PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 114, Number 3, MARCH 1992

GENERALIZED CONVEX FUNCTIONS AND BEST L_p APPROXIMATION

RONALD M. MATHSEN AND VASANT A. UBHAYA

(Communicated by Andrew M. Bruckner)

ABSTRACT. Some properties of generalized convex functions significant to approximation theory are obtained. The existence of a best L_p approximation $(1 \leq p \leq \infty)$ from subsets of these functions is established under certain conditions. Special cases of these functions include n-convex functions which are much investigated in the literature.

1. INTRODUCTION

Let $I = (a, b)$ with $-\infty < a < b < \infty$, and $C = C(I)$ be the space of real continuous functions on I. A family G of functions in C is said to be an n-parameter family $(n \geq 2)$ if for any n points x_i, $1 \leq i \leq n$, in I with $a = x_0 < x_1 < x_2 < \cdots < x_n < x_{n+1} = b$ and real numbers $y_1, y_2, \ldots, y_n$, there exists a unique function $g \in G$ satisfying $g(x_i) = y_i$, $1 \leq i \leq n$. A real function k on I is defined to be G-convex (or generalized convex with respect to G) if whenever $x_1 < x_2 < \cdots < x_n$ are points in I and $g \in G$ satisfies $g(x_i) = k(x_i)$, $1 \leq i \leq n$, then

$$(1.1) \qquad (-1)^{n+i-1}(k(s) - g(s)) \geq 0, \qquad s \in (x_{i-1}, x_i), \ 2 \leq i \leq n.$$

The unique g satisfying $g(x_i) = k(x_i)$ is said to interpolate k at $\{x_i\}$. We let K denote the set of all G-convex functions on I. Clearly, $G \subset K$. In general, K is not convex. If G is convex so is K, as may be easily verified. It is easy to show that $K \subset C$; a simple proof appears in [12], although this result was first proved in [14]. For completeness we present the following equivalent definition of G-convexity which is a part of the folklore: a real function k is G-convex if (1.1) holds for some fixed i where $1 \leq i \leq n$, and points $\{x_i\}$ and g are as in the above definition. For example, [9] (resp. [4]) requires that (1.1) hold with $i = n$ (resp. all $1 \leq i \leq n+1$). See [13] for a discussion of this point. We say that G is a linear family or a Tchebycheff system, if G is an n-parameter family which is a vector space of dimension n. The results of

Received by the editors September 18, 1990; presented by the second author at the 869 meeting of the AMS, Fargo, North Dakota, October 25, 1991.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 26A51, 41A30; Secondary 46E30.

The research of the second author was supported by the National Science Foundation under grant RII8610675.

this article, in their full generality, are applicable to G–convex functions even when G is a nonlinear family.

Various concepts of generalized convexity have evolved over several decades in the literature. See, e.g., [16] and other references given there; for generalized convexity induced by ECT–systems see [8, 10]. The above definitions for $n = 2$ appeared in [1,2] and, for an arbitrary n, in [4, 9, 21] following the lead of [15]. It was further extended in [12, 13]. If G is the set of algebraic polynomials of degree at most $n-1$, then functions in K are called n–convex. See, e.g., [3] and references in [16]. Note that 1–convex (resp. 2–convex) functions are monotone nondecreasing (resp. convex) on I. Much effort has been expended in the past to investigate the properties of generalized convex functions and, in particular, n–convex functions, but not mainly from the point of view of approximation theory. However, recently there has been considerable interest in approximation by n–convex functions [6, 20, 23–25] for $n \geq 2$, by generalized convex functions induced by ECT–systems [26], and the special case of monotone functions [5, 18, 19]. In this article, we investigate several properties of generalized convex function significant in approximation theory (§2). We then apply them to establish the existence of a best L_p–approximation $(1 \leq p \leq \infty)$ by nonconvex subsets of such functions and derive properties of L_p–convergent sequences (§3).

