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ON A GENERALIZATION OF THE NOTION OF CENTRALIZING MAPPINGS

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ABSTRACT. Let R be a ring with center Z. A mapping $f: R \to R$ is called centralizing (resp. commuting) if $[f(x), x] \in Z$ (resp. $[f(x), x] = 0$) for all $x \in R$. In this paper we consider a more general case where a mapping $f: R \to R$ satisfies $[[f(x), x], x] = 0$ for all $x \in R$; it is shown that if R is a prime ring of characteristic not 2, then every additive mapping with this property is commuting.

Let R be a ring with center Z. A mapping f of R into itself is called centralizing if $[f(x), x] \in Z$ for all $x \in R$, where $[u, v]$ denotes the commutator $uv - vu$. In the special case where $[f(x), x] = 0$ for all $x \in R$, the mapping f is said to be commuting. In [10] Posner initiated the study of centralizing mappings. He showed that if d is a centralizing derivation of a prime ring R, then either $d = 0$ or R is commutative. Over the last twenty years a lot of work has been done on centralizing and commuting mappings. We refer the reader to some recent papers [1, 2, 3, 4, 5, 7, 9, 11] where further references can be found.

Recently Vukman [11] extended Posner's theorem by showing that if d is a derivation of a prime ring R of characteristic not 2, such that $[[d(x), x], x] = 0$ for all $x \in R$, then $d = 0$ or R is commutative. In fact, in view of Posner's theorem he merely showed that d is commuting. It is our aim in this paper to prove that this conclusion holds for any additive mapping. More precisely, we prove the following result.

Theorem 1. *Let R be a prime ring of characteristic not 2. If an additive mapping $f: R \to R$ satisfies $[[f(x), x], x] = 0$ for all $x \in R$, then $[f(x), x] = 0$ for all $x \in R$ (i.e., f is commuting).*

In particular, this theorem implies that every additive centralizing mapping of a prime ring of characteristic not 2 is actually commuting. However, in our forthcoming paper [4] it is shown that this result is true under some weaker hypothesis. In [4] we also proved the following result: If f is an additive commuting mapping of a prime ring R, then there exist an element λ in C, the

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extended centroid of R, and an additive mapping $\zeta\colon R \to C$, such that $f(x) = \lambda x + \zeta(x)$ for all $x \in R$. Combining this result with Theorem 1, one obtains a characterization of additive mappings $f\colon R \to R$ satisfying $[[f(x), x], x] = 0$ for all $x \in R$, where R is a prime ring of characteristic not 2.

We also remark that in view of Theorem 1, several results in papers quoted above can now be stated in a more general form.

The statement of Theorem 1 can be expressed as follows: If R is a prime ring of characteristic not 2 and $f\colon R \to R$ is an additive mapping such that the mapping $x \to [f(x), x]$ is commuting, then f is commuting. The question arises whether the same conclusion holds if we assume that the mapping $x \to [f(x), x]$ is centralizing. Neglecting the fact that we have the additional assumption that R is 3-torsionfree, the following result gives much more than the affirmative answer to this question.

Theorem 2. *Let R be a ring, and let $B\colon R \times R \to R$ be a biadditive mapping. Suppose that the mapping $x \to B(x, x)$ is centralizing. If R is 2-torsionfree and 3-torsionfree, and if the center Z of R does not contain any nonzero nilpotents (in particular, if R is semiprime), then the mapping $x \to B(x, x)$ is commuting.*

Of course, Theorem 2 can be applied to the case where $f\colon R \to R$ is an additive mapping with the property that the mapping $x \to [f(x), x]$ is centralizing. Combining this fact with Theorem 1 we get

Corollary 1. *Let R be a prime ring of characteristic different from 2 and 3. If $f\colon R \to R$ is an additive mapping, such that the mapping $x \to [f(x), x]$ is centralizing, then f is commuting.*

As an immediate consequence of Corollary 1 and Posner's theorem, we obtain the main result in Vukman's paper [11], which states that if R is a noncommutative prime ring of characteristic different from 2 and 3 and d is a derivation of R such that the mapping $x \to [d(x), x]$ is centralizing, then $d = 0$.

