ A \circ (1 + T)|W_0 \) is the composite of

(i) inclusion of \( W_0 \) in \( H \) (independent of \( T \)),
(ii) \((1 + T) : H \to H \),
(iii) \( A|H : H \to H \) (independent of \( T \)),
(iv) \( 1 - p_{w_0} : H \to H \),
(v) \( 1 - p_{w_0} : H \to H \) (independent of \( T \)).

Therefore \((d\Psi_{w_0})(A, T)(0, D) \in L(W_0, W_0^\perp) \) is \((1 - p_{w_0}) \circ \{(1 - p_w) \circ A \circ D - (dp_w)_T(D) \circ A \circ (1 + T)|W_0 \}\), which we rewrite as \((1 - p_{w_0})p_{w_0} \circ \{(1 - p_w) \circ A \circ D - (dp_w)_T(D) \circ A \circ (1 + T)|W_0 - (dp_w)_T(D) \circ (1 - p_w) \circ A \circ (1 + T)|W_0 \}. \) From Lemma 1 we now obtain
Lemma 2. 

\[(d\Psi_{W_0})(A,T)(0, D) = (1 - p_{W_0}) \circ \{(1 - p_W) \circ A \circ D - \{(1 - p_W) \circ D \circ (1 + T)^{-1} \circ p_W
+ (1 + T^*)^{-1} \circ D^* \circ (1 - p_W)^* \circ (1 - p_W)\} \circ A \circ (1 + T)|W_0\}.\]

In particular, 

\[(d\Psi_{W_0})(A,0)(0, D) = (1 - p_{W_0}) \circ \{A \circ D - p_{W_0} \circ A|W_0 - D^* \circ (1 - p_{W_0})^* \circ (1 - p_{W_0}) \circ A|W_0\}.\]

Suppose now that \(W_0\) is an invariant subspace for \(A_0 \in L(H, H)\). Then 

\[(1 - p_{W_0}) \circ A_0|W_0 = 0,\]

and we have 

**Lemma 3.** 

\[(d\Psi_{W_0})(A_0,0)(0, D), D) = \Lambda(W_0, A_0)(D) = (1 - p_{W_0})\{A_0 \circ D - D^* \circ (1 - p_{W_0}) \circ A|W_0\}.\]

**3. Stability in an ideal relative to a closed invariant subspace**

Let \(W_0\) be a closed subspace of \(H\), and let \(\mathcal{F}\) be a closed two-sided ideal of the algebra \(L(H, H)\). (Note that \(H\) is not necessarily separable, so that \(\mathcal{F}\) is not necessarily the ideal of compact operators.)

**Lemma 4.** For some \(\varepsilon \in (0, 1]\), there is a \(C^\infty\) function \(\delta : U_\varepsilon \equiv B(0, \varepsilon) \cap \mathcal{F} \to \mathcal{F}\), such that \(\delta(T) = p_W - p_{W_0}\) for all \(T \in U_\varepsilon\). Here \(W = (1 + T)W_0\).

**Proof.** Define a \(C^\infty\) function \(\mathcal{G} : U_1 \times \mathcal{F} \to \mathcal{F} \cap L(W_0, H) \oplus \mathcal{F} \cap L(W_0^\perp, H)\) by

\[\mathcal{G}(T, d) = ((1 - p_{W_0} - d) \circ (1 + T)|W_0, (p_{W_0} + d) \circ (1 + T^*)^{-1}|W_0^\perp).\]

Note that \(\mathcal{G}(T, d) = 0 \iff p_{W_0} + d = p_W\). If \(\mathcal{G}(T, d) = 0\) then \((d\mathcal{G})_T, d)(0, e) = (-e \circ (1 + T)|W_0, e \circ (1 + T^*)^{-1}|W_0^\perp).\) Therefore, by the implicit function theorem, for some \(\varepsilon\), there is a \(C^\infty\) function \(\delta\) such that \(\mathcal{G}(T, \delta(T)) = 0\) for all \(T \in U_\varepsilon\). This proves Lemma 4.

Now suppose that \(W_0\) is a proper closed invariant subspace for \(A_0 \in L(H, H)\), and let \(V_0\) be a closed subspace of \(W_0\) (not necessarily proper), which is also invariant for \(A_0\). Write \(W_0 = V_0 \oplus W_1\). Let \(\mathcal{F}_{W_0, V_0, A_0}\) be the closed affine subspace \(\{A_0 + S : S \in \mathcal{F}, S|V_0 = 0\}\) of \(L(H, H)\).

