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ON THE PERIOD-ENERGY RELATION

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ABSTRACT. We give a new proof of the period-energy relation for Hamiltonian systems.

Suppose that a Hamiltonian system (P, ω, h) has a manifold of periodic solutions. This means that there is a free S^1 action and the trajectories have no isotropy. In other words, we have an S^1 fibre bundle where the fibre is the trajectory. With no loss of generality, we can assume that on P, X_h has only periodic solutions. Define an $S^1 = R/Z$ action Φ by

$$\Phi^t(p) = \Phi_h^{\tau t}(p). \tag{1}$$

Here Φ_h is the flow of the Hamiltonian and τ is the period of a trajectory. Now we may regard P as an S^1 principal bundle with the action Φ.

Suppose that j is the momentum map of the S^1 action Φ. Now $X_j = \tau X_h$ (this follows by differentiating equation (1)), and

$$dj = i_{X_j}\omega = i_{\tau X_h}\omega = \tau\, dh.$$

Hence $0 = d^2 j = d\tau \wedge dh$. We conclude that τ is a function only of h. Hence the period depends only on the energy if the energy surface is connected.

The only thing we need to do is show that Φ really does have a momentum map. To this end, denote this infinitesimal generator of Φ_h by X, of Φ by Y, and let Z be a vector at the point p. Extend Z to a vector field, also denoted by Z. We can arrange things so that $\Phi^t_* Z = Z$, or infinitesimally, $L_Y Z = 0$. Let ζ be a one-form with $\omega = -d\zeta$ (it suffices to do this in a neighborhood of a trajectory). Define

$$j(p) = \int_0^1 \langle \zeta, Y\rangle \circ \Phi^t(p)\, dt.$$

Thus, (dropping the reference to p)

$$\langle dj, Z\rangle = \int_0^1 \langle d\langle \zeta, Y\rangle, Z\rangle \circ \Phi^t\, dt,$$

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and since $d\langle\zeta, Y\rangle = L_Y\zeta - d\zeta(Y, \cdot)$ we have

$$\begin{aligned}\langle d\langle\zeta, Y\rangle, Z\rangle &= \langle L_Y\zeta, Z\rangle - d\zeta(Y, Z)\\ &= L_Y\langle\zeta, Z\rangle - \langle\zeta, L_Y Z\rangle - d\zeta(Y, Z).\end{aligned}$$

However,

$$\int_0^1 L_Y\langle\zeta, Z\rangle \circ \Phi^t\, dt = \int_0^1 \frac{d}{dt}\langle\zeta, Z\rangle \circ \Phi^t\, dt = 0,$$

and $L_Y Z = 0$, thus

$$\begin{aligned}\langle dj, Z\rangle &= -\int_0^1 d\zeta(Y, Z)\circ\Phi^t\, dt\\ &= -\tau\int_0^\tau d\zeta(X, Z)\circ\Phi_h^t\, dt.\end{aligned}$$

We show that the integrand is invariant under the flow Φ_h. Now

$$L_X(d\zeta(X, Z)) = (L_X\, d\zeta)(X, Z) + d\zeta(L_X X, Z) + d\zeta(X, L_X Z),$$

and $L_X\, d\zeta = 0$ since X is Hamiltonian, $L_X X = 0$, and $L_X Z = \langle d\ln\tau, Z\rangle X$. Hence

$$\langle dj, Z\rangle = -\tau\, d\zeta(X, Z) = \omega(Y, Z),$$

which is precisely what is required to show that j is a momentum map.

This relation can also be proved by using averaging, as in Gordon [1], or using integral invariants and Stokes' theorem as in Moser [2], or one can use canonical relations as in Weinstein [3].

References

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