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CURVATURE ESTIMATES FOR MINIMAL SURFACES

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ABSTRACT. In this article, we estimate the Gaussian curvature of nonparametric minimal surfaces by using the properties of univalent harmonic mappings.

1. INTRODUCTION

Recently, Hengartner and Schober [3] studied the class Σ_H of all complex-valued, harmonic, orientation-preserving, univalent mappings f defined on $\Delta = \{z\colon |z| > 1\}$ that are normalized at infinity by $f(\infty) = \infty$. Such functions admit the representation

$$f(z) = h(z) + \overline{g(z)} + A\log|z|, \tag{1.1}$$

where

$$h(z) = \alpha z + \sum_{k=0}^{\infty} a_k z^{-k} \quad \text{and} \quad g(z) = \beta z + \sum_{k=1}^{\infty} b_k z^{-k}$$

are analytic in Δ and $0 \le |\beta| < |\alpha|$. In addition, $\tilde{a} = \overline{f_{\bar{z}}}/f_z$ is analytic and satisfies $|\tilde{a}(z)| < 1$.

Now let Ω be a doubly connected domain in the extended w-plane having the point $w = \infty$ as one of its boundary continua. Let S be a nonparametric surface over Ω given by

$$S = \{(u, v, \varphi(u, v))\colon u + iv \in \Omega\}.$$

We shall associate with every solution $\varphi(u, v)$ of the classical equation of minimal surfaces,

$$(1 + \varphi_v^2)\varphi_{uu} - 2\varphi_u\varphi_v\varphi_{uv} + (1 + \varphi_u^2)\varphi_{vv} = 0, \tag{1.2}$$

the functions

$$\psi(u, v) = \int \frac{\varphi_u\, dv - \varphi_v\, du}{\sqrt{1 + \varphi_u^2 + \varphi_v^2}};$$

$$F = \varphi + i\psi;$$

$$\omega = \frac{\varphi_u - i\varphi_v}{1 + \sqrt{1 + \varphi_u^2 + \varphi_v^2}}.$$

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Note that the integral defining ψ is path-independent by virtue of (1.2) and that ψ is determined but for an additive constant. Also, we have $|\omega| < 1$. The functions F and ω are to be considered as defined on S. While ω is single-valued on this surface, F may be multi-valued, but its branches differ only by constants.

If $\varphi(u, v)$ is interpreted as the potential of a flow of a hypothetical "Chaplygin gas" whose density ρ and speed q are connected by the relation

$$\rho^2(1+q^2) = 1,$$

then ψ is the stream-function, F the complex potential, and $\varphi_u - i\varphi_v$ the conjugate complex velocity (cf. [1]).

From now on, assume that φ is nonconstant; that is, S is not a horizontal plane. Then S is a minimal surface if and only if S admits a conformal reparametrization of the form

$$S = \{(\operatorname{Re} G_1(z), \operatorname{Re} G_2(z), \operatorname{Re} F(z)) \colon z = x+iy \in \Delta\},$$

where

$$(1.3) \qquad G_1(z) = \frac{1}{2}\int F'\left(\frac{1}{\omega} - \omega\right) dz, \quad G_2(z) = -\frac{i}{2}\int F'\left(\frac{1}{\omega} + \omega\right) dz.$$

The function

$$(1.4) \qquad f(z) = \operatorname{Re} G_1(z) + i \operatorname{Re} G_2(z) = \frac{1}{2}\int \frac{F'(z)}{\omega(z)}\, dz - \frac{1}{2}\overline{\int \omega(z) F'(z)\, dz}$$

is a univalent harmonic mapping from Δ onto Ω with $f(\infty) = \infty$ and $\overline{f_{\bar z}}/f_z = -\omega^2$. In Δ, the variables F and ω considered as functions of z are regular analytic, dF/dz and ω are single-valued, and $(1/\omega)(dF/dz) \neq 0, \infty$. The function $\omega(z)$ is regular at $z = \infty$ and $|\omega(\infty)| < 1$. Furthermore, $F'(z)/\omega(z)$ is regular and different from zero at ∞. Also observe that we may assume f is orientation-preserving and that we may obtain any other set of isothermal parameters by applying a conformal mapping to Δ (cf. [1]).

Note that $\omega(u, v) = 0$ if and only if the unit normal vector to the surface $\mathbb{N} = (-\varphi_u, -\varphi_v, 1)/\sqrt{1+\varphi_u^2+\varphi_v^2}$ is $(0, 0, 1)$ at $(u, v, \varphi(u, v))$. Since $f(z) \in \Sigma_H$, f has the representation (1.1).

