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POSNER'S SECOND THEOREM DEDUCED FROM THE FIRST

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ABSTRACT. Posner's second theorem is derived as a consequence of his first theorem.

In a paper from 1957, Posner proved the following two theorems [5]. We say that a mapping f on a ring R is *centralizing* if $[a, f(a)] = af(a) - f(a)a \in Z(R)$, the center of R, for all $a \in R$.

Theorem 1. *Let R be a prime ring with characteristic different from 2. The product of two nonzero derivations of R is not a derivation.*

Theorem 2. *Let R be a prime ring. If there is a nonzero centralizing derivation of R, then R is commutative.*

These results were subsequently refined and extended by a number of authors; we refer to [4] for a state-of-the-art account and a comprehensive bibliography. While the proof of Theorem 1 consists in a straightforward computation, the second theorem has been considered to lie somewhat deeper although the arguments were simplified in [2]. In the present note we show how to obtain Theorem 2 directly from Theorem 1, at least where it can be done, viz. if $\operatorname{char} R \neq 2$. The exceptional case of characteristic 2 is, however, easily treated (see [1, p. 13]). In fact, since the arguments in the proof of Theorem 1 go over verbatim to the case where the hypothesis is only made on a nonzero ideal I of R, we also obtain Theorem 2 under the assumption that the derivation is centralizing on I only (cf. [4, Corollary, p. 283]).

Proof. Let I be a nonzero ideal of the prime ring R with $\operatorname{char} R \neq 2$, and let $\delta: R \to R$ be a derivation such that $[a, \delta a] \in Z(R)$ for all $a \in I$. Replacing a by $a + b$, $b \in I$, yields

$$[a, \delta b] + [b, \delta a] \in Z(R), \tag{1}$$

in particular,

$$[a, \delta(a^2)] + [a^2, \delta a] \in Z(R). \tag{2}$$

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As δ is a derivation, it is easily computed that

$$[a, \delta(a^2)] - [a^2, \delta a] = 0, \tag{3}$$

and therefore combining (2) and (3) gives $[a^2, \delta a] \in Z(R)$. Since $[a, \delta a]$ is central too, we thus obtain

$$2[a, \delta a]^2 = [2[a, \delta a]a, \delta a] = [[a^2, \delta a], \delta a] = 0,$$

and therefore $[a, \delta a] = 0$ as R is prime.

This shows that in (1) we in fact have

$$[a, \delta b] + [b, \delta a] = 0 \quad \text{for all } a, b \in I. \tag{1'}$$

Let δ_c denote the inner derivation $\delta_c(x) = [x, c]$, $x \in R$. Then (1′) says nothing but $\delta_b \circ \delta = \delta_{\delta b}$ on I for all $b \in I$. By Theorem 1, we conclude that $\delta = 0$ or $\delta_b = 0$ for all $b \in I$, i.e. $I \subseteq Z(R)$. Since R is prime, the latter case entails that R is commutative.

Remarks. 1. Centralizing derivations are commuting (that is $[a, \delta a] = 0$ for all a) under rather general assumptions [3, Lemma 4]; this was kindly pointed out to us by the referee.

2. If $\operatorname{char} R = 2$, then (1) implies (1′) even more easily, cf. [1]. Bergen's version of Theorem 1 in the characteristic 2 case [4, Theorem 4] thus yields an element c in the extended centroid of R, such that $\delta_b = c\delta$ for all $b \in I$, which is only possible if $\delta_b = 0$ for all b. Therefore, the exceptional case can be treated in the same manner.

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