We say that \((W_0, A_0)\) is stable in \(\mathcal{F}\) relative to \(V_0\), when there is an open neighborhood \(N_0\) of \(A_0\) in \(\mathcal{F}_{W_0, V_0, A_0}\) and a \(C^1\) function \(f : N_0 \to \mathcal{F} \cap L(W_1, W_0^\perp)\), such that \(f(A_0) = 0\), and \((1 + f(A))W_0\) is an invariant subspace for \(A\), for each \(A \in N_0\). When \(N_0\) can be chosen so that \(f\) is unique we call \((W_0, A_0)\) uniquely stable in \(\mathcal{F}\) relative to \(V_0\). Note that if \(\mu \in \mathbb{C}\) then \((W_0, A_0)\) is (uniquely) stable in \(\mathcal{F}\) relative to \(V_0\) \(\iff (W_0, A_0 - \mu I)\) is (uniquely) stable in \(\mathcal{F}\) relative to \(V_0\).

If \(A \in \mathcal{F}_{W_0, V_0, A_0}, T \in \mathcal{F} \cap L(W_1, W_0^\perp)\), write \(W = (1 + T)W_0\). Then

\[\Psi_{A, W_0}(T) = (1 - p_{W_0}) \circ (1 - p_W) \circ A_0 \circ (1 + T)|W_0\]

\[= (1 - p_{W_0}) \circ A_0 \circ (1 + T)|W_0 + (1 - p_{W_0}) \circ S \circ (1 + T)|W_0
- (1 - p_{W_0}) \circ p_W \circ (A_0 + S) \circ (1 + T)|W_0
= (1 - p_{W_0}) \circ A_0 \circ (1 + T)|W_0 - (1 - p_{W_0}) \circ p_W \circ (A_0 + S) \circ (1 + T)|W_0\]
because $W_0$ is an invariant subspace for $A_0$.

Choose $\epsilon$ as in Lemma 4, and suppose that $T \in U_\epsilon \cap L(W_1, W_0^\perp)$. Then $(1 - p_{W_0}) \circ p_W \in \mathcal{F}$, and therefore $\Psi_{A,W_0}(T) \in \mathcal{F} \cap L(W_0, W_0^\perp)$. If $v_0 \in V_0$ then

\[ \Psi_{A,W_0}(T)(v_0) = (1 - p_{W_0}) \circ (1 - p_W) \circ A(v_0) = (1 - p_{W_0}) \circ (1 - p_W) \circ A_0(v_0) \]

\[ = (1 - p_{W_0}) \circ (1 - p_W) \circ (1 + T) \circ A_0(v_0) = 0 \]

because $T(v_0) = 0, S(v_0) = 0, A_0(v_0) \in W_0$, and $T \circ A_0(v_0) = 0$. This, together with Lemma 3, proves

**Lemma 5.** $\Psi_{W_0}$ restricts to a $C^\infty$ map

\[ \Psi_{W_0} : \mathcal{H}_{W_0} \times V_0 \times \{U \in L(W_1, W_0^\perp)\} \rightarrow \mathcal{F} \cap L(W_1, W_0^\perp) \]

and

\[ (d\Psi_{W_0})_{(A_0,v_0)}(0, D) = A(W_0, A_0)(D) \in \mathcal{F} \cap L(W_1, W_0^\perp), \]

where $D \in \mathcal{F} \cap L(W_1, W_0^\perp)$.

4. A NECESSARY CONDITION FOR STABILITY

In the setting of §3 we have

**Theorem 1.** If $(W_0, A_0)$ is stable in $\mathcal{F}$ relative to $V_0$, then $A(W_0, A_0)|\mathcal{F} \cap L(W_1, W_0^\perp)$ maps onto the whole of $\mathcal{F} \cap L(W_1, W_0^\perp)$.

**Proof.** Suppose that $(W_0, A_0)$ is stable in $\mathcal{F}$ relative to $V_0$. For $A \in \mathcal{F} \cap \{U \in L(W_1, W_0^\perp)\}$, $\Psi_{W_0}(A, f(A)) = 0$. For $B \in \mathcal{F} \cap L(W_1, W_0^\perp)$,

\[ (d\Psi_{W_0})_{(A_0,v_0)}(0, D) + (d\Psi_{W_0})_{(A_0,v_0)}(B, 0) = 0. \]

Now

\[ (d\Psi_{W_0})_{(A_0,v_0)}(0, D) = \Lambda(W_0, A_0)(D), \text{ and } \Lambda(W_0, A_0) \circ (df)_{A_0}(B) = -(d\Psi_{W_0})_{(A_0,v_0)}(B, 0) = -\Psi_{W_0}(B, 0), \]

since $\Psi_{W_0}(A, T)$ is linear in the variable $A$. Let $\rho : \mathcal{F} \cap L(W_1, W_0^\perp) \rightarrow L(H, H)$ be the inclusion. Then $\Psi_{W_0}(\rho(T), 0) = T$ for all $T \in \mathcal{F} \cap L(W_1, W_0^\perp)$. Therefore $\Lambda(W_0, A_0) \circ \chi = 1_{\mathcal{F} \cap L(W_1, W_0^\perp)}$ where $\chi = -(df)_{A_0} \circ \rho$.