The proof of Theorem 2 is entirely elementary, while our method in proving Theorem 1 requires the use of some results on Martindale's extended centroid. Therefore we first recall a few facts concerning extended centroid and central closure of a prime ring (see [6] or [8] for details). Let R be a prime ring, and let M be the set of all pairs (U, f), where U is a nonzero ideal of R and $f\colon U \to R$ is a right R-module map of U into R. We define an equivalence relation on M by $(U, f) \sim (V, g)$ if $f = g$ on some nonzero ideal W contained in $U \cap V$. The set Q of all equivalence classes forms a ring under the operations induced by addition and composition of representatives of the equivalence classes. We embed R in Q via the left multiplications on R by the elements in R. The center C of Q is a field containing the centroid of R and is called the extended centroid of R. The C-algebra $S = RC + C$ is again a prime ring and is called the central closure of R. The extended centroid of S is C. Thus the center of S coincides with the extended centroid of S.

We make crucial use of the following result.

Lemma 1 [6, Lemma 1.3.2]. *Suppose that elements a_i, b_i in R satisfy $\sum a_i x b_i = 0$ for all $x \in R$. If the b_i 's are nonzero then the a_i 's are linearly independent over C.*

We also need

Lemma 2. *If $s \in S$ satisfies $[[s, x], x] = 0$ for all $x \in R$, then $s \in C$.*

If $s \in R$, then Lemma 2 is merely a special case of Posner's theorem. Fortunately, even if $s \notin R$ the same proof works, but we include it for the sake of completeness.

Proof of Lemma 2. A linearization of $[[s, x], x] = 0$ gives

$$(1) \qquad [[s, x], y] + [[s, y], x] = 0 \quad \text{for all } x, y \in R.$$

Replacing y by yx in (1) and using $[[s, x], x] = 0$, we obtain

$$\begin{aligned} 0 &= [[s, x], yx] + [[s, yx], x] = [[s, x], y]x + [[s, y]x + y[s, x], x] \\ &= [[s, x], y]x + [[s, y], x]x + [y, x][s, x]. \end{aligned}$$

Applying (1), we then get $[y, x][s, x] = 0$ for all $x, y \in R$. Taking yz for y in this relation and using $[yz, x] = [y, x]z + y[z, x]$, we see that $[y, x]R[s, x] = 0$ for all $x, y \in R$. But then also $[s, x]S[s, x] = 0$ for all $x \in R$. Since S is prime, it follows that s commutes with all elements in R; hence $s \in C$.

We now have enough information to prove Theorem 1.

Proof of Theorem 1. For the proof we need several steps. In the first lemma, we extend the mapping $(x, y) \to [f(x), y]$, where $x, y \in R$, to a biadditive mapping of $S \times S$ to S.

Lemma A. *There exists a biadditive mapping B: $S \times S \to S$ such that $B(x, y) = [f(x), y]$ for $x, y \in R$. Moreover, B has the following properties:*

(i) *$B(s, \lambda) = B(\lambda, s) = 0$ for all $\lambda \in C$, $s \in S$;*
(ii) *for $t \in S$, the mapping $s \to B(t, s)$ is an inner derivation;*
(iii) *$[B(s, s), s] = 0$ for all $s \in S$.*

Proof. Taking $x + y$ instead of x in $[[f(x), x], x] = 0$, we obtain

$$\begin{aligned} (2) \quad & [[f(x), x], y] + [[f(x), y], x] + [[f(x), y], y] \\ & \qquad + [[f(y), x], x] + [[f(y), x], y] + [[f(y), y], x] = 0 \end{aligned}$$

for all $x, y \in R$. Replacing y by $-y$ in (2), and comparing the relation so obtained with (2), we obtain, since R has a characteristic different from 2, that

$$(3) \quad [[f(x), x], y] + [[f(x), y], x] + [[f(y), x], x] = 0 \quad \text{for all } x, y \in R$$

