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FAY'S TRISECANT FORMULA AND CROSS-RATIOS

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ABSTRACT. This note considers Fay's trisecant formula as a relation between cross-ratio functions and thereby gives a simple proof of the trisecant formula. In this proof the expression for the cross-ratio function is lifted from the theta locus to the entire Jacobian. Variations of the formula as used by different authors are also given.

The most powerful of the Jacobian theta identities is the trisecant formula due to Fay [3]:

$$\begin{aligned} &\theta\left(w + \int_{z_2}^{z_1} \vec{\omega}, \Omega\right) \theta\left(w + \int_{a_1}^{a_2} \vec{\omega}, \Omega\right) p(z_1, z_2, a_1, a_2) \\ &\quad + \theta\left(w + \int_{a_1}^{z_1} \vec{\omega}, \Omega\right) \theta\left(w + \int_{z_2}^{a_2} \vec{\omega}, \Omega\right) p(z_1, a_1, z_2, a_2) \\ &= \theta(w, \Omega)\, \theta\left(w + \int_{z_2+a_1}^{z_1+a_2} \vec{\omega}, \Omega\right). \end{aligned} \tag{1-1}$$

Besides being an identity among theta functions, the trisecant formula is also an identity between the cross-ratio functions

$$p(z_1, z_2, a_1, a_2) \quad \text{and} \quad p(z_1, a_1, z_2, a_2).$$

The Riemann surface cross-ratio function, $p(z_1, z_2, a_1, a_2)$, is the generalization of the usual cross-ratio $(z_1, z_2, a_1, a_2) = \frac{(z_1-a_1)(z_2-a_2)}{(z_1-a_2)(z_2-a_1)}$ and satisfies all of the identities that (z_1, z_2, a_1, a_2) does, except $(z_1, z_2, a_1, a_2) + (z_1, a_1, z_2, a_2) = 1$. The trisecant identity can hence be viewed as the replacement for this identity as we move from $\mathbb{P}^1$ to a general Riemann surface. Taking the cross-ratio function as a starting point also provides an simple proof of the trisecant formula which therefore, fits into a natural exposition of Riemann surfaces such as given in Gunning [4, 5]. Proofs of this formula may be found in Fay [3], Mumford [7], Farkas [1], and Gunning [6].

1. NOTATION AND BACKGROUND

Let M be a compact Riemann surface of genus $g \geq 1$ along with a marking; that is, a base point z_0 on the universal cover $\widehat{M}$, and distinguished generators

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$\{A_i, B_i\}_{i=1}^g$ for the fundamental group $\Gamma = \pi_1(z_0, M)$ which will be identified with the group of covering transformations. Let $\vec{\omega}$ be a normalized basis of holomorphic 1-forms and let the $g \times g$ period matrix Ω be defined by $(I, \Omega) = (\int_{A_i} \vec{\omega}, \int_{B_i} \vec{\omega})$, so that the theta function associated to the marked Riemann surface is:

$$\theta(w, \Omega) = \sum_{n \in \mathbb{Z}^g} e^{2\pi i[\frac{1}{2} n\Omega n + n \cdot w]}.$$

The θ-function vanishes simply on its zero locus, $\Theta = \{w \in \mathbb{C}^g : \theta(w, \Omega) = 0\}$. The normalized differentials of the third kind, $\omega_{z_1 z_2}$, satisfy the bilinear relation [2]

$$\int_{a_2}^{a_1} \omega_{z_1 z_2} = \int_{z_2}^{z_1} \omega_{a_1 a_2} \qquad \bmod (2\pi i \mathbb{Z})$$

and so the cross-ratio function $p(z_1, z_2, a_1, a_2) = e^{\int_{a_2}^{a_1} \omega_{z_1 z_2}} = e^{\int_{z_2}^{z_1} \omega_{a_1 a_2}}$ is a well-defined meromorphic function on $\widehat{M}^4$ [4, 5]. The cross-ratio function is in fact characterized by its divisor, its symmetries, and its transformation properties.

The cross-ratio function. *$p(z_1, z_2, a_1, a_2)$ is the unique meromorphic function on $\widehat{M}^4$ such that:*

(1) $p(z_1, z_2, a_1, a_2) = p(a_1, a_2, z_1, z_2) = p(a_2, a_1, z_2, z_1)$
$= p(z_2, z_1, a_2, a_1)$;
(2) $p(z_2, z_1, a_1, a_2) = p(z_1, z_2, a_1, a_2)^{-1}$;
(3) $p(z_1, z_2, a_1, a_2)\ p(z_1, a_2, z_2, a_1)\ p(z_1, a_1, a_2, z_2) = -1$;
(4) $p(z_1, z_2, a_1, a_2)$ *has simple zeros on* $z_1 = a_1 \bmod \Gamma$, *simple poles on* $z_1 = a_2 \bmod \Gamma$, *and the value* 1 *when* $a_1 = a_2$; *and*
(5) $p(Tz_1, z_2, a_1, a_2) = \chi_{\vec{t}}(T) p(z_1, z_2, a_1, a_2)$, $T \in \Gamma$, *where* $\chi_{\vec{t}}$ *is the character defined by*: $\chi_{\vec{t}}(A_i) = 1$, $\chi_{\vec{t}}(B_i) = e^{-2\pi i t_i}$, *and* $\vec{t} = \int_{a_1}^{a_2} \vec{\omega}$.