In this article, we estimate the Gaussian curvature of nonparametric minimal surfaces over Ω by using the properties of univalent harmonic mappings f in Δ with $f(\infty) = \infty$. Our estimates are stated as inequalities (2.6) and (2.7) in Theorem 2.4.

2. Curvature estimates

We want to discuss the Gaussian curvature K for nonparametric minimal surfaces S. Since $G_1'(z)^2 + G_2'(z)^2 + F'(z)^2 = 0$ by virtue of (1.3), the Gaussian curvature at each point is given by

$$K = -\left[\frac{4|\mathscr{G}'|}{|\mathscr{F}|(1+|\mathscr{G}|^2)^2}\right]^2,$$

where $\mathscr{F} = G_1' - iG_2'$ and $\mathscr{G} = F'/\mathscr{F}$, by Lemma 9.1 in [4]. Since $\mathscr{F} = -F'\omega$ and $\mathscr{G} = -1/\omega$, we have

$$|K| = \frac{16|\omega'|^2}{|F'/\omega|^2(1+|\omega|^2)^4}.$$

From (1.4) and (1.1), we have that

$$2(h' - g' + (A - \overline{A})/(2z))/(1+\omega^2) = F'/\omega.$$

Hence,

$$|K| = \frac{4|\omega'|^2|1+\omega^2|^2}{|h' - g' + (A-\overline{A})/(2z)|^2(1+|\omega|^2)^4}.$$

Furthermore, the estimate $|\omega'|/(1-|\omega|^2) \le 1/(|z|^2-1)$ from Schwarz's lemma for $|z| > 1$ implies

$$|K| \le \frac{4T(z)}{|h' - g' + (A-\overline{A})/(2z)|^2(|z|^2-1)^2},$$

where $T(z) = (1-|\omega|^2)^2|1+\omega^2|^2/(1+|\omega|^2)^4$.

Now we restrict our attention to the case of $\Omega = \mathbb{C}\backslash[a, b]$, where $[a, b]$ is a real line segment in the complex plane.

For $f = u + iv$, we necessarily have $v = 0$ on $|z| = 1$, and $f(\infty) = \infty$ gives

$$v(z) = \operatorname{Im}\{\alpha z + a_0 + \overline{\beta z}\} + \tilde{r}(z) + (\operatorname{Im} A)\log|z|,$$

where $\tilde{r}$ is harmonic and vanishes at infinity (cf. [3]). Let $R(\zeta) = \tilde{r}(1/\zeta)$; then $R(\zeta)$ is harmonic on $|\zeta| < 1$ and $R(0) = 0$. So we have

$$0 = R(0) = \frac{1}{2\pi}\int_0^{2\pi} R(e^{i\phi})\,d\phi = \frac{1}{2\pi}\int_0^{2\pi} \operatorname{Im}\{\bar{\alpha}e^{i\phi} + \bar{a}_0 + \beta e^{-i\phi}\}\,d\phi,$$

since $v = 0$ on $|z| = 1$ and $-\operatorname{Im}\{\alpha e^{-i\phi} + a_0 + \bar{\beta}e^{i\phi}\} = \operatorname{Im}\{\bar{\alpha}e^{i\phi} + \bar{a}_0 + \beta e^{-i\phi}\}$. This implies $\operatorname{Im}\bar{a}_0 = 0$. Thus a_0 is real. Therefore, $v(z) = \operatorname{Im}\{\alpha z + \overline{\beta z}\} + \tilde{r}(z) + (\operatorname{Im} A)\log|z|$. Solving the Dirichlet problem for $\tilde{r}$, we conclude that $\tilde{r}(z) = \operatorname{Im}\{(\bar{\alpha}/z) + (\beta/\bar{z})\}$, and therefore

$$v(z) = \operatorname{Im}\{\alpha z + \overline{\beta z} + (\bar{\alpha}/z) + (\beta/\bar{z})\} + (\operatorname{Im} A)\log|z|. \tag{2.1}$$

In the representation (1.1), it follows that

$$\operatorname{Im}\{h(z) + \overline{g(z)}\} = \operatorname{Im}\{\alpha z + \overline{\beta z} + (\bar{\alpha}/z) + (\beta/\bar{z})\}.$$

