

## ON THE EXISTENCE OF FIXPOINTS OF COMPOSITE MEROMORPHIC FUNCTIONS

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**ABSTRACT.** Let  $f$  be a meromorphic function with at least two poles, and let  $g$  be an entire transcendental function. It is proved that the composite function  $f \circ g$  has infinitely many fixpoints.

### 1. INTRODUCTION AND MAIN RESULT

Let  $f$  be a transcendental meromorphic function, and let  $g$  be a transcendental entire function. Gross [3, p. 247, Problem 6] has asked whether the composite function  $f \circ g$  has infinitely many fixpoints. This was proved in [1] if  $f$  is entire, and it can be proved by the same method if  $f$  has exactly one pole. In this note, we shall give an affirmative answer to Gross's question.

**Theorem.** *Let  $f$  be a meromorphic function that has at least two different poles, and let  $g$  be a transcendental entire function. Then the composite function  $f \circ g$  has infinitely many fixpoints.*

The proof is very short and based on Picard's theorem, but completely elementary otherwise. In fact, the argument is quite simple, but it seems to have been overlooked before.

### 2. LEMMAS

We shall use the following elementary lemmas.

**Lemma 1.** *Let  $f$  be a meromorphic function, and let  $z_0$  be a pole of  $f$  of order  $p$ . Then there exists a function  $h$ , defined and analytic in a neighborhood of 0, such that  $h(0) = 0$  and  $f(h(z) + z_0) = z^{-p}$  for  $z \neq 0$ .*

To prove Lemma 1, we note that there exists a function  $k$ , defined and analytic in a neighborhood of 0, such that  $f(z + z_0) = k(z)^{-p}$  and  $k'(0) \neq 0$ . The conclusion follows if we define  $h$  as the inverse function of  $k$ .

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**Lemma 2.** *Let  $f$  and  $g$  be meromorphic functions. Then  $f \circ g$  has infinitely many fixpoints if and only if  $g \circ f$  does.*

Lemma 2 follows from the fact that  $g$  maps the set of fixpoints of  $f \circ g$  bijectively onto the set of fixpoints of  $g \circ f$ . Lemma 2 is not new and has been used in several papers on the subject. It seems to have been noticed first by Gross [2, Lemma 3].

### 3. PROOF OF THE THEOREM

Let  $z_1$  and  $z_2$  be poles of  $f$ . For  $j \in \{1, 2\}$ , denote the order of  $z_j$  by  $p_j$  and choose  $h_j$  according to Lemma 1. Define  $k_1(z) = h_1(z^{p_2}) + z_1$  and  $k_2(z) = h_2(z^{p_1}) + z_2$ . Then  $f(k_1(z)) = f(k_2(z)) = z^{-p_1 p_2}$  in a punctured neighborhood of 0. Now define  $u(z) = g(z^{-p_1 p_2})$ . Then 0 is an essential singularity of  $u$  and in a punctured neighborhood of 0 we have  $u(z) = g(f(k_1(z))) = g(f(k_2(z)))$ .

Suppose now that  $f \circ g$  has only finitely many fixpoints. Then  $g \circ f$  has only finitely many fixpoints by Lemma 2. It follows that  $u(z) \neq k_j(z)$  for  $j \in \{1, 2\}$  in a punctured neighborhood of 0. Moreover,  $k_1(z) \neq k_2(z)$  in a neighborhood of 0 since  $k_1(0) = z_1 \neq z_2 = k_2(0)$ . Define

$$v(z) = \frac{u(z) - k_1(z)}{k_2(z) - k_1(z)}.$$

Then 0 is an essential singularity of  $v$  and  $v$  does not take the values 0, 1, and  $\infty$  in a punctured neighborhood of 0. This contradicts Picard's Theorem and the theorem is proved.

### 4. REMARK

Similarly, we can prove that if  $f$  and  $g$  are transcendental meromorphic functions and if one of the two functions  $f$  and  $g$  has at least three poles, then  $f \circ g$  has infinitely many fixpoints.

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