This relation is needed in the sequel. Now define B: $S \times S \to S$ by

$$B\left(\sum_{i=1}^{n} \lambda_i x_i + \mu, s\right) = \left[\sum_{i=1}^{n} \lambda_i f(x_i), s\right].$$

In order to show that B is well defined, suppose $\sum_{i=1}^{n} \lambda_i x_i + \mu = 0$. We must verify that in this case, the element $\sum_{i=1}^{n} \lambda_i f(x_i)$ commutes with all elements in S. We may assume that $\lambda_1 \neq 0$. Hence $x_1 = \sum_{i=2}^{n} \mu_i x_i + \nu$ where $\mu_i = -\lambda_1^{-1}\lambda_i$

and $\nu = -\lambda_1^{-1}\mu$. By (3) it follows that for any $x \in R$, we have

$$\begin{aligned}
[x, [f(x_1), x]] &= [[f(x), x], x_1] + [[f(x), x_1], x] \\
&= \left[[f(x), x], \sum_{i=2}^{n} \mu_i x_i + \nu\right] + \left[\left[f(x), \sum_{i=2}^{n} \mu_i x_i + \nu\right], x\right] \\
&= \sum_{i=2}^{n} \mu_i \{[[f(x), x], x_i] + [[f(x), x_i], x]\} \\
&= \sum_{i=2}^{n} \mu_i [x, [f(x_i), x]] = \left[x, \left[\sum_{i=2}^{n} \mu_i f(x_i), x\right]\right].
\end{aligned}$$

Thus

$$\left[x, \left[f(x_1) - \sum_{i=2}^{n} \mu_i f(x_i), x\right]\right] = 0 \quad \text{for all } x \in R.$$

By Lemma 2 it follows that the element $f(x_1) - \sum_{i=2}^{n} \mu_i f(x_i)$ is contained in C. But then also $\sum_{i=1}^{n} \lambda_i f(x_i) \in C$. This means that B is well defined.

It is obvious that B is a biadditive mapping satisfying (i) and (ii). Using (3) and its linearized form, one shows by a direct computation that B also satisfies (iii). Lemma A is thereby proved.

We assume henceforth that S is any prime ring of characteristic not 2, such that its center C coincides with its extended centroid, and that B: $S \times S \to S$ is an additive mapping satisfying (i), (ii), and (iii). Our intention is to show that $B(s, s) = 0$ for every s in S. According to Lemma A, with this the theorem is proved.

First note that the same approach as in the proof of (3) also gives the following:

(4) $$[B(s, s), t] + [B(s, t), s] + [B(t, s), s] = 0 \quad \text{for all } s, t \in S.$$

We continue with a technical lemma.

Lemma B. *For all* $u, s \in S$,

$$u^2 s B(u, u) = usB(u^2, u) + s(u^2 B(u, u) - uB(u^2, u)).$$

Proof. Define a mapping D: $S \times S \times S \to S$ by

(5) $$D(s, t, u) = [B(s, t), u] + [B(s, u) + B(u, s), t].$$

Linearizing (4) we see that on the other hand

(6) $$D(s, t, u) = [u, B(t, s)] + [s, B(t, u) + B(u, t)].$$

Given $u, s \in S$ we compute $D(su, u^2, u)$ in two ways. The idea behind these computations is the observation that, in view of (5) and (ii), the mapping $t \to B(s, t, u)$ is equal to the sum of the composition of two derivations and the derivation; similarly, by (6) and (ii), the mapping $s \to B(s, t, u)$ has the same property.