The above definition of generalized convexity allows for many classes of functions other than the n–convex functions. Examples for $n = 2$ appear in [1]. For an arbitrary n, let $\alpha = (\alpha_0, \alpha_1, \dots, \alpha_{n-1}) \in R^n$ denote a parameter, and G consist of functions of the form (i) $g_\alpha(s) = \sum_{i=0}^{n-1} \alpha_i s^i + As^n + Bs^{n+1}$, where A and B are fixed constants, or (ii) $g_\alpha(s) = \sum_{i=0}^{n-1} \alpha_i \exp(-(\rho_i - s)^2)$, where ρ_i, $0 \leq i \leq n-1$, are fixed distinct numbers. Then, in each case, G is a linear family, (In (ii) above, the unique interpolation property follows as in [8, Example 5, p. 11].) Other examples are given in §2. Now let G be a linear family spanned by a basis e_i, $0 \leq i \leq n-1$, in C, and τ be a strictly increasing continuous function whose domain and range are the entire real line R. Define $f_\alpha(s) = \tau(\sum_{i=0}^{n-1} \alpha_i e_i)$, $\alpha \in R^n$, and $F = \{f_\alpha : \alpha \in R^n\}$. It is easy to see that F is a nonlinear family when τ is not the identity function; for example, if $\tau(t) = t^3$ or $\tau(t) = \log(1+t)$, $t \geq 0$, and $\tau(t) = -\tau(-t)$, $t < 0$.

2. Properties of G–convex functions

A point $x \in I$ is said to be a local maximum (resp. minimum) of $f \in C$ if there exists an $\varepsilon > 0$ such that $f(x) \geq f(s)$ (resp. $f(x) \leq f(s)$) for all $s \in (x-\varepsilon, x+\varepsilon) \cap I$. A local maximum or minimum is referred to as a local extremum. A function f in C is said to have r alternating local extrema if there exist points $x_1 < x_2 < \cdots < x_r$ in I such that exactly one of the following condition holds. (i) Points x_i with odd (resp. even) indices are local maxima (resp. minima) with $(-1)^i f(x_{i-1}) > (-1)^i f(x_i)$ for $2 \leq i \leq r$. (ii) Points x_i with odd (resp. even) indices are local minima (resp. maxima) with $(-1)^i f(x_{i-1}) < (-1)^i f(x_i)$ for $2 \leq i \leq r$. A constant function has zero alternating local extrema. If f has exactly r alternating local extrema, then the set E of all local extrema of f has r connected components in I where each component is a closed interval on which the function is constant. However, if E has r connected components, it does not necessarily follow that f has

r alternating local extrema. For example, consider a nondecreasing function which is constant on disjoint subintervals of I.

Theorem 2.1. *In the following* (a) *implies* (b) *which implies* (c).

(a) *Every g in G has at most $n-2$ alternating local extrema in I.*
(b) *Every k in K has at most $n-1$ alternating local extrema in I. If k in K has exactly $n-1$ alternating local extrema $x_1 < x_2 < \cdots < x_{n-1}$ in I, then x_{n-1} is a local minimum, k is nondecreasing on (x_{n-1}, b), and $(-1)^{n-1}k$ is nondecreasing on (a, x_1).*
(c) *The total variation $V(k, J)$ of every $k \in K$ is bounded on a compact interval $J \subset I$ with $V(k, J) \leq 2n \max\{|k(s)| : s \in J\}$.*