The first order theta function $\theta(w, \Omega)$ has a transformation rule, and is also characterized up to a constant multiple by that rule.

The first-order theta function. *$\theta(w + t, \Omega)$ is the unique (up to a constant factor) holomorphic function on $\mathbb{C}^g$ such that:*

For $p, q \in \mathbb{Z}^g$,

$$\theta(w + t + p + \Omega q, \Omega) = \bar{\chi}_t(q) e^{-2\pi i[\frac{1}{2} q\Omega q + q \cdot w]} \theta(w + t, \Omega)$$

where $\bar{\chi}_t$ is the character defined by: $\bar{\chi}_t(p) = 1$, $\bar{\chi}_t(q) = e^{-2\pi i t \cdot q}$

Restriction of the theta function to the image of the abelian integral $\int_a^z \vec{\omega}$ gives a well-defined divisor described by Riemann's vanishing theorem.

Riemann's vanishing theorem [2]. *$\exists r \in \mathbb{C}^g$ (the Riemann point) such that:*

$$\Theta = \left\{ t \in \mathbb{C}^g : t = r - \int_{z_0}^{y_1} \vec{\omega} - \cdots - \int_{z_0}^{y_{g-1}} \vec{\omega}, y_1, \dots y_{g-1} \in \widehat{M} \right\}.$$

Also, let $t = r - \int_{z_0}^{y_1} \vec{\omega} - \cdots - \int_{z_0}^{y_{g-1}} \vec{\omega}$ be a nonsingular point of Θ; then $\theta(t + \int_a^z \vec{\omega}, \Omega)$ has divisor $[y_1 + \cdots + y_{g-1+a}]$ on $M = \widehat{M}/\Gamma$.

Using Riemann's vanishing theorem, the transformation of the theta function, and the characterization of the cross-ratio function, we obtain:

$$p(z_1, z_2, a_1, a_2) = \frac{\theta(\alpha + \int_{a_1}^{z_1} \vec{\omega}, \Omega)\, \theta(\alpha + \int_{a_2}^{z_2} \vec{\omega}, \Omega)}{\theta(\alpha + \int_{a_2}^{z_1} \vec{\omega}, \Omega)\, \theta(\alpha + \int_{a_1}^{z_2} \vec{\omega}, \Omega)} \tag{1-2}$$

for any nonsingular $\alpha \in \Theta$ [5].

symmetries (1) and (2) of the theta expression are immediate and the divisor as a function of z_1 given in (4) $a_1 - a_2 = (a_1 + y_1 + \cdots + y_{g-1}) - (a_2 + y_1 + \cdots + y_{g-1})$ follows directly from the vanishing theorem. The theta expression transforms according to (5) because $(\alpha + \int_{a_1}^{z_1} \vec{\omega}) - (\alpha + \int_{a_2}^{z_1} \vec{\omega}) = \int_{a_1}^{a_2} \vec{\omega}$. Property (3) is a consequence of the others and was only listed for thoroughness. Because these properties characterize the cross-ratio function we have (1-2). The entire function theory of a Riemann surface is expressed by its cross-ratio function, $p(z_1, z_2, a_1, a_2)$, and (1-2) expresses this natural object on M in terms of the theta function on $\mathbb{C}^g$.

2. The proof

The establishment of equation (1-2) requires Riemann's vanishing theorem; it is a remarkable equation because α may be any nonsingular point of the theta locus. Hence we have a holomorphic function, $h(w)$, which vanishes on the theta locus.

$$\begin{aligned} h(w) = {} & \theta\left(w + \int_{a_2}^{z_1} \vec{\omega}, \Omega\right) \theta\left(w + \int_{a_1}^{z_2} \vec{\omega}, \Omega\right) p(z_1, z_2, a_1, a_2) \\ & - \theta\left(w + \int_{a_1}^{z_1} \vec{\omega}, \Omega\right) \theta\left(w + \int_{a_2}^{z_2} \vec{\omega}, \Omega\right). \end{aligned}$$

