UNIFORM CONVERGENCE OF ERGODIC LIMITS AND APPROXIMATE SOLUTIONS

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Abstract. Let \( A \) be a densely defined closed (linear) operator, and \( \{A_\alpha\} \), \( \{B_\alpha\} \) be two nets of bounded operators on a Banach space \( X \) such that \( \|A_\alpha\| = O(1) \), \( A_\alpha A \subset AA_\alpha \), \( \|AA_\alpha\| = o(1) \), and \( B_\alpha A \subset AB_\alpha = I - A_\alpha \). Denote the domain, range, and null space of an operator \( T \) by \( D(T) \), \( R(T) \), and \( N(T) \), respectively, and let \( P \) (resp. \( B \)) be the operator defined by \( Px = \lim_\alpha A_\alpha x \) (resp. \( By = \lim_\alpha B_\alpha y \)) for all those \( x \in X \) (resp. \( y \in R(A) \)) for which the limit exists. It is shown in a previous paper that \( D(P) = N(A) \cap R(A) \), \( R(P) = N(A) \), \( D(B) = A(D(A) \cap R(A)) \), \( R(B) = D(A) \cap R(A) \), and that \( B \) sends each \( y \in D(B) \) to the unique solution of \( Ax = y \) in \( R(A) \). In this paper, we prove that \( D(P) = X \) and \( \|P - B\| \to 0 \) if and only if \( \|B_\alpha D(B) - B\| \to 0 \), if and only if \( \|B_\alpha D(B)\| = O(1) \), if and only if \( R(A) \) is closed. Moreover, when \( X \) is a Grothendieck space with the Dunford-Pettis property, all these conditions are equivalent to the mere condition that \( D(P) = X \). The general result is then used to deduce uniform ergodic theorems for \( n \)-times integrated semigroups, \( (Y) \)-semigroups, and cosine operator functions.

1. Introduction

Let \( X \) be a Banach space and \( B(X) \) be the set of all bounded linear operators on \( X \). Let \( A : D(A) \subset X \to X \) be a densely defined closed linear operator, and let \( \{A_\alpha\} \) and \( \{B_\alpha\} \) be two nets in \( B(X) \) satisfying the conditions:

\begin{enumerate}
\item[(C1)] \( \|A_\alpha\| = O(1) \), i.e., there exist \( M \) and \( \alpha_0 \) such that \( \|A_\alpha\| \leq M \) for \( \alpha \geq \alpha_0 \),
\item[(C2)] \( R(B_\alpha) \subset D(A) \) and \( B_\alpha A \subset AB_\alpha = I - A_\alpha \) for all \( \alpha \),
\item[(C3)] \( R(A_\alpha) \subset D(A) \) and \( A_\alpha A \subset AA_\alpha \), and \( \|AA_\alpha\| \to 0 \).
\end{enumerate}

These two systems of operators have been employed in our earlier papers [13] and [14] to formulate an abstract mean ergodic theorem and to produce approximate solutions of the functional equation \( Ax = y \).

Let \( P \) be the operator defined by \( Px := s\text{-}\lim_\alpha A_\alpha x \) for \( x \in D(P) := \{x \in X ; s\text{-}\lim_\alpha A_\alpha x \text{ exists}\} \), and let \( B \) be the operator defined by \( By := s\text{-}\lim_\alpha B_\alpha y \) for \( y \in D(B) := \{y \in R(A) ; s\text{-}\lim_\alpha B_\alpha y \text{ exists}\} \). The following two...
strong convergence theorems were proved in [13]: (i) $P$ is a bounded linear projection with range $R(P) = N(A)$, null space $N(P) = \overline{R(A)}$, and domain $D(P) = N(A) \oplus \overline{R(A)} = \{x \in X; \{A_\alpha x\} \text{ has a weak cluster point}\}$; (ii) $B$ is the inverse operator of the restriction $A|\overline{R(A)}$ of $A$ to $\overline{R(A)}$; it has range $R(B) = D(A) \cap \overline{R(A)}$ and domain $D(B) = A(D(A) \cap \overline{R(A)}) = \{y \in \overline{R(A)}; \{B_\alpha y\} \text{ has a weak cluster point}\}$. Moreover, for any given $y \in D(B)$, the vector $By$ is the unique solution of the functional equation $Ax = y$ that lies in $\overline{R(A)}$. This closed operator $B$ is called the inner inverse of $A$.