Proof. We show (a) implies (b). Suppose (a) holds and $k \in K$. Assume that k has r alternating local extrema x_i in I with $x_1 < x_2 < \cdots < x_r$, where $r \geq n$. We reach a contradiction as shown below. Assume first that x_r is a local maximum. Set $z_{n-i} = x_{r-i+1}$, $1 \leq i \leq n-1$. Now choose $z_{n-1} < z_n < b$ with $k(z_n) \leq k(z_{n-1})$. Since $z_{n-1} = x_r$ is a local maximum, this is possible. Now let $g \in G$ so that $g(z_i) = k(z_i)$, $1 \leq i \leq n$, and apply (1.1) with $\{z_i : 1 \leq i \leq n\}$. We conclude that $g(s) \geq k(s)$ for $s \in (z_{n-1}, z_n)$ and $g(s) \leq k(s)$ for $s \in (z_{n-2}, z_{n-1})$. Since $g(z_n) = k(z_n)$ and $g(z_{n-1}) = k(z_{n-1})$, there exists a local maximum t_{n-1} of g with $t_{n-1} \in [z_{n-1}, z_n)$. (z_{n-1}, z_n). Similarly, there exists a local minimum t_{n-2} of g with $t_{n-2} \in [z_{n-2}, z_{n-1})$. Now $g(t_{n-2}) \leq k(z_{n-2}) < k(z_{n-1}) \leq g(t_{n-1})$. Applying this procedure to each interval $[z_{i-1}, z_i]$, $i \geq 2$, we obtain $t_{i-1} \in [z_{i-1}, z_i)$ with $g(t_{n-1}) > g(t_{n-2}) < g(t_{n-3}) \cdots$. Hence, t_i, $1 \leq i \leq n-1$, are $n-1$ alternating local extrema of g which is a contradiction. Now suppose that x_r is a local minimum. Then set $z_{n-i} = x_{r-i}$, $1 \leq i \leq n-1$. Now $z_{n-1} = x_{r-1}$ is a local maximum. Hence a contradiction is reached by arguments as above. Now let x_i, $1 \leq i \leq n-1$, be as in the second statement of (b), where x_{n-1} is a local maximum. Then exactly as above, by letting $z_i = x_i$, $1 \leq i \leq n-1$, we reach a contradiction. Hence, x_{n-1} is a local minimum. The monotonicity of k on (x_{n-1}, b) and (a, x_1) follows because without it there would be additional extrema of k. To show that (b) implies (c), we observe that if k has r alternating local extrema x_i in I, then k is monotone on each subinterval (x_{i-1}, x_i), $1 \leq i \leq r+1$, where $x_0 = a$ and $x_{r+1} = b$. Since $r \leq n-1$, the result follows. The proof is complete.

The next result will be established using Tornheim's convergence theorem [21, Theorem 5].

Proposition 2.2. *Let $K' \subset K$ be a nonempty set of functions which are pointwise bounded on a dense subset I' of I. Let $x_1 < x_2 < \cdots < x_n$ be n points in I'. Then*

$$G' = \{g \in G : g(x_i) = k(x_i),\ 1 \leq i \leq n,\ k \in K'\},$$

is pointwise bounded on I.

Proof. If for some t in I, $\{g(t) : g \in G'\}$ is not bounded above, then there exists a sequence $g_j \in G'$ such that $g_j(t) \to \infty$. If $k_j \in K'$ with $g_j(x_i) = k_j(x_i)$, $1 \leq i \leq n$, then, by hypothesis, the set of real numbers $A_i = \{k_j(x_i) : j \geq 1\}$ is bounded for each i. Hence, $t \neq x_i$ for any i. Since A_i is bounded, there exists a subsequence of g_j which converges at each x_i. Assume, for

convenience, that $g_j(x_i) \to y_i$ for each i where the y_i are real. Let g in K satisfy $g(x_i) = y_i$. Then by [21, Theorem 5], $g_j \to g$ pointwise on I. Hence, $g(t) = \infty$, which is a contradiction since g is real valued. Thus $\{g(t): g \in G'\}$ is bounded above. Similarly, it is bounded below. The proof is complete.

A subset F of C is called equi–Lipschitzian on a compact interval $J \subset I$ if $|f(s) - f(t)| \leq c|s - t|$ holds for all f in F, all s, t in J, and some $c > 0$ possibly depending upon J.

Theorem 2.3. *Let $J \subset I$ be a compact interval and I' be a dense subset of I. The following conditions are equivalent.*

(a) *Every subset G' of G, which is pointwise bounded on I', is equi–Lipschitzian on J.*
(b) *Every subset K' of K, which is pointwise bounded on I', is equi–Lipschitzian on J.*