**Remark.** $\Lambda(W_0, A_0)|\mathcal{F} \cap L(W_0, W_0^\perp)$ has nontrivial kernel $\iff A_0$ commutes with some $D \in \mathcal{F}$ for which $W_0$ is not an invariant subspace.

**Example 1.** Let $A_0 = 0$. Then $\Lambda(W_0, A_0)|\mathcal{F} \cap L(W_1, W_0^\perp)$ is trivial, and therefore $(W_0, 0)$ is not stable in $\mathcal{F}$ relative to $V_0$ unless $\mathcal{F} \cap L(W_1, W_0^\perp)$ is also trivial. Let $V_0 = \{0\}$.

Then there is no $C^1$ assignment of proper closed invariant subspaces to operators in a neighborhood of 0 in $\mathcal{F}$. The $C^1$ assignment would be by continuous isomorphisms, which are perturbations of 1 by elements of $\mathcal{F}$.

We have in mind the case where $H$ is infinite-dimensional, and $\mathcal{F}$ is $L(H,H)$, or perhaps the ideal of compact operators in $L(H,H)$, but when $H = \mathbb{R}^2$, with the Euclidean inner product, note the following alternative proof that $(0, W_0)$ is not stable in $L(H,H)$ relative to $(0)$.

Without loss of generality, $W_0 = \mathbb{R} \times \{0\}$. Suppose that, for some $\beta > 0$, and some continuous $f : B(0, \beta) \rightarrow L(\mathbb{R}^2, \mathbb{R}^2)$, we have $f(0) = 0$, and $(1 + f(A))W_0$ is an invariant subspace of $A$ whenever $A \in B(0, \beta)$.
Let $S^1$ be the unit circle in $\mathbb{R}^2 \cong \mathbb{C}$. For $z \in S^1$, let $R_z \in L(\mathbb{R}^2, \mathbb{R}^2)$ be a complex multiplication by $z$. Define $g(z) \in B(0, \beta)$ to be

$$R_z \circ \begin{bmatrix} 0 & \beta/2 \\ 0 & 0 \end{bmatrix} \circ R_z^{-1}.$$ 

The only one-dimensional invariant subspace for $g(z)$ is $R_z W_0$ and therefore, $(1 + f \circ g(z)) W_0 = R_z W_0$.

Define $f : B(0, \beta) \to \mathbb{R}P^1$ by $f(A) = (1 + f(A)) W_0$, and $R : S^1 \to \mathbb{R}P^1$ by $R(z) = R_z W_0$. Then $R = f \circ g : S^1 \to B(1, \beta) \to \mathbb{R}P^1$, which is a contradiction, since $R$ is the double cover, and $B(0, \beta)$ is contractible.

**Example 2.** Suppose $A_q \in \mathcal{J}$ where $\mathcal{J}$ is a two-sided ideal of $L(H, H)$ and $\mathcal{J} \cap L(W_1, W_0^\perp) \neq \mathcal{J} \cap L(W_1, W_0^\bot)$. Then $\Lambda(W_0, A_0)\{\mathcal{J} \cap L(W_1, W_0^\perp)\} \neq \mathcal{J} \cap L(W_1, W_0^\bot)$, and therefore $(W_0, A_0)$ is not stable in $\mathcal{J}$ relative to $V_0$. For instance, let $H$ be separable, and let $\mathcal{J}$ be the ideal of compact transformations in $L(H, H)$. Suppose neither $W_0$ nor $W_0^\perp$ is finite dimensional: then $W_0 \cong W_0^\perp \cong L^2[0, 1]$, and therefore, $\mathcal{J} \cap L(W_0, W_0^\perp) \neq L(W_0, W_0^\perp)$. So $(W_0, A_0)$ is not stable in $L(H, H)$ relative to $\{0\}$.

5. A SUFFICIENT CONDITION FOR UNIQUE STABILITY

Still in the setting of §3, we have

**Lemma 6.** If $\Lambda(W_0, A_0)\{\mathcal{J} \cap L(W_1, W_0^\perp)\} : \mathcal{J} \cap L(W_1, W_0^\bot) \to \mathcal{J} \cap L(W_1, W_0^\bot)$ has a continuous inverse then $(W_0, A_0)$ is uniquely stable in $\mathcal{J}$ relative to $V_0$.

**Proof.** If $\Lambda(W_0, A_0)\{\mathcal{J} \cap L(W_1, W_0^\bot)\}$ has a continuous inverse then, by Lemma 5 and the implicit function theorem, there is a unique $C^\infty$ function $f$ defined over some neighborhood $N_0$ of $A_0$ in $\mathcal{J}_{W_0, V_0, A_0}$ with $f(A_0) = 0$ and $\Psi'_{W_0}(A, f(A)) = 0$ for all $A \in f^{-1}\{U_\epsilon \cap L(W_1, W_0^\bot)\}$. Then $(1 + f(A)) W_0$ is an invariant subspace for $A$.

