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INTERPOLATING OPERATORS IN NEST ALGEBRAS

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ABSTRACT. Given two families $\{x_\alpha\}$, $\{y_\alpha\}$ of vectors and a nest $\mathcal{N}$ in a Hilbert space H, we provide a necessary and sufficient condition for the existence of an operator T in the nest algebra satisfying $Tx_\alpha = y_\alpha$ for every α.

Given two families of vectors $\{x_\alpha\}$, $\{y_\alpha\}$ in a Hilbert space, an interpolating operator is a bounded operator T such that $Tx_\alpha = y_\alpha$ for every α. In [5] Lance gives a necessary and sufficient condition on two vectors x and y for the existence of an operator T lying in a given nest algebra such that $Tx = y$. Hopenwasser extended this result to the case of a CSL algebra [3]. Other questions about interpolating operators are investigated in [1, 4]. In this note we give a necessary and sufficient condition on two families $\{x_\alpha\}$, $\{y_\alpha\}$ of vectors in a Hilbert space for the existence of an operator T lying in a given nest algebra, such that $Tx_\alpha = y_\alpha$ for every α.

Throughout this note H denotes a complex Hilbert space and $B(H)$ denotes the space of all bounded operators from H to itself. By a nest we mean a complete totally ordered set of orthogonal projections. If $\mathcal{N}$ is a nest in H, then $\operatorname{Alg}\mathcal{N}$ is the set of operators X in $B(H)$ satisfying $N^\perp X N = 0$ for every N in $\mathcal{N}$.

The main step in our proof is Proposition 3. We need two lemmas.

Lemma 1. *Let $x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_m$ be vectors in H. Let V be the span of the x_i's, $i = 1, 2, \ldots, m$. Assume that there exists a real number $M > 0$ such that:*

$$\left\| \sum_{i=1}^{m} c_i y_i \right\| \leq M \cdot \left\| \sum_{i=1}^{m} c_i x_i \right\| ,$$

for every m-tuple $(c_1, c_2, \ldots, c_m)$ of complex numbers. Then there exists a linear operator $T: V \to H$ such that $Tx_i = y_i$, for $i = 1, 2, \ldots, m$ and $\|T\| \leq M$.

Proof. We may assume that there exists an index k, $1 \leq k \leq m$ such that the set $\{x_1, x_2, \ldots, x_k\}$ is a basis of V. Define $T: V \to H$ such that $Tx_i = y_i$, $1 \leq i \leq k$. Then T is a linear operator from V to H, and by the condition of the lemma, it follows that $\|T\| \leq M$. We have to show that $Tx_i = y_i$ for every

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i, $1 \leq i \leq m$. Suppose, to the contrary, there exists l, $k < l \leq m$ such that $Tx_l \neq y_l$. There are complex numbers a_i, $1 \leq i \leq k$ such that $x_l = \sum_{i=1}^k a_i x_i$. Then,

$$\left\| \sum_{i=1}^k a_i y_i - y_l \right\| = \left\| \sum_{i=1}^k a_i T x_i - y_l \right\| = \left\| T \left(\sum_{i=1}^k a_i x_i \right) - y_l \right\| = \| T x_l - y_l \| \neq 0,$$

while $\| \sum_{i=1}^k a_i x_i - x_l \| = 0$, contradicting the hypothesis. □

Lemma 2. *Let A be a positive contraction in $B(H)$. We denote by B the positive semidefinite hermitian form on H defined by $B(x, y) = \langle x, Ay \rangle$ for x, y in H. Let V be a finite-dimensional subspace of H. Then there exists a closed subspace V_0 of H such that:*

(1) $V + V_0 = H$, $V \cap V_0 = \{0\}$;

(2) $B(x, y) = 0$ *for every x in V and y in V_0.*

Proof. Let $V_1 = \{x \in H: B(x, y) = 0, \ \forall y \in V\} = (AV)^\perp$. We show that $V + V_1 = H$. Let z be a vector in $(V + V_1)^\perp = V^\perp \cap V_1^\perp$. Since $V_1 = (AV)^\perp$, which has finite codimension, z is of the form Av for some v in V. We have:

$$\langle v, Av \rangle = 0 \Rightarrow A^{1/2} v = 0 \Rightarrow Av = 0 \Rightarrow z = 0.$$

Let $V_0 = V_1 \ominus (V \cap V_1)$. We have:

$$V + V_0 = V + V_1 = H, \qquad V \cap V_0 = (V \cap V_1) \cap V_0 = \{0\}.$$

It follows that the first condition of the lemma is satisfied. The second condition holds because V_0 is contained in V_1. □

Proposition 3. *Let $\mathcal{N} = \{0 = N_0, N_1, N_2, \dots, N_k = I\}$ be a finite nest in H. Let $x_1, x_2, \dots, x_m, y_1, y_2, \dots, y_m$ be vectors in H. Let $M > 0$. The following are equivalent:*