Proof. Since $G \subset K$, (b) implies (a). To show (a) implies (b), let K' be as in (b), $J = [c, d]$, and $c < s < t < d$. Choose points x_i in I' with $x_1 < x_2 < \cdots < x_{n-1} < s < x_n < t$. Define G' as in Proposition 2.2. Then, by that proposition, G' is pointwise bounded on I and, hence, equi–Lipschitzian on J. Suppose $k \in K'$, and $g \in G'$ with $g(x_i) = k(x_i)$, $1 \leq i \leq n$. Then we must have $k(t) \geq g(t)$ and $k(s) \leq g(s)$. Hence $k(s) - k(t) \leq g(s) - g(t) \leq c|s - t|$ for some $c > 0$ by the equi–Lipschitzian condition on G'. Again, choosing points x_i in I' with $x_1 < x_2 < \cdots < x_{n-2} < s < x_{n-1} < t < x_n$, we may show as above that $k(t) - k(s) \leq c|s - t|$. The proof is complete.

The proofs of the following theorems are identical to [17, Theorems 10.8 and 10.9].

Theorem 2.4. *Suppose that Theorems* 2.3(a) *holds for some compact interval $J \subset I$. Let (k_j) be a sequence in K which converges pointwise on a dense subset of I. The limit then exists for every s in I and the function k given by $k(s) = \text{limit } k_j(s)$ as $j \to \infty$ and s in I is in K. Moreover, (k_j) converges to k uniformly on J.*

Theorem 2.5. *Suppose that Theorem* 2.3(a) *holds for some compact interval $J \subset I$. Let (k_j) be a sequence in K which is pointwise bounded on I or a dense subset of I. Then there exists a subsequence of (k_j) which converges pointwise on I to some function in K, and it does so uniformly on J.*

The following are some examples to which Theorem 2.1 (a) and Theorem 2.3 (a) apply. Let $I = (0, 1)$ and $\alpha \in R^n$ be a parameter as in §1. Let G, in the respective examples, consist of functions of the form (i) $g_\alpha(s) = \alpha_0 + \sum_{i=1}^{n-1} \alpha_i s^i$; (ii) $g_\alpha(s) = \alpha_0 + \sum_{i=1}^{n-1} \alpha_i(\rho_i + s)^{-1}$, where ρ_i are fixed distinct numbers in I; and (iii) $g_\alpha(s) = \alpha_0 + \sum_{i=1}^{n-1} \alpha_i \exp(\rho_i s)$, where ρ_i are as in (ii). The fact that in each case G has the unique interpolating property and Theorem 2.1 (a) applies may be shown by arguments as in [8, Examples, p. 9]. It is easy to see that Theorem 2.3 (a) holds for the cited linear families G of differentiable functions and every compact $J \subset I$. Indeed, in this case, G' is uniformly bounded on any fixed n points in J, and n interpolating values uniquely determine the parameter α of a $g \in G'$. It follows that these parameters are bounded for $g \in G'$, and, hence, the derivative of $g \in G'$ is

also bounded on J. Consequently, Theorem 2.3(a) holds. As was observed in §1, K in example (i) above is the well-known n-convex functions. The equi-Lipschitzian property of Theorem 2.3 (b) then applies to K on every compact $J \subset I$. For convex functions this result and the conclusions of Theorems 2.4 and 2.5 are established in [17], and for n-convex functions in [25]. Several results on the number and properties of components of the set of local extrema of n-convex functions are mentioned in [15, §1.4]. Now let τ be a strictly increasing nonidentity function as in §1 but also differentiable with its derivative $\tau' > 0$ on R. Then $F = \{\tau(g_\alpha) : \alpha \in R^n\}$, where g_α, is as in any of the above examples, is an n-parameter nonlinear family which satisfies Theorems 2.1 (a) and 2.3 (a).

3. Applications to L_p-approximation

Let H be the set of all extended real functions on I. Let L_p, $1 \le p < \infty$, denote the Banach space of all (equivalence classes of) Lebesgue measurable functions f in H with $\int |f|^p < \infty$ and the norm $\|f\|_p = (\int |f|^p)^{1/p}$. Similarly, let L_∞ be the Banach space of (equivalence classes of) essentially bounded functions f with norm $\|f\|_\infty = \operatorname{ess\,sup} |f|$. Let $P \subset K$ be any nonempty set. Given $f \in L_p$, define $\Delta = \inf\{\|f - k\|_p : k \in P \cap L_p\}$. The approximation problem is to find $h \in P \cap L_p$ so that $\Delta = \|f - h\|_p$; such an h is called a best approximation to f from K in the given norm.