Actually, the convergence of $B_\alpha y$ to $By = (A|\overline{R(A)})^{-1}y$ for $y \in D(B)$ is seen from the following computation using (C2) and (i).

\[
B_\alpha y - By = B_\alpha AB_\alpha y - By = (B_\alpha A - I)By
\]

\[
= \|A_\alpha By\| = \|(A_\alpha - P)By\| \to 0.
\]

{\{A_\alpha\}} is said to be strongly ergodic if $D(P) = X$. In this case, we have $R(A) = A(D(A) \cap X) = A(D(A) \cap [N(A) \oplus \overline{R(A)}]) = A(D(A) \cap \overline{R(A)}) = D(B)$.

Conversely, the equality $D(B) = R(A)$ implies the strong ergodicity because, if not, there would be a $z \in D(A) \setminus D(P)$ and an $x \in D(A) \cap \overline{R(A)}$ such that $Az = Ax$, which leads to $z = (z - x) + x \in N(A) \oplus \overline{R(A)} = D(P)$, a contradiction.

The purpose of this paper is to prove the following two uniform convergence theorems for the two systems {\{A_\alpha\}} and {\{B_\alpha\}}. Applications to concrete examples are to be given in §3.

**Theorem 1.** Let $A$ be a densely defined closed linear operator in $X$, and let {\{A_\alpha\}} and {\{B_\alpha\}} be two nets in $B(X)$ which satisfy (C1), (C2), and (C3). Then the following statements are equivalent:

1. $\|A_\alpha|D(P) - P\| \to 0$,
2. $D(P) = X$ and $\|A_\alpha - P\| \to 0$,
3. $R(A)$ is closed,
4. $R(A^2)$ is closed,
5. $X = N(A) \oplus R(A)$,
6. $\|B_\alpha R(A)\| = O(1)$,
7. $B$ is bounded and $\|B_\alpha|D(B) - B\| \to 0$.

Moreover, if (1)–(7) hold, then $D(B) = R(A^2) = R(A)$, $\|A_\alpha - P\| \leq (M + 1) \times \|A_\alpha A_\alpha\| \|B\|$ and $\|B_\alpha|D(B) - B\| \leq (M + 1) \|A_\alpha\| \|B\|^2$.

A Banach space $X$ is called a Grothendieck space if every weakly* convergent sequence in the dual space $X^*$ is weakly convergent, and is said to have the Dunford-Pettis property if $\langle x_n, x_n^* \rangle \to 0$ whenever $x_n \to 0$ weakly in $X$ and $x_n^* \to 0$ weakly in $X^*$. Among examples of Grothendieck spaces with the Dunford-Pettis property are $L^\infty$, $B(s, \Sigma)$, $H^\infty(D)$, etc. (See [7].)

**Theorem 2.** Let $X$ be a Grothendieck space with the Dunford-Pettis property, and let $A$, {\{A_\alpha\}}, and {\{B_\alpha\}} be as in Theorem 1. If {\{A_\alpha\}} is strongly ergodic, then it is uniformly ergodic.

Thus, in this case the conditions (1)–(7) all are equivalent to each of $D(P) = X$ and $D(B) = R(A)$. In view of the next theorem, we have another equivalent condition: $\overline{R(A^*)} = w^* - \text{cl}(R(A^*))$. 

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Theorem 3. Let $X$ be a Grothendieck space, and let $A$, $\{A_\alpha\}$, and $\{B_\alpha\}$ be as in Theorem 1. Then $\{A_\alpha\}$ is strongly ergodic if and only if $\overline{R(A^*)} = w^* - \text{cl}(R(A^*))$.