According to (5) we have

$$D(su, u^2, u) = [B(su, u^2), u] + [B(su, u) + B(u, su), u^2].$$

By assumption, the mapping $t \to B(su, t)$ is a derivation, so it follows that

$$\begin{aligned} D(su, u^2, u) &= [B(su, u)u + uB(su, u), u] + [B(su, u) + B(u, su), u^2] \\ &= [B(su, u), u]u + u[B(su, u), u] + [B(su, u) + B(u, su), u]u \\ &\quad + u[B(su, u) + B(u, su), u]; \end{aligned}$$

by (5) this relation can be written in the form

$$D(su, u^2, u) = D(su, u, u)u + uD(su, u, u). \tag{7}$$

We now look at the expression $D(su, u, u)$. By (6) we have

$$D(su, u, u) = [u, B(u, su)] + [su, 2B(u, u)].$$

Since the mapping $t \to B(u, t)$ is a derivation, it follows that

$$D(su, u, u) = [u, B(u, s)u + sB(u, u)] + [su, 2B(u, u)].$$

According to $[B(u, u), u] = 0$, we then get

$$\begin{aligned} D(su, u, u) &= [u, B(u, s)]u + [u, s]B(u, u) + [s, 2B(u, u)]u \\ &= D(s, u, u)u + [u, s]B(u, u). \end{aligned}$$

Using this relatin in (7), we obtain

$$\begin{aligned} D(su, u^2, u) = D(s, u, u)u^2 &+ [u, s]B(u, u)u \\ &+ uD(s, u, u)u + u[u, s]B(u, u). \end{aligned} \tag{8}$$

The computation on the first way is thereby finished. We begin the computation on the second way by applying (6):

$$D(su, u^2, u) = [u, B(u^2, su)] + [su, B(u^2, u) + B(u, u^2)].$$

The mapping $t \to B(u^2, t)$ is a derivation, hence

$$D(su, u^2, u) = [u, B(u^2, s)u + sB(u^2, u)] + [su, B(u^2, u) + B(u, u^2)] \tag{9}$$

Since $B(u, u^2) = B(u, u)u+uB(u, u)$ and since $[B(u, u), u] = 0$, it follows that $[B(u, u^2), u] = 0$. By (4) we then also have $[B(u^2, u), u] = 0$. Therefore (9) can be written in the form

$$D(su, u^2, u) = [u, B(u^2, s)]u + [u, s]B(u^2, u) + [s, B(u^2, u) + B(u, u^2)]u.$$

Thus

$$D(su, u^2, u) = D(s, u^2, u)u + [u, s]B(u^2, u). \tag{10}$$

Using (5) we have

$$\begin{aligned} D(s, u^2, u) &= [B(s, u^2), u] + [B(s, u) + B(u, s), u^2] \\ &= [B(s, u)u + uB(s, u), u] + [B(s, u) + B(u, s), u^2] \\ &= [B(s, u), u]u + u[B(s, u), u] + [B(s, u) + B(u, s), u]u \\ &\quad + u[B(s, u) + B(u, s), u] \\ &= D(s, u, u)u + uD(s, u, u); \end{aligned}$$

hence (10) yields

$$D(su, u^2, u) = D(s, u, u)u^2 + uD(s, u, u)u + [u, s]B(u^2, u). \tag{11}$$

Comparing (8) and (11), we get

$$[u, s]B(u, u)u + u[u, s]B(u, u) = [u, s]B(u^2, u).$$

Since $B(u, u)u = uB(u, u)$, this relation can be written in the form $u^2sB(u, u) = usB(u^2, u) + s(u^2B(u, u) - uB(u^2, u))$, which proves the lemma.

We show that if $B(u, u) \neq 0$ for some $u \in S$, then u is rather special.