Givn $P \subset H$, we define $\overline{P}$ to be the set of all functions f in H such that $f_j \to f$ pointwise on I for some sequence (f_j) in P. Such sets are useful in approximation [24]. Later we shall apply the results of [24]. The definition of $\overline{P}$ given here is weaker than the one in [24]; however, it will be seen that all the results of [24] hold with this change. Note that if $P \subset K$, then $\overline{P}$ is not necessarily a subset of K since the functions in $\overline{P}$ may take the values $\pm\infty$.

Proposition 3.1. *Let (k_j) be a sequence in K which converges pointwise to an extended real valued function k on I. Suppose $x_1 < x_2 < \cdots < x_{n+1}$ are points in I at which k is finite. Then k is finite on $[x_1, x_{n+1}]$.*

Proof. Let $g_j \in G$ interpolate k_j at x_i, $2 \le i \le n+1$, and let $h_j \in G$ interpolate k_j at x_i, $1 \le i \le n$. Let g and h, respectively, interpolate k at x_i, $2 \le i \le n+1$, and x_i, $1 \le i \le n$. Then by [21, Theorem 5], $g_j \to g$ and $h_j \to h$ pointwise on I. Since (1.1) holds for k_j and g_j, in the limit it must hold for k and g. We therefore obtain $(-1)^{n+i}(k(s) - g(s)) \ge 0$ for $s \in (x_{i-1}, x_i)$, $3 \le i \le n+1$, and $(-1)^n(k(s)) - g(s) \ge 0$ for $s \in (a, x_2)$. Similarly, considering k_j and h_j we obtain $(-1)^{n+i-1}(k(s) - h(s)) \ge 0$ for $s \in (x_{i-1}, x_i)$, $1 \le i \le n$, and $k(s) - h(s) \ge 0$ for $s \in (x_n, b)$. We therefore conclude that $(-1)^{n+i} g(s) \le (-1)^{n+i} k(s) \le (-1)^{n+i} h(s)$ for $s \in (x_{i-1}, x_i)$, $3 \le i \le n$. Also, $g(s) \ge k(s) \ge h(s)$ for $s \in (x_n, x_{n+1})$, and $(-1)^n g(s) \le (-1)^n k(s) \le (-1)^n h(s)$ for $s \in (x_1, x_2)$. It follows that k is finite on $[x_1, x_{n+1}]$. The proof is complete.

Proposition 3.2. $K \cap L_p = \overline{K} \cap L_p$ *for* $1 \le p \le \infty$.

Proof. Let $k \in \overline{K} \cap L_p$. Then there exists a sequence (k_j) in K such that $k_j \to k$ pointwise on I. Let $c, d \in I$ with $c < d$. Since $k \in L_p$, the set $\{s \in I : |k(s)| = \infty\}$ has Lebesque measure zero. Hence, we can find points $x_1 < x_2 < \cdots < x_{n+1}$ in I with $x_1 < c < d < x_{n+1}$ such that $|k(x_i)| < \infty$,

$1 \le i \le n+1$. By Proposition 3.1, k is finite on $[c, d]$ and hence on I since c and d are arbitrary. Since each k_j is in K so is k. Hence, $k \in K \cap L_p$ and the proof is complete.

Theorem 3.3. *Suppose that Theorem* 2.1(a) *holds. Let* $1 \le p \le \infty$ *and* $P \subset K$ *be nonempty satisfying* $P \cap L_p = \overline{P} \cap L_p$.

(i) *Let* (k_j) *be a sequence of functions in* $P \cap L_p$ *such that* $\|k_j\|_p \le D$ *for all* j *and some* $D > 0$. *Then there exists a subsequence* (h_j) *of* (k_j) *and* h *in* $P \cap L_p$ *such that* $h_j \to h$ *pointwise on* I *and* $\|h\|_p \le D$. *In particular, the above holds for* $P = K$.