2. Proof of Main Result

Proof of Theorem 1. We prove the implication $s$: (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (2), (1) $\Rightarrow$ (3) $\Rightarrow$ (6) $\Rightarrow$ (1) + (7).

(2) $\Rightarrow$ (3) (C3) implies that $\overline{R(A)}$ is invariant under $A_\alpha$ and $PA = 0$, so that $\|A_\alpha|\overline{R(A)}\| = \|(A_\alpha - P)|\overline{R(A)}\| \leq \|A_\alpha - P\| \rightarrow 0$. Hence for some $\alpha$, $(A_\alpha - I)|\overline{R(A)}$ is invertible and so $\overline{R(A)} \subset R(A_\alpha - I) \subset R(A)$.

(3) $\Rightarrow$ (4). By the open mapping theorem (cf. [15, p. 213]), there is some $m > 0$ such that for each $x \in R(A)$ is equal to $Ay$ for some $y \in D(A)$ with $\|y\| \leq m\|x\|$. Hence $\|A_\alpha x\| = \|AA_\alpha y\| \leq \|A_\alpha\|\|x\|$ , showing that $\|A_\alpha|\overline{R(A)}\| \leq m\|A_\alpha\| \rightarrow 0$ and so $(A_\alpha - I)|\overline{R(A)}$ is invertible for some $\alpha$. This, together with (C2) and (C3), implies that $R(A) = (A_\alpha - I)AD(A) = A(A_\alpha - I)D(A) \subset A(D(A) \cap R(A)) = R(A^2)$. Hence $R(A^2) = R(A)$ and is closed.

(4) $\Rightarrow$ (5). If $x \in D(A)$, then $Ax = \lim_{a \rightarrow \infty} A_\alpha x - A(A_\alpha - I)x = -\lim A_\alpha x \in R(A^2) = R(A^2)$. This shows that $R(A) = R(A^2)$ is closed and $D(A) \subset N(A) + R(A)$. Next, let $x \in X$, and let $\{x_\alpha\}$ be a sequence in $D(A)$ such that $x_\alpha \rightarrow x$. Then $A_\alpha x_\alpha \in D(A)$ and $(A_\alpha - I)x = \lim_{\alpha \rightarrow \infty}(A_\alpha - I)x_\alpha \in \overline{R(A)} = R(A)$ so that $x = A_\alpha x - (A_\alpha - I)x \in D(A) + R(A) \subset N(A) + R(A)$. Hence $X = N(A) + R(A)$. To see that this is a direct sum, let $x \in N(A) \cap R(A)$. Then there is a $y$ such that $Ay = x \in N(A) \subset N(A_\alpha - I)$ for all $\alpha$. But then $x = A_\alpha x = A_\alpha Ay \rightarrow 0$.

(5) $\Rightarrow$ (2). The closedness of $A$ and assumption (5) imply $R(A)$ is closed (see [16, p. 217]). Then, as was shown in (3) $\Rightarrow$ (4), we have $\|(A_\alpha - P)|\overline{R(A)}\| = \|A_\alpha|\overline{R(A)}\| \leq m\|A_\alpha\| \rightarrow 0$. Because $(A_\alpha - P)|N(A) = 0$, it follows that $\|A_\alpha - P\| \rightarrow 0$.

(1) $\Rightarrow$ (3). Since $A_\alpha A \subset A A_\alpha$, the space $D(P) = N(A) + \overline{R(A)}$ is invariant under $A_\alpha$, and $A|D(P)$ is a densely defined closed operator in $D(P)$. Applying (2) $\Rightarrow$ (5) to $\{A_\alpha |D(P)\}$ and $A|D(P)$, we infer that $D(P) = N(A|D(P)) \oplus R(A|D(P))$. Since $N(A|D(P)) = N(A)$ and $R(A|D(P)) \subset R(A)$, it follows from the two expressions of $D(P)$ that $\overline{R(A)} = R(A|D(P))$ and hence $\overline{R(A)} = R(A)$.