Lemma C. *If $u \in S$ and $B(u, u) \neq 0$, then there exists $\lambda \in C$ such that $(2u - \lambda)^2 = 0$.*

Proof. Since $B(u, u) \neq 0$, it follows from Lemmas B and 1 that there exist $\lambda_1, \lambda_2, \lambda_3$ not all zero in C such that $\lambda_1 u^2 + \lambda_2 u + \lambda_3 = 0$. Suppose that $\lambda_1 = 0$. Obviously in this case, λ_2 and λ_3 are different from zero, and so $\lambda_2 u + \lambda_3 = 0$ implies that $u \in C$. But then $B(u, u) = 0$ by (i). Thus we may assume that $\lambda_1 \neq 0$. Hence $u^2 = \lambda u + \mu$ where $\lambda = -\lambda_1^{-1}\lambda_2 \in C$ and $\mu = -\lambda_1^{-1}\lambda_3 \in C$.

By assumption, there exists $a \in S$ such that $B(u, t) = [a, t]$ for all $t \in S$. Thus the relation $[B(u, u), u] = 0$ can be written in the form $au^2 + u^2a = 2uau$. Since $u^2 = \lambda u + \mu$ it follows that

$$\lambda au + 2\mu a + \lambda ua = 2uau. \tag{12}$$

Multiply (12) from the right by u, we then get

$$\lambda a(\lambda u + \mu) + 2\mu au + \lambda uau = 2ua(\lambda u + \mu).$$

That is,

$$(\lambda^2 + 2\mu)au + \lambda\mu a - 2\mu ua = \lambda uau. \tag{13}$$

Multiply the relation (10) by λ, the relation (11) by 2, and compare the relations so obtained. Then we arrive at $(\lambda^2 + 4\mu)au = (\lambda^2 + 4\mu)ua$, i.e., $(\lambda^2 + 4\mu)B(u, u) = 0$. Since we have assumed that $B(u, u) \neq 0$, it follows that $\lambda^2 + 4\mu = 0$. Hence the relation $u^2 = \lambda u + \mu$ yields $(2u - \lambda)^2 = 0$.

Lemma D. *If $v \in S$ and $v^2 = 0$, then $B(v, v) = 0$.*

Proof. By assumption, for every $t \in S$ there exists $b(t) \in S$ such that $B(t, s) = [b(t), s]$ for all $s \in S$. We denote the element $b(v)$ by b. Taking v for s in (4) we obtain

$$[[b, v], t] + [[b, t], v] + [[b(t), v], v] = 0 \quad \text{for all } t \in S. \tag{14}$$

Since $B(v, v) = [b, v]$ commutes with v and since $v^2 = 0$, it follows that $2vbv = 0$; thus $vbv = 0$ and therefore, also $v[b, v] = 0$. Now multiply (14) from the left by v. Using $v^2 = 0$ and the last statement, we obtain

$$-vt[b, v] + v[b, t]v = 0 \quad \text{for all } t \in S.$$

This relation can be written in the form

$$vbtv = vt(2bv - vb) \quad \text{for all } t \in S. \tag{15}$$

Replacing t by tvr in (15), we obtain $vbtvrv = vtvr(2bv - vb)$. But on the other hand, again according to (15), we have $vbtvrv = vt(2bv - vb)rv$ and $vtvr(2bv-vb) = vtvbrv$. Comparing these relations, we get $vt(2bv-2vb)rv =$

0 where t and r are arbitrary elements in S. Therefore $2[b, v] = 2B(v, v) = 0$ by the primeness of S. But then $B(v, v) = 0$.

Now, suppose that $B(u, u) \neq 0$ for some $u \in S$. By Lemma C there exists $\lambda \in C$ such that $(2u - \lambda)^2 = 0$. Thus $B(2u - \lambda, 2u - \lambda) = 0$ by Lemma D. But on the other hand, since B is biadditive and since (i) holds, $B(2u - \lambda, 2u - \lambda) = 4B(u, u)$. Hence $B(u, u) = 0$. With this contradiction, Theorem 1 is proved.