Taking derivatives with respect to z, we obtain

$$h' - g' = (\alpha - \beta) - (\overline{\alpha - \beta})z^{-2}.$$

We can easily show that we are free to normalize $\alpha - \beta$ to be real and positive. In that case,

$$h' - g' = (\alpha - \beta)(1 - z^{-2}). \tag{2.2}$$

This implies $h - g = (\alpha-\beta)(z + z^{-1}) +$ constant. But $h - g = (\alpha - \beta)z + a_0 + (a_1 - b_1)z^{-1} + \sum_{k=2}^{\infty}(a_k - b_k)z^{-k}$. By comparing these two equations, we get $a_1 - b_1 = \alpha - \beta$ and $a_k = b_k$ for $k \ge 2$. Thus we have $h - g = (\alpha - \beta)(z + z^{-1}) + a_0$.

Now we want to find, at a fixed point of Ω, a bound for $|K|$ for the family of nonparametric minimal surfaces that lie over $\Omega = \mathbb{C}\backslash[a, b]$.

Let $p = p_1 + ip_2 \in \Omega$; then there exists $z = re^{i\theta}$ such that $f(z) = p$. From (2.1) we get

$$p_2 = (\alpha - \beta)(r - r^{-1})\sin\theta + (\operatorname{Im} A)\log r.$$

Equation (2.2) implies

$$|h' - g' + (A - \overline{A})(2z)^{-1}|^2 = r^{-2}(\operatorname{Im} A + (\alpha - \beta)(r^2+1)r^{-1}\sin\theta)^2 + (\alpha-\beta)^2(1-r^{-2})^2\cos^2\theta.$$

So,

$$|K| \le \frac{4}{(r-r^{-1})^2(\operatorname{Im} A + (\alpha-\beta)(r+r^{-1})\sin\theta)^2 + (r-r^{-1})^4(\alpha-\beta)^2\cos^2\theta},$$

since $T(re^{i\theta}) \le 1$.

If A is real, then $p_2 = (\alpha-\beta)(r-r^{-1})\sin\theta$. In this case,

$$|K| \le \frac{4}{(r-r^{-1})^4(\alpha-\beta)^2 + 4p_2^2}.$$

And the necessary and sufficient condition for $p_2 = 0$ is $\sin\theta = 0$, since $\alpha - \beta$ and $r - r^{-1}$ are positive real for $r > 1$. So we obtain the estimates

$$|K| \le \frac{4}{p_2^2(4 + p_2^2(\alpha-\beta)^{-2}\sin^{-4}\theta)} \quad \text{if } p_2 \ne 0 \tag{2.3}$$

and

$$|K| \le \frac{4}{(\alpha-\beta)^2(r-r^{-1})^4} \quad \text{if } p_2 = 0. \tag{2.4}$$

But the parameters r and θ in these estimates are not geometric quantities. So we want to replace these by estimates of a more geometric nature, for example, in terms of p.

The following result is due to Hengartner and Schober [3].

Lemma 2.1 [3, Theorem 4.3]. *Let* $\Sigma_{HR} = \{f \in \Sigma_H : \alpha = 1,\ a_0 = \beta = 0,\ \mathbb{C}\backslash f(\Delta) \subset \mathbb{R}\}$. *Then the diameter* D_f *of* $\mathbb{C}\backslash f(\Delta)$ *satisfies* $\max_{\Sigma_{HR}} D_f = 2\pi$. *Equality occurs if and only if* $f(z) = z - \bar{z}^{-1} + 2\arg\left(\frac{1+i/z}{1-i/z}\right)$.

In order to obtain estimates of a more geometric nature, the following lemmas will be useful.

Lemma 2.2. *If* $f \in \Sigma_H$ *and* f *extends to be of bounded variation on* $|z| = 1$, *then*

$|\alpha + \bar{b}_1| \le L/(2\pi)$, $|b_n| \le L/(2n\pi)$ *for* $n \ge 2$,
$|\beta + \bar{a}_1| \le L/(2\pi)$, *and* $|a_n| \le L/(2n\pi)$ *for* $n \ge 2$,

where L *is the length of* $f(|z| = 1)$.

