PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 114, Number 3, MARCH 1992

RULED SUBMANIFOLDS OF FINITE TYPE

FRANKI DILLEN

(Communicated by Jonathan M. Rosenberg)

ABSTRACT. We show that a ruled submanifold of finite type in a Euclidean space is a cylinder on a curve of finite type or a generalized helicoid.

INTRODUCTION

The well-known theorem of Catalan states that the only ruled minimal surfaces in Euclidean 3-space are the plane and the helicoid. This classical result can be generalized in two directions.

First one can say that a submanifold M^{n+1} of a Riemannian manifold M^{n+p} is ruled, if M^{n+1} is foliated by n-dimensional totally geodesic subspaces of M^{n+p}, and one can try to classify ruled minimal submanifolds of a Euclidean space. This is done independently by Ü. Lumiste in [7], by C. Thas in [9], and by J. M. Barbosa, M. Dacjzer, and L. P. Jorge in [2]. They show that a minimal ruled submanifold of a Euclidean space is a generalized helicoid. More details will be given in §1. This extends the theorem of Catalan in one direction.

On the other hand, minimal submanifolds of a Euclidean space can be considered as a special case of submanifolds of finite type. Submanifolds of finite type are introduced by B.-Y. Chen in [4]. A submanifold M^n of a Euclidean space $\mathbf{E}^{n+p}$ is said to be of finite type if each component of its position vector field X can be written as a finite sum of eigenfunctions of the Laplacian Δ of M^n, i.e. if $X = X_0 + X_1 + \cdots + X_k$, where X_0 is a constant vector and $\Delta X_i = \lambda_i X_i$ for $i = 1, \ldots, k$. If in particular all eigenvalues $\{\lambda_1, \lambda_2, \ldots, \lambda_k\}$ are mutually different, then M^n is said to be of k-type. Note that every minimal submanifold of a Euclidean space is of 1-type, since $\Delta X = 0$.

In [5] the Catalan theorem was extended to surfaces of finite type in a Euclidean space $\mathbf{E}^n$. There it is proved that a ruled surface of finite type in a Euclidean space is either a part of a cylinder over a curve of finite type or a helicoid in $\mathbf{E}^3$. In particular it follows that a ruled surface of finite type in $\mathbf{E}^3$ is a part of a plane, a circular cylinder or a helicoid.

In this paper we extend the Catalan theorem in both directions at the same time by showing the following theorem.

Received by the editors October 4, 1990.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 53A07, 53B25.

Key words and phrases. Finite type submanifolds, ruled submanifolds, generalized helicoid.

Supported by a post-doctoral fellowship of the Research Council of the Katholieke Universiteit Leuven.

0002-9939/92 \$1.00 + \$.25 per page

Theorem. *A ruled submanifold M^{n+1} in $\mathbf{E}^{n+p}$ is of finite type if and only if M^{n+1} is a part of a cylinder on a curve of finite type or a part of a generalized helicoid.*

When restricting to hypersurfaces, we have the following corollary.

Corollary. *A ruled hypersurface M^{n+1} in $\mathbf{E}^{n+2}$ is of finite type if and only if M^{n+1} is a part of a hyperplane, a circular cylinder, a helicoid H^2 in $\mathbf{E}^3$, a cone H^3 with top P on a minimal ruled surface in a 3-sphere, centered at P, or a cylinder on H^2 or H^3.*

1. Ruled minimal submanifolds of a Euclidean space

In 1958 Ü. Lumiste showed in [7] that a minimal ruled submanifold M^{n+1} of a Euclidean space is either

(i) generated by an n-dimensional affine subspace P under a screw motion in $\mathbf{E}^{2n+1}$ such that the axis cuts P orthogonal,
(ii) generated by an n-dimensional affine subspace P under a rotation in $\mathbf{E}^{2n}$ around a point in P,
(iii) a cylinder on a submanifold of the type (i) or (ii).