Proof of Theorem 2. Since R is 2-torsionfree, we may assume that B is symmetric (i.e., $B(x, y) = B(y, x)$ for $x, y \in R$); otherwise replace B by the mapping $(x, y) \to B(x, y) + B(y, x)$.

By assumption, for every $x \in R$ we have $[B(x, x), x] \in Z$. A linearization yields

$$(16) \qquad [B(x, x), y] + 2[B(x, y), x] + 2[B(x, y), y] + [B(y, y), x] \in Z.$$

Replace x by $-x$ in (16); comparing the relation so obtained with (16), we then get $2([B(x, x), y] + 2[B(x, y), x]) \in Z$. Since R is 2-torsionfree, it follows that

$$(17) \qquad [B(x, x), y] + 2[B(x, y), x] \in Z \quad \text{for all } x, y \in R.$$

Now fix $x \in R$ and let us show that the element $c = [B(x, x), x] \in Z$ is equal to 0. Take x^2 for y in (17); since $[B(x, x), x^2] = [B(x, x), x]x + x[B(x, x), x] = cx + xc = 2cx$, we then get $2cx + 2[B(x, x^2), x] \in Z$. R is 2-torsionfree, so it follows that

$$(18) \qquad cx + u \in Z,$$

where u denotes the element $[B(x, x^2), x]$. In particular, $cx + u$ commutes with x, which implies that u and x commute.

Replacing y by x and x by x^2 in (17), we obtain

$$[B(x^2, x^2), x] + 2[B(x^2, x), x]x + 2x[B(x^2, x), x] \in Z.$$

Since B is symmetric, this relation can be written as $v + 2ux + 2xu \in Z$ where $v = [B(x^2, x^2), x]$. However, u and x commute, so we have

$$(19) \qquad v + 4ux \in Z.$$

In particular, $v + 4ux$ commutes with x; hence we see that v and x commute. Hence the element $2xv = vx + xv = [B(x^2, x^2), x^2]$ lies in Z by assumption. Therefore

$$(20) \qquad vx \in Z.$$

The relation (18) yields

$$0 = [B(x, x), cx + u] = c[B(x, x), x] + [B(x, x), u].$$

Thus

$$(21) \qquad [B(x, x), u] = -c^2.$$

According to (19) we have

$$\begin{aligned} 0 &= [B(x, x), v + 4ux] \\ &= [B(x, x), v] + 4[B(x, x), u]x + 4u[B(x, x), x]; \end{aligned}$$

applying (21) we then get

$$[B(x, x), v] = 4c^2x - 4cu. \tag{22}$$

By (20),

$$0 = [B(x, x), vx] = [B(x, x), v]x + v[B(x, x), x] = [B(x, x), v]x + cv.$$

Using this relation and (22) we obtain

$$4cux = 4c^2x^2 + cv. \tag{23}$$

Consider $w = [B(x, x), 4cux]$. We have

$$w = 4c[B(x, x), ux] = 4c[B(x, x), u]x + 4cu[B(x, x), x],$$

and so it follows from (21) that

$$w = -4c^3x + 4c^2u. \tag{24}$$

On the other hand, according to (23) we have

$$\begin{aligned} w &= [B(x, x), 4c^2x^2 + cv] \\ &= 4c^2([B(x, x), x]x + x[B(x, x), x]) + c[B(x, x), v] \\ &= 4c^2(2cx) + c[B(x, x), v], \end{aligned}$$

which together with (22) gives $w = 12c^3x - 4c^2u$. Comparing this expression with (24) one obtains $8c^2u = 16c^3x$; thus $c^2u = 2c^3x$. Hence it follows from (21) that

$$c^4 = -[B(x, x), c^2u] = -[B(x, x), 2c^3x] = -2c^3[B(x, x), x] = -2c^4.$$

Thus $3c^4 = 0$ and so $c^4 = 0$ since R is 3-torsionfree. By assumption, this relation implies that $c = 0$. This proves the theorem.

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