Proof. By integration by parts, we have

$$\left|\int_{|z|=1} z^n\, df\right| = \begin{cases} 2\pi|\alpha + \bar{b}_1| & \text{if } n = -1, \\ 2\pi|-n\bar{b}_{-n}| & \text{if } n \le -2, \\ 2\pi|a_1 + \bar{\beta}| & \text{if } n = 1, \\ 2\pi n|a_n| & \text{if } n \ge 2. \end{cases}$$

Since $|\int_{|z|=1} z^n\, df| \le \int_{|z|=1} |df| = L$, we have

$$|\alpha + \bar{b}_1| \le L/(2\pi), \qquad |a_1 + \bar{\beta}| \le L/(2\pi),$$

and

$$n|b_n| \le L/(2\pi), \qquad n|a_n| \le L/(2\pi) \quad \text{for } n \ge 2.$$

These inequalities are equivalent to the desired ones. Q.E.D.

Lemma 2.3. *If* $\omega(\infty) = 0$, *then* $\beta = A = 0$ *and*

$$\frac{b-a}{2\pi} \leq \alpha \leq \frac{b-a}{\pi}.$$

Proof. Since $\overline{f_{\bar z}}/f_z = -\omega^2$, it follows that $\overline{f_{\bar z}}(\infty) = \beta = 0$. $\tilde{f}(z) = (f(z) - a_0)/\alpha \in \Sigma_{HR}$ and $\mathbb{C}\backslash\tilde{f}(\Delta) = [(a-a_0)/\alpha, (b-a_0)/\alpha]$ imply that $(b-a)/\alpha \leq 2\pi$ by Lemma 2.1. So we have $(b-a)/2\pi \leq \alpha$. We can easily show that $\lim_{r\to 1^+} f(re^{i\theta})$ exits a.e. and belongs to $[a, b]$. Also, f is orientation-preserving near $\partial\Delta$. So f extends to be of bounded variation on $|z| = 1$. Now apply Lemma 2.2; then we obtain

$$|\alpha + \bar{b}_1| \leq (b-a)/\pi. \tag{2.5}$$

In order to obtain an upper bound for α, we need to know the coefficients of f in terms of the coefficients of ω and F'. Since $\omega(\infty) = 0$ and $F'/\omega \neq 0, \infty$, we have series expansions

$$\omega(z) = \sum_{\nu=n}^{\infty} x_\nu z^{-\nu} \quad \text{and} \quad F'(z) = \sum_{\nu=n}^{\infty} y_\nu z^{-\nu},$$

where $x_n y_n \neq 0$ and $n \geq 1$. Substitute these two series into (1.4) and obtain

$$f(z) = \frac{1}{2}(y_n x_n^{-1} z + (y_{n+1}x_n^{-1} - y_n x_{n+1} x_n^{-2}) \log z + \cdots)$$
$$\overline{-\frac{1}{2}((1-2n)^{-1} x_n y_n z^{1-2n} - (2n)^{-1}(y_n x_{n+1} + x_n y_{n+1}) z^{-2n} + \cdots)} + \text{constant}.$$

Since $f(z)$ must be a single-valued function, we must have that

$$y_{n+1}x_n^{-1} - y_n x_{n+1} x_n^{-2} = 0.$$

So we do not have a logarithmic term in $f(z)$, i.e., $A = 0$, and

$$\alpha = y_n/(2x_n), \qquad b_1 = \begin{cases} (x_1 y_1)/2 & \text{if } n = 1, \\ 0 & \text{if } n > 1. \end{cases}$$

Note that y_1 must be real because $\varphi = \operatorname{Re} F(z)$ and φ is single-valued. Thus, we have

$$|\alpha + \bar{b}_1| = \begin{cases} \alpha(1 + |x_1|^2) & \text{if } n = 1, \\ \alpha & \text{if } n > 1. \end{cases}$$

From (2.5), we obtain $\alpha(1 + |x_1|^2) \leq (b-a)/\pi$. This implies that $\alpha \leq (b-a)/\pi$. Q.E.D.

Remark. $\omega(\infty) = 0$ if and only if the unit normal vector to the surface with the standard orientation at ∞

$$\mathbb{N}_\infty = \left(\frac{2\operatorname{Re}\{-1/\omega\}}{1 + |\omega|^{-2}}, \frac{2\operatorname{Im}\{-1/\omega\}}{1 + |\omega|^{-2}}, \frac{1 - |\omega|^2}{1 + |\omega|^2}\right)$$

is $(0, 0, 1)$.