(1) *There exists an operator T in* Alg$\mathcal{N}$ *such that $Tx_i = y_i$, for $i = 1, 2, \dots, m$, and $\|T\| \leq M$.*

(2) $\|N^\perp(\sum_{i=1}^m c_i y_i)\| \leq M \cdot \|N^\perp(\sum_{i=1}^m c_i x_i)\|$, *for every m-tuple of complex numbers $(c_1, c_2, \dots, c_m)$, and every projection N in $\mathcal{N}$.*

Proof. The implication (1) $\Rightarrow$ (2) is easy:

$$\begin{aligned} \left\| N^\perp \left(\sum_{i=1}^m a_i y_i \right) \right\| &= \left\| N^\perp T \left(\sum_{i=1}^m a_i x_i \right) \right\| = \left\| N^\perp T N^\perp \left(\sum_{i=1}^m a_i x_i \right) \right\| \\ &\leq \|N^\perp T\| \cdot \left\| N^\perp \left(\sum_{i=1}^m a_i x_i \right) \right\| \leq M \cdot \left\| N^\perp \left(\sum_{i=1}^m a_i x_i \right) \right\| . \end{aligned}$$

The converse is shown by induction on the length k of the nest $\mathcal{N}$. We may assume that $M = 1$.

First suppose that $k = 1$. It follows from Lemma 1 that there exists a contraction T_0 from the span V of the x_i's, $i = 1, 2, \dots, m$ into H such that $T_0 x_i = y_i$, $i = 1, 2, \dots, m$. We may extend T_0 to an operator T on H by defining T to be 0 on $V^\perp$. Now assume that the length of $\mathcal{N}$ is $k > 1$. Let H_0 be the Hilbert space $N_1^\perp H$, and let $\mathcal{N}_0$ be the nest in H_0 consisting of all projections of the form $N_1^\perp N$ with N in $\mathscr{N}$. It is easy to see that if L belongs to $\mathscr{N}_0$ then:

$$\left\|L^\perp\left(\sum_{i=1}^m c_i N_1^\perp y_i\right)\right\| \le M\cdot\left\|L^\perp\left(\sum_{i=1}^m c_i N_1^\perp x_i\right)\right\|,$$

for every m-tuple $(c_1, c_2, \dots, c_m)$ of complex numbers. Hence, by the induction hypothesis, there exists a contraction A_0 in $\operatorname{Alg}\mathscr{N}_0$ such that $A_0 N_1^\perp x_i = N_1^\perp y_i$ for every i, $1 \le i \le m$. We denote by T_0 the operator defined by $T_0 x = A_0 x$ if $x \in N_1^\perp H$, $T_0 x = 0$ if $x \in N_1 H$. It is easy to see that T_0 belongs to $\operatorname{Alg}\mathscr{N}$. For each $(c_1, c_2, \dots, c_m)$ in $\mathbb{C}^m$ we have:

$$\left\|N_1\left(\sum_{i=1}^m c_i y_i\right)\right\| \le \left\|\sum_{i=1}^m c_i y_i\right\| \le 1\cdot\left\|\sum_{i=1}^m c_i x_i\right\|.$$

Thus by Lemma 1, there exists a contraction $A_1\colon V \to H$ such that $A_1 x_i = N_1 y_i$ for every i, $1 \le i \le m$. For each $(c_1, c_2, \dots, c_m)$ in $\mathbb{C}^m$, we have:

$$\left\|N_1\left(\sum_{i=1}^m c_i y_i\right)\right\|^2 + \left\|N_1^\perp\left(\sum_{i=1}^m c_i y_i\right)\right\|^2 \le \left\|\sum_{i=1}^m c_i x_i\right\|^2$$

$$\Rightarrow \left\|A_1\left(\sum_{i=1}^m c_i x_i\right)\right\|^2 \le \left\|\sum_{i=1}^m c_i x_i\right\|^2 - \left\|N_1^\perp\left(\sum_{i=1}^m c_i y_i\right)\right\|^2$$

$$= \left\|\sum_{i=1}^m c_i x_i\right\|^2 - \left\|N_1^\perp T_0 N_1^\perp\left(\sum_{i=1}^m c_i x_i\right)\right\|^2,$$

since $T_0 N_1^\perp x_i = N_1^\perp y_i$ for $1 \le i \le m$ by construction. It follows that for every v in V, we have:

$$\|A_1 v\|^2 \le \|v\|^2 - \|N_1 T_0 N_1^\perp v\|^2.$$

Rewrite this as:

$$(*)\qquad \|A_1 v\|^2 \le B(v, v), \qquad \text{where } B(x, y) = \langle x, (I - N_1^\perp T_0^* N_1^\perp T_0 N_1^\perp) y\rangle$$

for x, y in H. Since $\|T_0\| \le 1$, $I - N_1^\perp T_0^* N_1^\perp T_0 N_1^\perp$ is a positive contraction. By Lemma 2, there exist a closed subspace V_0 of H such that $V + V_0 = H$, $V \cap V_0 = \{0\}$, and $B(x, y) = 0$ for every x in V and y in V_0. We define an operator $T_1\colon H \to H$ by: $T_1 x = A_1 x$ if $x \in V$, $T_1 x = 0$ if $x \in V_0$.