(3) $\Rightarrow$ (6). If $R(A)$ is closed, then as shown in (3) $\Rightarrow$ (4), we have $R(A^2) = R(A)$, so that $D(B) = A(D(A) \cap R(A)) = R(A^2) = R(A)$ is closed. Since $B_\alpha y \rightarrow By$ for all $y \in D(B)$, the uniform boundedness principle implies (6).

(6) $\Rightarrow$ (1) + (7). (6) implies that $B$ is bounded and hence $R(A|D(P)) = A(D(A) \cap \overline{R(A)}) = D(B)$ is closed. An application of (3) $\Rightarrow$ (2) to $\{A_\alpha |D(P)\}$ asserts that $\|A_\alpha |D(P) - P\| \rightarrow 0$, from which, together with (7), we see that $\|B_\alpha |D(B) - B\| \leq \|A_\alpha |D(P) - P\| \|B\| \rightarrow 0$.

Finally, if (1)–(7) hold, then for any $x \in X$, we have $x - Px \in R(A) = R(A^2) = D(B)$ so that $AB(x - Px) = (x - Px)$ and

$$\|A_\alpha x - Px\| = \|A_\alpha (x - Px)\| = \|A_\alpha AB(I - P)x\| \leq \|AA_\alpha\|\|B\||(M + 1)\|x\|.$$  
Hence $\|A_\alpha - P\| \leq (M + 1)\|AA_\alpha\|\|B\|$ and $\|B_\alpha |D(B) - B\| \leq (M + 1)\|AA_\alpha\|\|B\|^2$.

To prove Theorems 2 and 3 we need the following lemma.
Lemma. If \( A, \{A_\alpha\}, \) and \( \{B_\alpha\} \) satisfy conditions (C1), (C2), and (C3), then \( A^*, \{A^*_\alpha\}, \) and \( \{B^*_\alpha\} \) also satisfy these conditions.

Proof. It suffices to show that if \( E \in B(X) \) is such that \( R(E) \subset D(A) \) and \( EA \subset AE \), then \( R(E^*) \subset D(A^*) \) and \( E^*A^* \subset A^*E^* = (AE)^* \). If \( x^* \in D(A^*) \), then \( \langle Ax, E^*x^* \rangle = \langle EAx, x^* \rangle = \langle x, E^*A^*x^* \rangle \) for all \( x \in D(A) \), so that \( E^*x^* \in D(A^*) \) and \( A^*E^*x^* = E^*A^*x^* = (AE)^*x^* \). Hence \( E^*A^* \subset A^*E^* \). To show \( R(E^*) \subset D(A^*) \) and \( A^*E^* = (AE)^* \), we use the fact that \( A^* \) is weakly* densely defined and \( w^* - w^* \)-closed. For any \( x^* \in X^* \), let \( \{x^*_n\} \) be a net in \( D(A^*) \) such that \( x^*_n \to x^* \) weakly*. Then \( E^*x^*_n \) and \( A^*E^*x^*_n \) converge weakly* to \( E^*x^* \) and \( (AE)^*x^* \), respectively. This implies that \( E^*x^* \in D(A^*) \) and \( A^*E^*x^* = (AE)^*x^* \).

Proof of Theorem 2. Since \( A_\alpha|P(R) = I \), we may assume \( A_\alpha \to 0 \) strongly and show that \( \|A_\alpha\| \to 0 \), without loss of generality. Take a sequence \( A_n \equiv A_{\alpha_n} \to 0 \). Then \( A_nx \) converges strongly to 0 for all \( x \in X \). This implies that \( \{A^*_n x^*_n\} \) converges weakly* and hence weakly to 0 whenever \( \{x^*_n\} \) is bounded. In particular, \( \{A^*_n x^*_n\} \to 0 \) weakly for all \( x^* \in X^* \). The convergence actually holds in the strong topology, by the strong ergodic theorem (applied to \( \{A^*_n\} \)). This fact in turn implies that \( A_n x_n \) converges weakly* to 0 whenever \( \{x_n\} \) is bounded. Now, it follows from a lemma of Lotz [7] that \( \|A^*_n\| \to 0 \). Thus \( I - A_m \) is invertible for a sufficiently large \( m \). By (C2) and (C3) we obtain that

\[
\|A_n\| = \|A_n(I - A_m)(I - A_m)^{-1}\| = \|A_n A B_m (I - A_m)^{-1}\| \\
\leq \|A A_n\| \|B_m\| \|(I - A_m)^{-1}\| \to 0 \quad \text{as } n \to \infty.
\]