If $A \in L(H, H)$ write

$$A_1 = p_{W_0} \circ A|_{W_0} : W_0 \to W_0; \quad A_2 = p_{W_0} \circ A|_{W_0^\bot} : W_0^\bot \to W_0;$$

$$A_3 = (1 - p_{W_0}) \circ A|_{W_0} : W_0 \to W_0^\perp; \quad A_4 = (1 - p_{W_0}) \circ A|_{W_0^\bot} : W_0^\bot \to W_0^\perp.$$ 

The spectrum of a linear operator $B$ is denoted by $\sigma(B)$.

**Theorem 2.** If $\sigma(p_{W_0} \circ A_0.4|W_1) \cap \sigma(A_0.4)$ is empty then $(W_0, A_0)$ is uniquely stable in $\mathcal{J}$ relative to $V_0$.

First, recall Rosenblum's theorem [2, Theorem 0.12]: Let $J, K$ be Hilbert spaces. Given $B \in L(J, J)$, $C \in L(K, K)$, define $B^* \ast C_* : L(J, K) \to L(J, K)$ by $B^* \ast D = D \circ B$ and $C_* \ast D = C \circ D$. Then $\sigma (C_* \ast B^*) \subseteq \sigma (C) \ast \sigma (B)$ and $\sigma (C_* \ast B^*) \subseteq \sigma (B)$. It follows that if $M$ is a closed subspace of $L(J, K)$, which is invariant for both $B^*$ and $C_*$, then $\sigma (C_* \ast M - B^* \ast M) \subseteq \sigma (B)$.

To prove Theorem 2, take $J = W_1$, $K = W_0^\perp$, $B = p_{W_1} \circ A_{0.4}|W_1$, $C = A_0.4$, $M = \mathcal{J} \cap L(W_1, W_0^\bot)$. By hypothesis, $0 \notin \sigma (C) \ast \sigma (B)$ and, therefore $L(W_0, A_0)$ is not stable in $L(H, H)$ relative to $\{0\}$.
Example 3. When $H$ is a complex Hilbert space, a disconnection of $\sigma(A_0)$ leads to the Riesz decomposition of $H$, as a direct sum of invariant subspaces $W_0, Y_0$ for $A_0$ [2, Theorem 2.10]. If $\sigma(A_0)$ is disconnected and $\|A - A_0\|$ is small, then $\sigma(A)$ is also disconnected; therefore $H$ is a direct sum of invariant subspaces $W, Y$ for $A$. A stronger form of stability follows from Theorem 2; namely, $(W_0, A_0)$ is uniquely stable in $L(H, H)$ relative to $\{0\}$.

To apply Theorem 2 when $\sigma(A_0)$ is not necessarily disconnected, we take $V_0 \neq \{0\}$.

Example 4. Let $A_0 \in L(H, H)$ be selfadjoint, where $H$ is a complex Hilbert space. Let $\{E_\lambda : \lambda \in \mathbb{R}\} \subseteq P(H)$ be its resolution of the identity: then $A_0 = \int_{\mathbb{R}} \lambda dE_\lambda$. Let $\lambda_0 \in \sigma(A_0)$. Then $W_0 \equiv E_{\lambda_0}(H)$ is a closed invariant subspace for $A_0$.

Suppose that, for some $\alpha > 0$, $(\lambda_0 - \alpha, \lambda_0 + \alpha) \subseteq \sigma(A_0)$. Let $V_0$ be the orthogonal complement of $E_{\lambda_0-\alpha}(H)$ in $W_0 \equiv E_{\lambda_0}(H)$. By Theorem 2, $(W_0, A_0)$ is uniquely stable in $L(H, H)$ relative to $V_0$.

For instance, let $H = L^2[-1, 1]$ and define $A_0 \in L(H, H)$ by $(A_0(f))(x) = xf(x)$ a.e. Let $W_0 = L^2[-1/2, 1/2]$ and $V_0 = L^2[1/2 - \alpha, 1/2]$ where $\alpha \in (0, 1/2)$. Then $(W_0, A_0)$ is uniquely stable in $L(H, H)$ relative to $V_0$.

In particular, if we perturb $A_0$ to a nearly $A \in L(L^2[-1, 1], L^2[-1, 1])$, which agrees with $A_0$ on $L^2[1/2 - \alpha, 1/2]$, then $A$ has an invariant subspace of the form $B(L^2[-1, 1/2])$ where $B : L^2[-1, 1] \to L^2[-1, 1]$ is continuous linear and close in norm to 1.

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