Theorem 2.4. *If S is a nonparametric minimal surface over the slit plane $\Omega = \mathbb{C}\backslash[a, b]$ and if the unit normal to the surface at ∞ is $(0, 0, 1)$, then we have*

$$|K(p)| \leq \frac{4(b-a)^2}{p_2^2[\pi^2 p_2^2 + 4(b-a)^2]} \quad \text{if } p_2 \neq 0, \tag{2.6}$$

$$|K(p)| \leq \frac{4(b-a)^2}{\pi^2 d^4} \quad \text{if } p_2 = 0, \tag{2.7}$$

where $p = p_1 + ip_2 \in \Omega$ and d is the distance from p to $[a, b]$.

Proof. Let $f(re^{i\theta}) = p$. If $p_2 \neq 0$, then we have (2.3). By applying Lemma 2.3, we obtain

$$|K(p)| \leq \frac{4(b-a)^2}{p_2^2[\pi^2 p_2^2 + 4(b-a)^2]}.$$

If $p_2 = 0$, then we have $\sin\theta = 0$, and so $f(s) = p$ for some real s. From $\beta = A = 0$, we have $f(z) = \alpha z + a_0 + (b_1 + \alpha)z^{-1} + \bar{b}_1 \bar{z}^{-1} + 2\operatorname{Re}\sum_{k=2}^{\infty} a_k z^{-k}$. If z is real, then $f(z)$ is real because a_0 is real. Since α is positive real, we have $f(r) > b$ for sufficiently large $r > 1$. This implies that $\lim_{r\to 1^+} f(r) = b$. From $\operatorname{Im}\{f\} = \operatorname{Im}\{h - g\}$, $\operatorname{Re}\{f\} = \operatorname{Re}\{h + g\}$, and $h' + g' = \frac{1-\omega^2}{1+\omega^2}(h' - g')$, it follows that we have the representation

$$\operatorname{Re} f(z) = \operatorname{Re}\left\{\alpha z + \int_\infty^z \left(\frac{1-\omega^2}{1+\omega^2}(h' - g') - \alpha\right) dz\right\} + a_0.$$

If $f(r) = p$ $(r > 1)$, then $p > b$ and

$$\begin{aligned} p - b &= \operatorname{Re}\int_1^r \frac{1-\omega^2}{1+\omega^2}(h' - g')\,dz \\ &= \operatorname{Re}\int_1^r \frac{1-\omega^2}{1+\omega^2}\alpha(1 - z^{-2})\,dz \quad \text{(from (2.2))} \\ &= \alpha \int_1^r \operatorname{Re}\left\{\frac{1-\omega^2}{1+\omega^2}\right\}(1 - x^{-2})\,dx \\ &\leq \alpha \int_1^r \frac{x^2+1}{x^2-1}(1 - x^{-2})\,dx \quad \text{(from Schwarz's lemma for } |z| > 1) \\ &= \alpha(r - r^{-1}). \end{aligned}$$

Thus $p - b \leq \alpha(r - r^{-1})$. From (2.4) and this, we have $|K| \leq 4\alpha^2/(p-b)^4$. Apply Lemma 2.3; then we have $|K| \leq 4(b-a)^2/(p-b)^4\pi^2$ for $p > b$. Similarly, we obtain $|K| \leq 4(b-a)^2/(p-a)^4\pi^2$ for $p < a$. Q.E.D.

Unfortunately, the estimates of $|K(p)|$ are not sharp. In order to obtain estimates (2.6) and (2.7), inequalities $T(z) \leq 1$ and $\alpha \leq (b-a)/\pi$ are used. In the inequality $T(z) \leq 1$, equality for $z \neq \infty$ occurs if and only if $\omega(z) = c/z$ $(|c| = 1)$. Assume that $\omega(z) = e^{i\gamma}/z$ for some γ; then $|\alpha + \bar{b}_1| = 2\alpha$ (cf. Proof of Lemma 2.3), and $|\alpha + \bar{b}_1| \leq (b-a)/\pi$ (cf. Lemma 2.2). Therefore, the strict inequality $\alpha < (b-a)/\pi$ holds in this case.

But the estimates (2.6) and (2.7) are good in the following sense. As $b - a \to \infty$, the estimate (2.6) reduces to the sharp estimate $|K(p)| \leq 1/p_2^2$ from Hengartner and Schober [2]. For $b - a < \infty$, it is even smaller.

If $b = 0$, then the estimate (2.7) is $4a^2/\pi^2 p^4$ when $p > 0$. This bound is better than Hengartner and Schober's when $a^2/p^2 < \pi^2(3 + 2\sqrt{3})/48$.

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