FRÉCHET VS. GÂTEAUX DIFFERENTIABILITY
OF LIPSCHITZIAN FUNCTIONS

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Abstract. Examples have been given of Lipschitzian functions that are Gâteaux-differentiable everywhere, but nowhere Fréchet-differentiable. One such example has been reported, mistakenly, in several papers as having domain in $L^2([0, \pi])$, when it should have been $L^1([0, \pi])$. We discuss this example.

The purpose of this note is to point out a misunderstanding that has been perpetuated about an example due to Sova [7]. In order to do this, we consider two mappings: (1) $f : L^1([0, \pi]) \rightarrow \mathbb{R}$ defined by $f(x) = \int_0^\pi x(t) \, dt$, and (2) $g : L^2([0, \pi]) \rightarrow \mathbb{R}$ defined by $g(x) = \int_0^\pi x(t) \, dt$. Clearly, $g$ is the restriction of $f$ to $L^2([0, \pi]) \subseteq L^1([0, \pi])$. The mapping $f$ is an example of a Lipschitzian real-valued function that is everywhere Gâteaux-differentiable, but nowhere Fréchet-differentiable. In fact, $f$ is a special case of a whole class of mappings, from the space $L^1(X, \Sigma, \mu)$ of all $\Sigma$-measurable, $\mu$-integrable functions from $X$ to $\mathbb{R}$, defined by Sova in [7] that are Gâteaux-differentiable, but not Fréchet-differentiable.

On the other hand, the function $g$ is Fréchet-differentiable everywhere, but is given in some papers, in error, as an example of a mapping that is nowhere Fréchet-differentiable; cf. [3, p. 205], [4, p. 124], and [5, p. 125]. Earlier, Phelps [6, pp. 981–982] gave an example of an equivalent norm on $l^1$ that is Gâteaux-differentiable everywhere (except at the origin) and nowhere Fréchet-differentiable. Other examples of Lipschitzian real-valued functions that are nowhere Fréchet-differentiable were given by Aronszajn in [1]; his functions are on the space $l^1$.

In order to consider the differentiability of $f$ and $g$, let $x, v \in L^1([0, \pi])$,
$v \neq 0$, $h > 0$. Then
\[
\lim_{h \to 0} \frac{1}{h} \int_0^\pi \left[ \sin(x(t) + hv(t)) - \sin x(t) \right] dt
\]
\[
= \lim_{h \to 0} \int_0^\pi \sin \left( \frac{hv(t)}{2} \right) \cos \left( x(t) + \frac{hv(t)}{2} \right) dt
\]
\[
= \int_0^\pi v(t) \cos x(t) dt,
\]
since the integrand $((\sin \frac{hv(t)}{2})/(h/2)) \cos(x(t) + \frac{hv(t)}{2})$ is dominated by $v \in L^1([0, \pi])$. Hence, the Gâteau $x$ derivative of the mapping $f$ at $x$ is $D_G f(x) = \cos x$. It is clear that the mapping $g$, which is the restriction of $f$ to $L^2([0, \pi])$, is also Gâteaux-differentiable and that the Gâteaux derivative $D_G g(x) = \cos x$ is a continuous mapping from $L^2([0, \pi])$ into the norm topologies. Therefore, $g$ is Fréchet-differentiable everywhere (see [2, Examples 1 and 2, pp. 18–22] for a more general class having the two properties described previously). Actually, in our case, $g$ is uniformly Fréchet-differentiable. (Apply Taylor’s formula to the sine function, and conclude that $|\sin(a + b) - \sin a - b \cos a| = \frac{b^2}{2} \sin z$ for some $z$ between $a$ and $a + b$.

Thus,
\[
\int_0^\pi |\sin(x(t) + y(t)) - \sin x(t) - y(t) \cos x(t)| dt \leq \int_0^\pi \frac{1}{2} y(t)^2 dt = \frac{1}{2} ||y||^2.
\]
Hence, if $||y||_2 < 2\varepsilon$, we see that $|g(x + y) - g(x) - (y, \cos x)| \leq \varepsilon ||y||_2$.

To prove that $f$ is not Fréchet-differentiable at any point $x \in L^1([0, \pi])$, we will follow Sova’s proof of [7, Theorem 2.1.6]. First, we show that for each $x \in L^1([0, \pi])$, there exists $v \in L^1([0, \pi])$ such that the Lebesgue measure of the set $\{t \in \mathbb{R} | 0 \leq t \leq \pi \text{ and } \sin(x(t) + v(t)) - \sin x(t) - v(t) \cos x(t) \neq 0\}$ is positive. If not, let $q$ be a rational number and define $v_q$ by $v_q(t) = q$ for all $t \in [0, \pi]$. Then $v_q \in L^1([0, \pi])$ and the set $N_q = \{t \in [0, \pi] | \sin(x(t) + q) - \sin x(t) = q \cos x(t)\}$ has Lebesgue measure 0. Hence, the union $N = \bigcup\{N_q | q \text{ rational}\}$ also has measure 0. Thus, for all rational numbers $q$ and all $t \notin N$, $\sin(x(t) + q) - \sin x(t) = q \cos x(t)$. This is a contradiction since the mapping $q \cos x(t)$ is a linear function of $q$, but $\sin(x(t) + q) - \sin x(t)$ is not.

Next, choose $v_0 \in L^1([0, \pi])$ such that $\mu(\{t \in [0, \pi] | \sin(x(t) + v_0(t)) - \sin x(t) - v_0(t) \cos x(t) \neq 0\}) > 0$, where $\mu$ denotes Lebesgue measure. Then we can find $\alpha > 0$ such that the set $Z = \{t \in [0, \pi] | \sin(x(t) + v_0(t)) - \sin x(t) - v_0(t) \cos x(t) > \alpha\}$ satisfies $\mu(Z) > 0$. Further, there exists a $\beta$ and a measurable subset $Z_0$ of $Z$ such that $\mu(Z_0) > 0$ and $|v(t)| < \beta$ for $t \in Z_0$. Choose a sequence $\{Z_n\}_{n=1}^\infty$ of measurable subsets of $Z_0$ of $Z$ such that $Z_{n+1} \subset Z_n$, $\mu(Z_n) > 0$ for $n = 1, 2, \ldots$, and $\bigcap_{n=1}^\infty Z_n = \emptyset$, and define a sequence $\{h_n\}_{n=1}^\infty$ of functions in $L^1([0, \pi])$ by
\[
h_n(t) = \begin{cases} 
v_0(t) & \text{if } t \in Z_n \\
0 & \text{if } t \notin Z_n.
\end{cases}
\]
It is easy to check that $||h_n||_1 \to 0$ as $n \to \infty$, but
\[
\frac{1}{||h_n||_1} \int_0^\pi [\sin(x(t) + h_n(t)) - \sin x(t) - h_n(t) \cos x(t)] dt \geq \frac{\alpha \mu(Z_n)}{\beta \mu(Z_n)} = \frac{\alpha}{\beta} > 0.
\]
This shows that \( f \) is not Fréchet-differentiable at \( x \in L^1([0, \pi]) \).

*Added in proof.* It should be noted that the authors were not the first to discover the difficulties discussed in this paper. See R. R. Phelps, *Convex functions, monotone operators and differentiability*, Lecture Notes in Math., vol. 1364, Springer-Verlag, New York, 1989, p. 105. See also D. Preiss, *Fréchet derivatives of Lipschitz functions*, J. Funct. Anal. 91 (1990), 312–345, for a very strong positive result on differentiability of Lipschitz functions.

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**REFERENCES**


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