(ii) $P \cap L_p$ *is closed in* L_p, *and a best approximation to* f *in* L_p *from* $P \cap L_p$ *exists if* $P \cap L_p$ *is nonempty. In particular, the above holds for* $P = K$.

Proof. Since (a) of Theorem 2.1 implies (b), if $k \in K$, then k has $r \le n-1$ alternating local extrema $x_1 < x_2 < \cdots < x_r$ in I. Consequently, k is monotone (nondecreasing or nonincreasing) on each interval (x_{i-1}, x_i), $1 \le i \le r+1$, where $x_0 = a$ and $x_{r+1} = b$. Hence, conditions (1) and (2) of [24, p. 224] hold. The required conclusions (i) and (ii) for $P \cap L_p$ then follow from [24, Theorems 2.1 and 2.2]. Since, by Proposition 3.1, $P \cap L_p = \overline{P} \cap L_p$ holds when $P = K$, the required conclusions also hold for $K \cap L_p$. The proof is complete.

The special case of the above theorem as applied to n–convex functions appears in [24, p. 235]. The existence of a best L_1–(resp. L_∞) approximation by n–convex functions is also established in [6] (resp. [25]) by different methods. For the problem of L_∞–approximation by convex functions, [23] characterizes the maximal best approximation to f as the shift of the greatest convex minorant of f, and develops efficient algorithm for its computation. The existence of a best L_p–approximation, $1 \le p \le \infty$, to f in $C([a, b])$ from G and certain uniqueness results are established in [21, 22].

Lemma 3.4. *Suppose that Theorem* 2.1(a) *holds. Let* $1 \le p \le \infty$ *and* $K' \subset K \cap L_p$ *be nonempty such that* $\|k\|_p \le D$ *for all* $k \in K'$ *and some* $D > 0$. *Then* K' *is pointwise bounded on* I.

Proof. Suppose K' is not bounded above for some $t \in I$. Then there exists a subsequence (k_j) in K' such that $k_j(t) \to \infty$. By Theorem 3.3 (i) with $P = K$, there exists a subsequence (h_j) of (k_j) and $h \in K \cap L_p$ such that $h_j \to h$ pointwise on I. It follows that $h(t) = \infty$, which is a contradiction since h is real valued. Similarly, K' is bounded below. The proof is complete.

Theorem 3.5. *Let* $J \subset I$ *be a compact interval. Suppose that Theorem* 2.1(a) *and Theorem* 2.3(a) *hold. Let* $1 \le p \le \infty$, $k \in C$, *and* (k_j) *be a sequence in* $K \cap L_p$. *If* $\|k_j - k\|_p \to 0$, *then* $k_j \to k$ *uniformly on* J.

Proof. There exists $D > 0$ such that $\|k_j\|_p \le D$ for all j. Hence, by Lemma 3.4, $K' = \{k_j\}$ is pointwise bounded on I. Theorem 2.3(b) then applies to give $|k_j(s) - k_j(t)| \le c|s-t|$ for all j, for all $s, t \in J$, and some $c > 0$. We first show that $k_j \to k$ on J. Suppose $s \in J$, $\varepsilon > 0$, and $\theta = \varepsilon/(2c)$. By the continuity of k at s, there exists $0 < \delta < \theta$ such that if $J' = J \cap (s-\delta, s+\delta)$, then $|k(s) - k(t)| \le \varepsilon/2$ for all $t \in J'$. Hence, $|k_j(t) - k(t)| \ge |k_j(s) - k(s)| - \varepsilon$ for all $t \in J'$, for all j. If χ denotes the indicator function of J', then

$\|k_j - k\|_p (k_j - k)\chi\|_p \geq \max\{|k_j(s) - k(s)| - \varepsilon, 0\}\mu(J')^{1/p}$. Letting $j \to \infty$, we conclude that $k_j(s) \to k(s)$ on J. It follows that $|k(s) - k(t)| \leq c|s - t|$ for all $s, t \in J$. The result now follows by Theorem 2.4. The proof is complete.

The special case of the above theorem as applied to n-convex functions is established by different methods in [11].