Analytically any minimal submanifold therefore can be given by

$$(*) \quad \begin{aligned} X(s, t_1, \ldots, t_n) = (&t_1 \cos(a_1 s), t_1 \sin(a_1 s), \ldots, \\ &t_k \cos(a_k s), t_k \sin(a_k s), t_{k+1}, \ldots, t_n, bs), \end{aligned}$$

where $a_1, \ldots, a_k$ and b are real numbers. A submanifold with this kind of parameterization is called a generalized helicoid. If $b \neq 0$, resp. $b = 0$, then this gives a cylinder on a submanifold of the type (i), resp. (ii). This analytic description was given independently by C. Thas in [9] in 1979, who proved directly that any minimal ruled submanifold can be parameterized by $(*)$. In 1984 this result was proved again by J. M. Barbosa, M. Dacjzer, and L. P. Jorge in [2]. They show that any minimal ruled submanifold is generated by an affine subspace P under a one-parameter subgroup A of rigid motions of the Euclidean space, such that P is orthogonal to the orbits of A. Then they show that the resulting submanifold (at least if it is minimal) has a parameterization $(*)$. They also extend their result to ruled submanifolds of real space forms.

I would like to point out that a ruled submanifold of the type (ii) is a cone on a minimal ruled submanifold of some hypersphere of the Euclidean space. Indeed, putting $k = n$ and $b = 0$ in $(*)$ gives that M^{n+1} is a cone with as top the origin of $\mathbf{E}^{2n}$. In particular, the rulings of M^{n+1} pass through the origin. Then the intersection of M^{n+1} and a sphere S^{2n-1}, centered at the origin, is a ruled submanifold M' in S^{2n-1}. Using the fact that a submanifold M' of a sphere S^m is minimal (in the sphere) if and only if the cone shaped on M' with as top the center of the sphere is minimal in $\mathbf{E}^{m+1}$, cf. [8, Proposition 6.1.1], we obtain that M' is minimal in S^{2n-1}. Note that this argumentation reduces the classification of minimal ruled submanifolds of a sphere to the classification of minimal ruled submanifolds of the Euclidean space whose rulings pass through a fixed point.

Finally, if we look for ruled minimal hypersurfaces M^{n+1} of a Euclidean space $\mathbf{E}^{n+2}$, we see that there are essentially (up to cylinders, built on those submanifolds) three possibilities, namely M^{n+1} is of the type (i) and $n = 1$, in which case we obtain a helicoid in $\mathbf{E}^3$, or M^{n+1} is of the type (ii) and $n = 1$, in which case we obtain a plane, or M^{n+1} is of the type (ii) and $n = 2$, in which case we obtain cone (with top 0) on a minimal ruled surface in a 3-dimensional sphere, centered at 0. We remark that ruled minimal surfaces in a 3-sphere were classified by H. B. Lawson in [6], that a classification of ruled minimal hypersurfaces of a Euclidean space was also given by G. Aumann in [1], and that complete minimal hypersurfaces of a Euclidean space were classified by D. Blair and J. R. Vanstone in [3].

2. Ruled submanifolds of finite type

Let M^{n+1} be a ruled submanifold of $\mathbf{E}^{n+p}$. Let α be an orthogonal trajectory of M^{n+1} and assume α to be parameterized by arc length. Let $\{e_1(s), \dots, e_n(s)\}$ be a set of orthonormal vector fields along α such that $\{e_1(s), \dots, e_n(s)\}$ span the ruling of M^{n+1} through $\alpha(s)$. The set $\{e_1(s), \dots, e_n(s)\}$ can be chosen such that $\langle e_i'(s), e_j(s)\rangle = 0$ for all i and j, see [2, Lemma 2.2]. So we can give a parameterization of M^{n+1} by

$$X(s, t_1, \dots, t_n) = \alpha(s) + \sum_{i=1}^{n} t_i e_i.$$

If we define a function q on M^{n+1} by

$$q = ||X_s||^2 = 1 + 2\sum_{i=1}^{n} t_i \langle \alpha', e_i'\rangle + \sum_{i,j=1}^{n} t_i t_j \langle e_i', e_j'\rangle,$$

then the Laplacian Δ of M^{n+1} can be expressed as follows:

$$\Delta = -\sum_{i=1}^{n} \frac{\partial^2}{\partial t_i^2} - \frac{1}{q}\frac{\partial^2}{\partial s^2} + \frac{1}{2}\frac{\partial q}{\partial s}\frac{1}{q^2}\frac{\partial}{\partial s} - \frac{1}{2}\frac{1}{q}\sum_{i=1}^{n}\frac{\partial q}{\partial t_i}\frac{\partial}{\partial t_i}.$$