Claim. $\|T_1 x\|^2 \le B(x, x)$, for every x in H.

Indeed, let x be in H. Then x is of the form $x = v + v_0$ with $v \in V$ and $v_0 \in V_0$. We have:

$$\begin{aligned}\|T_1 x\|^2 &= \|T_1(v + v_0)\|^2 = \|A_1 v\|^2 \le B(v, v), \quad \text{by } (*)\\ &\le B(v, v) + B(v_0, v_0) = B(v + v_0, v + v_0) = B(x, x).\end{aligned}$$

The range of T_1 being contained in $N_1 H$, T_1 belongs to $\operatorname{Alg}\mathscr{N}$. We may now define $T = T_1 + T_0$. Then $T \in \operatorname{Alg}\mathscr{N}$ and $T x_i = y_i$ for every i, $1 \le i \le m$. It remains to see that T is a contraction. Indeed, for every x in H, we have:

$$\begin{aligned}\|Tx\|^2 &= \|T_1 x + T_0 x\|^2 = \|T_1 x\|^2 + \|T_0 x\|^2\\ &\le B(x, x) + \|T_0 x\|^2 = \|x\|^2 - \|N_1^\perp T_0 N_1^\perp x\|^2 + \|T_0 x\|^2 = \|x\|^2,\end{aligned}$$

where we have used the fact that $T_0 = N_1^\perp T_0 N_1^\perp$. □

Theorem 4. *Let $\mathcal{N}$ be a nest in H. Let $\{x_\alpha\}$, $\{y_\alpha\}$ be two families of vectors in H. Let $M \geq 0$. The following are equivalent*:

(1) *There exist T in $\operatorname{Alg}\mathcal{N}$ such that $Tx_\alpha = y_\alpha$ for every α and $\|T\| \leq M$.*

(2) $\|N^\perp(\sum c_\alpha y_\alpha)\| \leq M \cdot \|N^\perp(\sum c_\alpha x_\alpha)\|$, *for every N in $\mathcal{N}$ and for every family $\{c_\alpha\}$ of complex numbers, that is 0 except for finitely many indices.*

Proof. The implication $(1) \Rightarrow (2)$ is as easy as in the finite nest case.

For the converse, first assume that the families $\{x_\alpha\}$, $\{y_\alpha\}$ are finite.

Let $\mathcal{L}$ be the set of finite subnests of $\mathcal{N}$ directed by inclusion. It follows from the proposition that for every $\mathcal{F}$ in $\mathcal{L}$ there exists an operator $T_\mathcal{F}$ in $\operatorname{Alg}\mathcal{F}$ such that $T_\mathcal{F} x_\alpha = y_\alpha$ for every α and $\|T_\mathcal{F}\| \leq M$. The net $\{T_\mathcal{F} : \mathcal{F} \in \mathcal{L}\}$ is contained in the set $\{T \in B(H) : \|T\| \leq M\}$, which is compact in the weak operator topology. Let T be a weak operator limit of the net $\{T_\mathcal{F} : \mathcal{F} \in \mathcal{L}\}$. Clearly, $\|T\| \leq M$ and $Tx_\alpha = y_\alpha$ for every α. Also it is easy to see that T leaves every N in $\mathcal{N}$ invariant.

For the general case, we may assume that the families $\{x_\alpha\}$, $\{y_\alpha\}$ are indexed by an infinite set A. Then it follows from above that for every finite subset U of A there exists an operator T_U in $\operatorname{Alg}\mathcal{N}$ such that $T_U x_\alpha = y_\alpha$ for every α in U and $\|T_U\| \leq M$. Let T be a weak operator limit point of the net $\{T_U\}$. Then $\|T\| \leq M$, $Tx_\alpha = y_\alpha$ for every α in A and T belongs to $\operatorname{Alg}\mathcal{N}$ since $\operatorname{Alg}\mathcal{N}$ is closed in the weak operator topology. □

The following corollary generalizes Theorem 4.3 of [2].

Corollary 5. *Let A and B be operators in $B(H)$, and let $\mathcal{N}$ be a nest in H. Let $M \geq 0$. The following are equivalent*:

(1) *There exists an operator X in $\operatorname{Alg}\mathcal{N}$ such that $A = XB$ and $\|X\| \leq M$.*

(2) $\|N^\perp Ax\| \leq M \cdot \|N^\perp Bx\|$, *for every N in $\mathcal{N}$ and x in H.*

Proof. Let $\{e_\alpha\}$ be an orthonormal basis of H. We set $y_\alpha = Ae_\alpha$, $x_\alpha = Be_\alpha$ and apply the Theorem. □

Remark. After this work was completed, it came to our attention (private communication with E. Katsoulis) that E. Katsoulis, R. Moore, and T. Trent have arrived at similar results.

References

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