References

1. E. F. Beckenbach, *Generalized convex functions*, Bull. Amer. Math. Soc. **43** (1937), 363–371.
2. E. F. Beckenbach and R. H. Bing, *On generalized convex functions*, Trans. Amer. Math. Soc. **58** (1945), 220–230.
3. P. S. Bullen, *A criterion for n-convexity*, Pacific J. Math. **36** (1971), 81–98.
4. P. Hartman, *Unrestricted n-parameter families*, Rend. Circ. Mat. Palermo (2) **7** (1958), 123–142.
5. R. Huotari, D. Legg, A. D. Meyerowitz, and D. Townsend, *The natural best L_1-approximation by nondecreasing functions*, J. Approx. Theory **52** (1988), 132–140.
6. R. Huotari, R. Legg, and D. Townsend, *Existence of best n-convex approximants in L_1*, Approx. Theory Appl. **5** (1989), 51–57.
7. D. Landers and L. Rogge, *Isotonic approximation in L_s*, J. Approx. Theory **31** (1981), 199–223.
8. S. Karlin and W. J. Studden, *Tchebycheff systems: with applications in analysis and statistics*, Interscience, New York, 1966.
9. J. H. B. Kemperman, *On the regularity of generalized convex functions*, Trans. Amer. Math. Soc. **135** (1969), 69–93.
10. M. G. Krein and A. A. Nudel'man, *The Markov moment problem and extremal problems*, Trans. Math. Monographs, vol. 50, Amer. Math. Soc., Providence, RI, 1977.
11. J. T. Lewis and O. Shisha, *L_p convergence of monotone functions and their uniform convergence*, J. Approx. Theory **14** (1975), 281–284.
12. R. M. Mathsen, *$\lambda(n)$-convex functions*, Rocky Mountain J. Math. **2** (1972), 31–43.
13. ——, *Hereditary $\lambda(n, k)$-families and generalized convexity of functions*, Rocky Mountain J. Math. **12** (1982), 753–756.
14. E. Moldovan, *Sur une généralisation des fonctions convexes*, Matematica (Cluj) (2) **1** (1959), 49–80.
15. T. Popoviciu, *Les fonctions convexes*, Hermann, Paris, 1944.
16. A. W. Roberts and D. E. Varberg, *Convex functions*, Academic Press, New York, 1973.
17. R. T. Rockafellar, *Convex analysis*, Princeton Univ. Press, Princeton, NJ, 1970.
18. J. J. Swetits, S. E. Weinstein, and Y. Xu, *On the characterization and computation of best monotone approximation in $L_p[0, 1]$ for $1 \leq p < \infty$*, J. Approx. Theory **60** (1990), 58–68.
19. J. J. Swetits and S. E. Weinstein, *Construction of best monotone approximation on $L_p[0, 1]$*, J. Approx. Theory **61** (1990), 118–130.
20. J. J. Swetits, S. E. Weinstein, and Y. Xu, *Approximation in $L_p[0, 1]$ by n-convex functions*, Numer. Funct. Anal. Optim. **11** (1990), 167–179.
21. L. Tornheim, *On n-parameter families of functions and associated convex functions*, Trans. Amer. Math. Soc. **69** (1950), 457–467.
22. ——, *Approximation by families of functions*, Trans, Amer. Math. Soc. **7** (1956), 641–643.
23. V. A. Ubhaya, *An $O(n)$ algorithm for discrete n-point convex approximation with applications to continuous case*, J. Math. Anal. Appl. **72** (1979), 338–354.
24. ——, *L_p approximation from nonconvex subsets of special classes of functions*, J. Approx. Theory **57** (1989), 223–238.

25. D. Zwick, *Existence of best n-convex approximations*, Proc. Amer. Math. Soc. **97** (1986), 273–276.

26. ——, *Best L_1-approximation by generalized convex functions*, J. Approx. Theory **59** (1989), 116–123.

DEPARTMENT OF MATHEMATICS, NORTH DAKOTA STATE UNIVERSITY, FARGO, NORTH DAKOTA 58105

DEPARTMENT OF COMPUTER SCIENCE AND OPERATIONS RESEARCH, 300 MINARD HALL, NORTH DAKOTA STATE UNIVERSITY, FARGO, NORTH DAKOTA 58105