

RULED SUBMANIFOLDS OF FINITE TYPE

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ABSTRACT. We show that a ruled submanifold of finite type in a Euclidean space is a cylinder on a curve of finite type or a generalized helicoid.

INTRODUCTION

The well-known theorem of Catalan states that the only ruled minimal surfaces in Euclidean 3-space are the plane and the helicoid. This classical result can be generalized in two directions.

First one can say that a submanifold M^{n+1} of a Riemannian manifold M^{n+p} is ruled, if M^{n+1} is foliated by n -dimensional totally geodesic subspaces of M^{n+p} , and one can try to classify ruled minimal submanifolds of a Euclidean space. This is done independently by Ü. Lumiste in [7], by C. Thas in [9], and by J. M. Barbosa, M. Dacjzer, and L. P. Jorge in [2]. They show that a minimal ruled submanifold of a Euclidean space is a generalized helicoid. More details will be given in §1. This extends the theorem of Catalan in one direction.

On the other hand, minimal submanifolds of a Euclidean space can be considered as a special case of submanifolds of finite type. Submanifolds of finite type are introduced by B.-Y. Chen in [4]. A submanifold M^n of a Euclidean space E^{n+p} is said to be of finite type if each component of its position vector field X can be written as a finite sum of eigenfunctions of the Laplacian Δ of M^n , i.e. if $X = X_0 + X_1 + \cdots + X_k$, where X_0 is a constant vector and $\Delta X_i = \lambda_i X_i$ for $i = 1, \dots, k$. If in particular all eigenvalues $\{\lambda_1, \lambda_2, \dots, \lambda_k\}$ are mutually different, then M^n is said to be of k -type. Note that every minimal submanifold of a Euclidean space is of 1-type, since $\Delta X = 0$.

In [5] the Catalan theorem was extended to surfaces of finite type in a Euclidean space E^n . There it is proved that a ruled surface of finite type in a Euclidean space is either a part of a cylinder over a curve of finite type or a helicoid in E^3 . In particular it follows that a ruled surface of finite type in E^3 is a part of a plane, a circular cylinder or a helicoid.

In this paper we extend the Catalan theorem in both directions at the same time by showing the following theorem.

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Theorem. *A ruled submanifold M^{n+1} in E^{n+p} is of finite type if and only if M^{n+1} is a part of a cylinder on a curve of finite type or a part of a generalized helicoid.*

When restricting to hypersurfaces, we have the following corollary.

Corollary. *A ruled hypersurface M^{n+1} in E^{n+2} is of finite type if and only if M^{n+1} is a part of a hyperplane, a circular cylinder, a helicoid H^2 in E^3 , a cone H^3 with top P on a minimal ruled surface in a 3-sphere, centered at P , or a cylinder on H^2 or H^3 .*

1. RULED MINIMAL SUBMANIFOLDS OF A EUCLIDEAN SPACE

In 1958 Ü. Lumiste showed in [7] that a minimal ruled submanifold M^{n+1} of a Euclidean space is either

- (i) generated by an n -dimensional affine subspace P under a screw motion in E^{2n+1} such that the axis cuts P orthogonal,
- (ii) generated by an n -dimensional affine subspace P under a rotation in E^{2n} around a point in P ,
- (iii) a cylinder on a submanifold of the type (i) or (ii).

Analytically any minimal submanifold therefore can be given by

$$(*) \quad X(s, t_1, \dots, t_n) = (t_1 \cos(a_1 s), t_1 \sin(a_1 s), \dots, \\ t_k \cos(a_k s), t_k \sin(a_k s), t_{k+1}, \dots, t_n, bs),$$

where a_1, \dots, a_k and b are real numbers. A submanifold with this kind of parameterization is called a generalized helicoid. If $b \neq 0$, resp. $b = 0$, then this gives a cylinder on a submanifold of the type (i), resp. (ii). This analytic description was given independently by C. Thas in [9] in 1979, who proved directly that any minimal ruled submanifold can be parameterized by (*). In 1984 this result was proved again by J. M. Barbosa, M. Dacjzer, and L. P. Jorge in [2]. They show that any minimal ruled submanifold is generated by an affine subspace P under a one-parameter subgroup A of rigid motions of the Euclidean space, such that P is orthogonal to the orbits of A . Then they show that the resulting submanifold (at least if it is minimal) has a parameterization (*). They also extend their result to ruled submanifolds of real space forms.

I would like to point out that a ruled submanifold of the type (ii) is a cone on a minimal ruled submanifold of some hypersphere of the Euclidean space. Indeed, putting $k = n$ and $b = 0$ in (*) gives that M^{n+1} is a cone with as top the origin of E^{2n} . In particular, the rulings of M^{n+1} pass through the origin. Then the intersection of M^{n+1} and a sphere S^{2n-1} , centered at the origin, is a ruled submanifold M' in S^{2n-1} . Using the fact that a submanifold M' of a sphere S^m is minimal (in the sphere) if and only if the cone shaped on M' with as top the center of the sphere is minimal in E^{m+1} , cf. [8, Proposition 6.1.1], we obtain that M' is minimal in S^{2n-1} . Note that this argumentation reduces the classification of minimal ruled submanifolds of a sphere to the classification of minimal ruled submanifolds of the Euclidean space whose rulings pass through a fixed point.

