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PSEUDO-RIEMANNIAN METRICS AND HIRZEBRUCH SIGNATURE

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ABSTRACT. On compact, orientable, $4k$-dimensional manifolds, nonvanishing Hirzebruch signature is shown to be an obstruction to the existence of certain kinds of pseudo-Riemannian metrics.

Consider a compact, orientable, $4k$-dimensional manifold M. The bilinear mapping

$$Q(\alpha, \beta) := \int_M \alpha \wedge \beta$$

on the de Rham cohomology group $H^{2k}(M, \mathbf{R})$ is a symmetric, nondegenerate bilinear form. Suppose it is of signature type (p, q). The number $\tau := p - q$ is called the Hirzebruch signature and is a topological invariant of M. (Consult, for example, [2, p. 224] for properties of τ, and translate into differential forms via the de Rham isomorphism of cohomology algebras.) With respect to a Riemannian metric on M, the Hodge theory of harmonic differential forms allows one to replace the space $H^{2k}(M, \mathbf{R})$ by the space H^{2k} of harmonic $2k$-forms on M. Since the Hodge star operator $*$ commutes with the Hodge Laplacian Δ, then $*$ induces a linear isomorphism of H^{2k}.

A pseudo-Riemannian metric g on M also defines a Hodge star operator and a Hodge Laplacian in an analogous fashion; but the Hodge theory of the Riemannian case need not be valid with respect to these operators. For the remainder of this article, all metric-dependent quantities are with respect to g. If E^p denotes the space of global p-forms on M, let B^p, D^p, Z^p, and C^p denote the subspaces of d-exact, δ-exact, d-closed, and δ-closed forms respectively. Avez [1, pp. 168–175] calls (M, g) "weakly de Rham" if $A^p := B^p + D^p + (Z^p \cap C^p)$ is dense (with respect to the weak topology defined by the natural inner product on E^p) in E^p. Avez shows that this condition entails that $H^p = (Z^p \cap C^p)$, whence every harmonic form is d-closed, and that if a cohomology class contains a harmonic representative it contains a unique such. If $A^p = E^p$, Avez calls (M, g) "strongly de Rham" and demonstrates [1, p. 181], at least in two dimensions, that the former notion is weaker than the latter. The argument in the opening paragraph is applicable to (M, g) if (M, g) is strongly de Rham. More generally, it is applicable if (M, g) is only weakly de Rham and each cohomology class contains a harmonic representative (that

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this condition is weaker than (M, g) being strongly de Rham is again shown by Avez's two-dimensional examples). Call such an (M, g) "weakly de Rham with harmonic cohomology."

Theorem. *If M is a compact, orientable, $4k$-dimensional manifold M carrying a pseudo-Riemannian metric g of signature (r, s), s odd (equivalently r odd), such that (M, g) is weakly de Rham with harmonic cohomology then the Hirzebruch signature of M vanishes.*

Proof. For any two forms of equal degree, $\alpha \wedge *\beta = \beta \wedge *\alpha$, so for α and β in H^{2k} one has $Q(*\alpha, *\beta) = \int_M *\alpha \wedge *\beta = \int_M \beta \wedge **\alpha = (-1)^{2k(2k)+s} \int_M \beta \wedge \alpha = (-1)^s \int_M \alpha \wedge \beta = (-1)^s Q(\alpha, \beta)$, i.e. on H^{2k}, $Q(*\alpha, *\beta) = (-1)^s Q(\alpha, \beta)$. Thus, for s odd, $*$ is an anti-isometry of (H^{2k}, Q), which must therefore be neutral (cf. [3, p. 164]), i.e. $\tau = 0$.

Except in two dimensions, Avez [1] gave no results concerning the existence or impossibility of weakly or strongly de Rham pseudo-Riemannian metrics.

Example. The complex projective spaces CP^{2k} do not admit any pseudo-Riemannian metrics of the type stated in the theorem.

Proof. $\tau(\mathbf{CP}^{2k}) = 1$ [2, p. 225].

As an indication of the limitations of the theorem presented above, suppose M admits a Riemannian metric of constant curvature. Then the Pontryagin classes $p_1, \ldots, p_k$ all vanish (cf. [4, p. 516]). By the Hirzebruch signature theorem [2, p. 224], it follows that τ vanishes. Thus, the theorem provides no information in such a case.