Any such function must be equal to $\lambda \theta(w, \Omega)\, \theta(w + \int_{a_1+a_2}^{z_1+z_2} \vec{\omega}, \Omega)$! For $\theta(w, \Omega)$ vanishes *simply* on Θ and so $h(w)/\theta(w, \Omega)$ is a holomorphic first order theta function for the fixed character $\bar{\chi}_t$, $t = \int_{a_1+a_2}^{z_1+z_2} \vec{\omega}$, which is thereby determined up to a multiplicative constant $\lambda(z_1, z_2, a_1, a_2)$. Therefore:

$$\begin{aligned} & \theta\left(w + \int_{a_2}^{z_1} \vec{\omega}, \Omega\right) \theta\left(w + \int_{a_1}^{z_2} \vec{\omega}, \Omega\right) p(z_1, z_2, a_1, a_2) \\ & \quad - \theta\left(w + \int_{a_1}^{z_1} \vec{\omega}, \Omega\right) \theta\left(w + \int_{a_2}^{z_2} \vec{\omega}, \Omega\right) \\ & \quad = \lambda(z_1, z_2, a_1, a_2) \theta(w, \Omega)\, \theta\left(w + \int_{a_1+a_2}^{z_1+z_2} \vec{\omega}, \Omega\right). \end{aligned}$$

We may evaluate the constant λ by setting $w = \alpha + \int_{z_1}^{a_2} \vec{\omega}$ for nonsingular $\alpha \in \Theta$.

$$\lambda(z_1, z_2, a_1, a_2) = -\frac{\theta(\alpha + \int_{a_1}^{a_2} \vec{\omega}, \Omega)\, \theta(\alpha + \int_{z_1}^{z_2} \vec{\omega}, \Omega)}{\theta(\alpha + \int_{z_1}^{a_2} \vec{\omega}, \Omega)\, \theta(\alpha + \int_{a_1}^{z_2} \vec{\omega}, \Omega)} = -p(a_2, z_2, a_1, z_1).$$

Substitution for λ gives Fay's trisecant formula.

$$\begin{aligned}\theta\left(w+\int_{a_2}^{z_1}\vec{\omega},\Omega\right)\theta\left(w+\int_{a_1}^{z_2}\vec{\omega},\Omega\right)&p(z_1, z_2, a_1, a_2)\\ +\theta(w,\Omega)\,\theta\left(w+\int_{a_1+a_2}^{z_1+z_2}\vec{\omega},\Omega\right)&p(a_2, z_2, a_1, z_1)\\ =\theta\left(w+\int_{a_1}^{z_1}\vec{\omega},\Omega\right)\theta\left(w+\int_{a_2}^{z_2}\vec{\omega},\Omega\right)&.\end{aligned}$$

The symmetries of p give the prettier form (1-1).

3. Other forms of Fay's trisecant formula

It may be of service to point out the various forms of the trisecant identity which appear in the literature. The paper of Farkas [1] proves the identity by exploiting divisors on the Riemann surface, whereas this note uses divisors on the Jacobian. Replacing the cross-ratios $p(z_1, z_2, a_1, a_2)$ and $p(z_1, a_1, z_2, a_2)$ via (1-2) one obtains the formula proven by Farkas.

If one employs the prime function $E(x, y)$ on $\widehat{M}^2$ that vanishes simply when $x = y \mod \Gamma$; then substitution via

$$p(z_1, z_2, a_1, a_2) = \frac{E(z_1, a_1)E(z_2, a_2)}{E(z_1, a_2)E(z_2, a_1)}$$

will give the formula in Fay and Mumford [3, 6]. To obtain the formula as proven by Gunning, introduce the vector of basis elements for the second order theta functions $\vec{\theta}_2(w, \Omega) = \{\theta[\nu/2|0](2w, 2\Omega),\ \nu \in (\mathbb{Z}/2\mathbb{Z})^g\}$ and recall the addition theorem:

$${}^t\vec{\theta}_2(w_1, \Omega)\cdot\vec{\theta}_2(w_2, \Omega) = \theta(w_1 - w_2, \Omega)\,\theta(w_1 + w_2, \Omega).$$

An eight line calcualation using $(1-1)$ will then reveal:

$$\begin{aligned}\vec{\theta}_2\left(\frac{1}{2}\int_{z_1+a_2}^{z_2+a_1}\vec{\omega},\Omega\right) &= p(z_1, z_2, a_1, a_2)\vec{\theta}_2\left(\frac{1}{2}\int_{z_2+a_2}^{z_1+a_1}\vec{\omega},\Omega\right)\\ &\quad + p(z_1, a_1, z_2, a_2)\vec{\theta}_2\left(\frac{1}{2}\int_{a_1+a_2}^{z_1+z_2}\vec{\omega},\Omega\right).\end{aligned}$$

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