Application of Theorem 1 to \( \{A_n\} \) and \( \{A_\alpha\} \) shows first that \( R(A) \) is closed and then that \( \|A_\alpha\| \to 0 \).

Proof of Theorem 3. If \( D(P) = X \), then for every \( x^* \in X^* \) we have

\[
\lim_{n \to \infty} A^*_n x^* = \lim_{n \to \infty} A^*_\alpha_n x^* = P^* x^*,
\]

where \( \{A_\alpha_n\} \) is any subsequence of \( \{A_\alpha\} \). The strong ergodic theorem applied to \( \{A^*_\alpha_n\} \), shows that \( \lim_{n \to \infty} A^*_\alpha_n x^* \) for all \( x^* \in X^* \), \( N(P^*) = \overline{R(A^*)} \). Hence \( \overline{R(A^*)} = R(P)^\perp = N(A)^\perp = [^\perp R(A^*^\perp)]^\perp = w^*-\text{cl}(R(A^*)). \)

Conversely, if \( \overline{R(A^*)} = w^*-\text{cl}(R(A^*)) \), then \( D(P)^\perp = \{N(A) \oplus \overline{R(A)}\}^\perp = [^\perp R(A^*)]\perp \cap R(A)^\perp = w^*-\text{cl}(R(A^*)) \cap N(A^*) = \overline{R(A^*)} \cap N(A^*) = \{0\} \), again following from the strong ergodic theorem applied to \( \{A^*_\alpha\} \). Since \( D(P) \) is closed, it must be equal to \( X \).

3. Examples

We consider applications to \( n \)-times integrated semigroups, \( (Y) \)-semigroups, and cosine operator functions.

3.1. \( n \)-times integrated semigroups. Let \( n \) be a positive integer. A strongly continuous family \( \{T(t); t \geq 0\} \) in \( B(X) \) is called an \( n \)-times integrated semigroup (see [1, 15]) if \( T(0) = I \) and

\[
T(t)T(s) = \frac{1}{(n - 1)!} \left( \int_s^{t+s} (t + s - r)^{n-1} T(r) \, dr - \int_0^s (t + s - r)^{n-1} T(r) \, dr \right) \\
(s, t \geq 0).
\]
A \( C_0 \)-semigroup is called an \( o \)-times integrated semigroup. It is known that the integrals over \([0, t], \; t \geq 0\), of an \( n \)-times (\( n \geq 0 \)) integrated semigroup form an \((n + 1)\)-times integrated semigroup, but not conversely.

\( T(\cdot) \) is called nondegenerate if \( T(t)x = 0 \) for all \( t > 0 \) implies \( x = 0 \). It is called exponentially bounded if there are \( M \geq 0, \; w \in \mathbb{R} \) such that \( \|T(t)\| \leq Me^{wt} \) for all \( t \geq 0 \). If \( T(\cdot) \) is nondegenerate and exponentially bounded, then there exists a unique closed operator \( A \) satisfying \( (w, \infty) \subset \rho(A) \) and \( (\lambda - A)^{-1}x = \int_0^\infty e^{-\lambda t}T(t)dt \) for \( x \in X \) and \( \lambda > w \). This operator is called the generator of \( T(\cdot) \). It is not necessarily densely defined. We only consider the case when \( A \) is densely defined. This includes all \( C_0 \)-semigroups.