Of special interest is dimension four, in which case the metric must be Lorentzian for the theorem to be relevant and so the Euler characteristic χ must vanish. Examples of four-dimensional, compact, orientable manifolds with χ zero but τ nonzero appear to require a little effort to produce. Let S be a compact, orientable, two-dimensional manifold of genus two so that $\chi(S) = -2$. Let X_n denote the connected sum of n copies of $\mathbf{CP}^2$ and Y_m the connected sum of m copies of $\mathbf{CP}^1 \times S$. Using the formula

$$\chi(M \sharp N) = \chi(M) + \chi(N) - 2,$$

valid for connected, orientable manifolds M and N, properties of τ, and the fact that $\tau(\mathbf{CP}^{2k}) = 1$, one easily determines:

$$\chi(X_n) = n + 2, \qquad \tau(X_n) = n,$$
$$\chi(Y_m) = 2 - 6m, \qquad \tau(Y_m) = 0.$$

Hence, one finds

$$\chi(X_n \sharp Y_m) = n - 6m + 2, \qquad \tau(X_n \sharp Y_m) = n.$$

Reversing the orientation of $\mathbf{CP}^2$ reverses the sign of τ in these formulae.

Example. $M_m := X_n \sharp Y_m$, with $n = 6m - 2$ and m any positive integer, are compact, orientable, four-dimensional manifolds admitting Lorentz metrics that cannot be weakly de Rham with harmonic cohomology.

An example of a compact, orientable, four-dimensional Lorentz manifold that is weakly de Rham with harmonic cohomology can be produced by extending Avez's two-dimensional example [1, pp. 175–177].

Example. Let M be $\mathbf{S}^1 \times \mathbf{S}^1 \times \mathbf{S}^1 \times \mathbf{S}^1$ equipped with the "pseudo-Euclidean" metric

$$g := (\sigma^1)^2 - \mu^2[(\sigma^2)^2 + (\sigma^3)^2 + (\sigma^4)^2],$$

where σ^i is the pullback of the standard global 1-form on $\mathbf{S}^1$ by the projection onto the ith factor and μ^2 is an irrational constant. (M, g) is weakly de Rham with harmonic cohomology.

Proof. The 1-forms $\sigma^1, \dots, \sigma^4$ provide a global frame for the cotangent bundle T^* of M (indeed, a pseudo-orthonormal frame when suitably normalized) and global frames for each $\Lambda^k(T^*)$ may be formed from wedge products of the σ^i. Moreover, by the Künneth formula, these various forms may be taken as representatives of the distinct cohomology classes of M. The action of the Hodge star operator with respect to g on these forms is easily determined and one finds that they lie in $Z^p \cap C^p$ for the appropriate p, and thus are harmonic.

Let $L^p := \Delta(E^p)$ and denote by R^p the subspace of E^p whose elements have constant components when expressed in terms of the global frame derived from the σ^i. The Fourier series argument of Avez [1, p. 176] extends to the present example with the same conclusion that $L^0 + R^0$ is dense in E^0. It is in this argument that the irrationality of μ^2 is vital. Of course, L^0 is contained in $B^0 + D^0$ and R^0 in $Z^0 \cap C^0$. Now $E^1 \simeq (E^0)^4$, via the global frame mentioned above. Writing $\Phi = \sum_{i=1}^4 f_i \sigma^i$ for Φ in E^1 then, because $\sigma^i = d\theta^i$ for obvious local coordinates $\{\theta^1, \dots, \theta^4\}$ in a neighborhood of any point and because the metric is pseudo-Euclidean, the Weitzenböck formula yields $\Delta\Phi = \sum_{i=1}^4 (\Delta f_i)\sigma^i$. Thus, L^1 may be identified with $(L^0)^4$ and R^1 with $(R^0)^4$ under the linear homeomorphism $E^1 \simeq (E^0)^4$. Thus the density of $L^0 + R^0$ in E^0 entails the density of $L^1 + R^1$ in E^1, and consequently the density of $B^1 + D^1 + (Z^1 \cap C^1)$ in E^1. The same argument may be applied to E^p for $p = 2, 3, 4$. Note also that the density result for a given p follows from the density result for $n - p$ by application of $*$.

Finally, note that the proof of the theorem requires only that one can replace $H^{2k}(M, \mathbf{R})$ by H^{2k}. This is not just a condition on forms of degree $2k$, however. If α is a harmonic $(2k-1)$-form that is δ-closed but not d-closed, then $\Delta(d\alpha) = d\delta\, d\alpha = d(\Delta\alpha) = 0$, whence harmonic representatives of elements of $H^{2k}(M, \mathbf{R})$ will not be unique.

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