Note that q is a polynomial in $t = (t_1, \dots, t_n)$ with functions in s as coefficients. The degree of q is 2, unless all e_i' are zero. If all e_i' are identically zero, then M^{n+1} is a cylinder. If M^{n+1} is a cylinder, then, similarly as in [5], one can show that M^{n+1} is a cylinder on a curve of finite type. So from now on we assume that M^{n+1} is not cylindrical. Then we can assume, by restricting to a sufficiently small piece of α, that q has degree 2 everywhere. The proof of the following lemma is straightforward.

Lemma. *If P is a polynomial in $t = (t_1, \dots, t_n)$ with functions in s as coefficients and $\deg(P) = d$, then*

$$\Delta\left(\frac{P(t)}{q^m}\right) = \frac{\widetilde{P}(t)}{q^{m+3}}$$

where $\widetilde{P}$ is a polynomial in t with functions in s as coefficients and $\deg(\widetilde{P}) \leq d + 4$.

The theorem now can be proved similarly as in [5]. Since the argumentation is short, we repeat it here. If M^{n+1} is of k-type, it follows, cf. [4, p. 256], that there exist numbers $c_1, \dots, c_k$, such that

$$\Delta^{k+1}X + c_1\Delta^k X + \cdots + c_k \Delta X = 0. \qquad (**)$$

Let X_i be any component of X. We know that X_i is a linear function in t with functions in s as coefficients. By applying the lemma, we easily obtain that

$$\Delta^r X_i = \frac{P_r(t)}{q^{3r-1}} \quad \text{and} \quad \deg(P_r) \leq 1 + 4r.$$

Hence if r goes up by one, the degree of the numerator of $\Delta^r X_i$ goes up by at most 4, while the degree of the denominator goes up by 6. Hence the sum $(**)$ can never be zero, unless $\Delta X_i = 0$. Therefore M^{n+1} is minimal, so that M^{n+1} is a generalized helicoid.

In order to prove the corollary, it is sufficient to remark that a plane curve of finite type is a part of a circle or a straight line [5, Theorem 3].

References

1. G. Aumann, *Die Minimalhyperregelflächen*, Manuscripta Math. **34** (1981), 293–304.
2. J. M. Barbosa, M. Dajczer, and L. P. Jorge, *Minimal ruled submanifolds in spaces of constant curvature*, Indiana Univ. Math. J. **33** (1984), 531–547.
3. D. Blair and J. R. Vanstone, *A generalization of the helicoid*, Minimal Submanifolds and Geodesics, Kaigai Publications, Tokyo, 1978, pp. 13–16.
4. B.-Y. Chen, *Total mean curvature and submanifolds of finite type*, World Scientific, Singapore, 1984.
5. B.-Y. Chen, F. Dillen, L. Verstraelen, and L. Vrancken, *Ruled surfaces of finite type*, Bull. Austral. Math. Soc. **42** (1990), 447–453.
6. H. B. Lawson, Jr., *Complete minimal surfaces in* S^3, Ann. of Math. (2) **92** (1970), 335–374.
7. Ü. Lumiste, *Die* n*-dimensionalen Minimalflächen mit einer* $(n-1)$*-dimensionalen asymptotischen Richtung im jedem Punkte*, Tartu Riikl. Ül. Toimetised **62** (1958), 117–141. (Russian, German summary)
8. J. Simons, *Minimal varieties in riemannian manifolds*, Ann. of Math. (2) **8** (1968), 62–105.
9. C. Thas, *Minimal generalized ruled surfaces in the Euclidean space* $\mathbf{E}^m$, Internal Report Seminar of Higher Geometry, State University of Ghent.

Katholieke Universiteit Leuven, Departement Wiskunde, Celestijnenlaan 200 B, B-3001 Leuven, Belgium

E-mail address: fgaba01@blekul11.bitnet