It is known \([1, \text{Proposition 3.3}]\) that \( \int_0^t T(s)x\,ds \in D(A) \) and \( A \int_0^t T(s)x\,ds = T(t)x = (t^n/n!x) \) for all \( x \in X \), and \( \int_0^t T(s)Ax\,ds = T(t)x - (t^n/n!)x \) for all \( x \in D(A) \). Since \( A \) is closed, taking integration gives that

\[
\int_0^t T(s)x\,ds - (t^{n+1}/(n+1)!x) = \begin{cases} 
A\int_0^t T(u)dudu & \text{for } x \in X, \\
\int_0^t T(u)Ax\,du & \text{for } x \in D(A).
\end{cases}
\]

Let \( A_t := (n + 1)!t^{-n-1}\int_0^t T(s)\,ds \) and \( B_t := -(n + 1)!t^{-n-1}\int_0^t T(u)dudu \) for \( t > 0 \). Then \( B_tA \subset AB_t = I - A_t \) and \( A_tA \subset AA_t = (n+1)!T(t)/t^{n+1}-(n+1)I/t \). Suppose \( \|T(t)\| = O(t^n) \) (\( t \to \infty \)). Then \( A, \{A_t\} \), \( \{B_t\} \) satisfy (C1), (C2), and (C3) as \( t \to \infty \). On the other hand, the systems \( \{\lambda(\lambda - A)^{-1}\}, \{-(\lambda - A)^{-1}\} \) clearly satisfy (C1), (C2), and (C3) as \( \lambda \to 0 \) too. Hence the strong ergodic theorem and the theorems in §1 are applicable to \( \{A_t\} \) with \( \{B_t\} \) and \( \{\lambda(\lambda - A)^{-1}\} \) with \( \{-(\lambda - A)^{-1}\} \), and the next two theorems follow immediately.

**Theorem 4.** Let \( \{T(t); \; t \geq 0\} \) be a nondegenerate \( n \)-times integrated semigroup with generator \( A \) densely defined. Suppose \( \|T(t)\| = O(t^n) \) (\( t \to \infty \)). Let \( A_t \) and \( B_t \) be as previously defined. Then \( \lim_{t \to \infty} A_t x \) and \( \lim_{\lambda \to 0^+} \lambda(\lambda - A)^{-1}x \) exist and are equal if one of them exists, and the limits define a bounded linear projection \( P \) onto \( N(A) \) along \( R(A) \). For \( y \in R(A) \), \( \lim_{t \to \infty} B_t y \) and \( \lim_{\lambda \to 0^+} (A - \lambda)^{-1}y \) exist and are equal if one of them exists, and the limits define an operator \( B \) which sends each \( y \in A(D(A) \cap R(A)) \) to the unique solution \( x = By \) of \( Ax = y \) in \( R(A) \).

**Theorem 5.** Under the hypothesis of Theorem 4, the following statements are equivalent:

1. \( \|A_t - P\| \to 0 \) as \( t \to \infty \),
2. \( \|\lambda(\lambda - A)^{-1} - P\| \to 0 \) as \( t \to 0^+ \),
3. \( R(A) \) is closed,
4. \( \|B_t\|R(A)\| = O(1) \) (\( t \to \infty \)),
5. \( \|B_t\|R(A) - B\| \to 0 \) as \( t \to \infty \),
6. \( \|\lambda - A\| - R(A) - B\| \to 0 \) as \( \lambda \to 0^+ \).

Moreover, when \( X \) is a Grothendieck space with the Dunford-Pettis property, \( D(P) = X \) and \( R(A^*) = w^*-\text{cl}(R(A^*)) \) are two more equivalent conditions.

**Remarks.** (i) When (1)-(6) hold, we have \( \|A_t - P\| = O(1/t) \), \( \|B_t\|R(A) - B\| = O(1/t) \) (\( t \to \infty \)), and \( \|\lambda(\lambda - A)^{-1} - P\| = O(\lambda) \), \( \|\lambda - A\| - R(A) - B\| = O(\lambda) \), \( \lambda \to 0^+ \).
(ii) In the case $n = 0$, Theorem 4 is well known (see [3, pp. 58–60] for the first part, and [4] for the second part), the equivalence of (1), (2), and (3) in Theorem 5 is proved in [6] (see also [10]), the equivalence of strong ergodicity and $R(A^*) = w^*\text{-cl}(RA^*)$ in a Grothendieck space is proved in [9], and the equivalence of strong ergodicity and uniform ergodicity in a Grothendieck space with the Dunford-Pettis property is proved in [7]. The theorems with $n \geq 1$ are new.