Finally, if we look for ruled minimal hypersurfaces M^{n+1} of a Euclidean space E^{n+2} , we see that there are essentially (up to cylinders, built on those submanifolds) three possibilities, namely M^{n+1} is of the type (i) and $n = 1$, in

which case we obtain a helicoid in E^3 , or M^{n+1} is of the type (ii) and $n = 1$, in which case we obtain a plane, or M^{n+1} is of the type (ii) and $n = 2$, in which case we obtain cone (with top 0) on a minimal ruled surface in a 3-dimensional sphere, centered at 0. We remark that ruled minimal surfaces in a 3-sphere were classified by H. B. Lawson in [6], that a classification of ruled minimal hypersurfaces of a Euclidean space was also given by G. Aumann in [1], and that complete minimal hypersurfaces of a Euclidean space were classified by D. Blair and J. R. Vanstone in [3].

2. RULED SUBMANIFOLDS OF FINITE TYPE

Let M^{n+1} be a ruled submanifold of E^{n+p} . Let α be an orthogonal trajectory of M^{n+1} and assume α to be parameterized by arc length. Let $\{e_1(s), \dots, e_n(s)\}$ be a set of orthonormal vector fields along α such that $\{e_1(s), \dots, e_n(s)\}$ span the ruling of M^{n+1} through $\alpha(s)$. The set $\{e_1(s), \dots, e_n(s)\}$ can be chosen such that $\langle e'_i(s), e_j(s) \rangle = 0$ for all i and j , see [2, Lemma 2.2]. So we can give a parameterization of M^{n+1} by

$$X(s, t_1, \dots, t_n) = \alpha(s) + \sum_{i=1}^n t_i e_i.$$

If we define a function q on M^{n+1} by

$$q = \|X_s\|^2 = 1 + 2 \sum_{i=1}^n t_i \langle \alpha', e'_i \rangle + \sum_{i,j=1}^n t_i t_j \langle e'_i, e'_j \rangle,$$

then the Laplacian Δ of M^{n+1} can be expressed as follows:

$$\Delta = - \sum_{i=1}^n \frac{\partial^2}{\partial t_i^2} - \frac{1}{q} \frac{\partial^2}{\partial s^2} + \frac{1}{2} \frac{\partial q}{\partial s} \frac{1}{q^2} \frac{\partial}{\partial s} - \frac{1}{2} \frac{1}{q} \sum_{i=1}^n \frac{\partial q}{\partial t_i} \frac{\partial}{\partial t_i}.$$

Note that q is a polynomial in $t = (t_1, \dots, t_n)$ with functions in s as coefficients. The degree of q is 2, unless all e'_i are zero. If all e'_i are identically zero, then M^{n+1} is a cylinder. If M^{n+1} is a cylinder, then, similarly as in [5], one can show that M^{n+1} is a cylinder on a curve of finite type. So from now on we assume that M^{n+1} is not cylindrical. Then we can assume, by restricting to a sufficiently small piece of α , that q has degree 2 everywhere. The proof of the following lemma is straightforward.

Lemma. *If P is a polynomial in $t = (t_1, \dots, t_n)$ with functions in s as coefficients and $\deg(P) = d$, then*

$$\Delta \left(\frac{P(t)}{q^m} \right) = \frac{\tilde{P}(t)}{q^{m+3}}$$

where \tilde{P} is a polynomial in t with functions in s as coefficients and $\deg(\tilde{P}) \leq d + 4$.

The theorem now can be proved similarly as in [5]. Since the argumentation is short, we repeat it here. If M^{n+1} is of k -type, it follows, cf. [4, p. 256], that there exist numbers c_1, \dots, c_k , such that

$$(**) \quad \Delta^{k+1} X + c_1 \Delta^k X + \dots + c_k \Delta X = 0.$$

Let X_i be any component of X . We know that X_i is a linear function in t with functions in s as coefficients. By applying the lemma, we easily obtain that

$$\Delta^r X_i = \frac{P_r(t)}{q^{3r-1}} \quad \text{and} \quad \deg(P_r) \leq 1 + 4r.$$

Hence if r goes up by one, the degree of the numerator of $\Delta^r X_i$ goes up by at most 4, while the degree of the denominator goes up by 6. Hence the sum (***) can never be zero, unless $\Delta X_i = 0$. Therefore M^{n+1} is minimal, so that M^{n+1} is a generalized helicoid.

In order to prove the corollary, it is sufficient to remark that a plane curve of finite type is a part of a circle or a straight line [5, Theorem 3].

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