3.2. $(Y)$-semigroups. Let $Y$ be a closed subspace of $X^*$ such that the canonical imbedding of $X$ into $Y^*$ is isometric. A semigroup $\{T(t); t \geq 0\}$ of operators on $X$ is called a $(Y)$-semigroup (cf. [8, 11]) if $Y$ is invariant under $T^*(t)$ for all $t \geq 0$ and $T(\cdot)x$ is $\sigma(X, Y)$-continuous on $[0, \infty)$ and locally $\sigma(X, Y)$-Pettis integrable for all $x \in X$. The generator $A$ of $T(\cdot)$ is defined by $Ax := \sigma(X, Y)\text{-lim}_{t \to 0^+} t^{-1}(T(t) - I)x$. A $C_0$-semigroup on $X$ is a $(X^*)$-semigroup, and its dual semigroup is a $(X)$-semigroup. The tensor product $T(t)$ of two $C_0$-semigroups $e^{tA}$ and $e^{-tB}$ on $X$ is a $(Y)$-semigroup on $B(X)$ for some suitable subspace $Y$ of $B(X)^*$; its generator is the operator $A: C \to AC - CB$.

The strong convergence of ergodic limits of a $(Y)$-semigroup and that of approximate solutions of the corresponding equation $Ax = y$ have been discussed in [13, Example VI]. The result is the same as Theorem 4 with $n = 0$. By applying Theorems 1 and 2 one can easily see that Theorem 5 with $n = 0$ holds for $(Y)$-semigroups too. Since $S(t) := \int_{0}^{t} T(s) ds$, $t \geq 0$, forms a once-integrated semigroup, we can apply Theorems 4 and 5 to $S(\cdot)$ to obtain ergodic theorems for $(C, 2)$-means of $T(\cdot)$; they are Theorems 4 and 5 with $A_t = 2t^{-2} \int_{0}^{t} \int_{0}^{s} T(u) du ds$ and $B_t = -2t^{-2} \int_{0}^{t} \int_{0}^{u} T(v) dv ds$.

3.3. Cosine operator functions. A strongly continuous family $\{C(t); t \in \mathbb{R}\}$ in $B(X)$ is called a cosine operator function if $C(0) = I$ and $C(t + s) + C(t - s) = 2C(t)C(s), s, t \in \mathbb{R}$. The generator $A$, defined by $Ax := C''(0)x$, is a densely defined closed operator.

For $t > 0$ let

$$A_t := 2t^{-2} \int_{0}^{t} \int_{0}^{s} C(u) du ds$$

and

$$B_t = -2t^{-2} \int_{0}^{t} \int_{0}^{s} \int_{0}^{u} C(w) dw dv du ds.$$ 

Then we have $B_tA \subset AB_t \subset I_t - A$ and $A_tA \subset AA_t = 2t^{-2}(C(t) - I)$. The strong convergence of $A_t x$ and $B_t y$ as $t \to \infty$ has been discussed in [13, Example VII]. We now deduce from Theorems 1 and 2 the following theorem about uniform convergence.

**Theorem 6.** Suppose that $\| \int_{0}^{t} \int_{0}^{s} C(u) du ds \| = O(t^2)$ $(t \to \infty)$ and $\| C(t) \| = o(t^2)$ $(t \to \infty)$. Then, with $A_t$ and $B_t$ defined as above, the conclusion of Theorem 5 remains valid.

**Concluding remark**

Our Theorems 1, 2, and 3 can also be used to deduce uniform ergodic theorems for discrete semigroups (cf. [5, 7]) and uniform ergodic theorems for pseudoresolvents (cf. [10, 12]).
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