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REMARKS ON QUASICONVEXITY AND STABILITY OF EQUILIBRIA FOR VARIATIONAL INTEGRALS

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ABSTRACT. Let $F: \mathbf{R}^{nN} \to \mathbf{R}$ be a uniformly strictly quasiconvex function (see [3, 4]) of class $C^{2+\alpha}$, $(0 < \alpha < 1)$, and be of polynomial growth. Then every smooth solution of the Euler-Lagrangian equation of the multiple integral $I(u; \Omega) = \int_\Omega F(Du(x))\,dx$ is a minimum of I for variations of sufficiently small supports contained in Ω.

This note establishes the stability of solutions to the equilibrium equations for variational integrals under the constitutive assumption of uniformly strictly quasiconvexity. We show that, for a class of quasiconvex integrands, all equilibria are strong local minimizers of sufficiently small support. This work could be compared with that of Sivaloganathan [9] and Zhang [10], where the similar problems are studied under the constitutive assumption of polyconvexity.

Let $\Omega \subset \mathbf{R}^n$ be bounded and open. To any given map $u: \Omega \mapsto \mathbf{R}^N$, we associate an energy

$$I(u; \Omega) = \int_\Omega F(Du(x))\,dx. \tag{1}$$

It is well known that any smooth minimizer of I satisfies the corresponding Euler-Lagrange equations:

$$\frac{\partial}{\partial x^\alpha}\left[\frac{\partial F}{\partial P^i_\alpha}(Du(x))\right] = 0, \quad \text{for all } x \in \Omega,\ i = 1, 2, \dots, N. \tag{2}$$

We define the set of admissible maps

$$A_R(x_0) = \{u \in W^{1,p}(B_R(x_0); \mathbf{R}^N): u|_{\partial\Omega} = u_0|_{\partial\Omega},\ B_R(x_0) \subset\subset \Omega\} \tag{3}$$

and consider the question of whether a given solution u_0 of (2) is a strong local minimizer of I, in the sense that u_0 minimizes I in $A_R(x_0)$ for some $x_0 \in \Omega$ and for some $R > 0$ (where we use $B_R(x_0)$, to denote the ball in $\mathbf{R}^n$ centered at x_0 with radius $R > 0$). We study this question in the case where the integrand F is uniformly strictly quasiconvex and of class $C^{2,\alpha}$.

This problem has been studied by many authors (see, e.g., Cesari [2], Rund [8] in the case that F is strictly convex and may depend on x, u as well, Sivaloganathan [9] for polyconvexity case, and the references therein). However, in

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the present case nonuniqueness phenomenon of solutions may occur (see Ball [1], John [6], Knops and Stuart [7]). It is interesting to study the behaviour of other solutions of (2) than the minimizers of (1).

Throughout this paper, summation convention is applied. For a function $F: \mathbf{R}^{Nn} \mapsto \mathbf{R}$, denote by $DF(P)$, $D^2F(P)$ the first and the second order derivatives of F while $\|\cdot\|_C$ denotes the supremum norm on the space of continuous functions on some $\overline{B}_R(x) \subset \Omega$. We use various C to denote positive constants independent of the variables.

Theorem. *Suppose $F: \mathbf{R}^{Nn} \mapsto \mathbf{R}$ is of class $C^{2,\alpha}_{\mathrm{loc}}$ with $0 < \alpha \leq 1$ and satisfies*

(1) *for some $p \geq 2 + \alpha$,*

$$|D^2F(P+Q) - D^2F(P)| \leq C_1(1 + |P|^{p-2-\alpha} + |Q|^{p-2-\alpha})|Q|^\alpha;$$

(2) *(uniformly strictly quasiconvexity (see Evans [3], Giaquinta and Modica [5], Fusco and Hutchinson [4])). For every open bounded set $G \subset \mathbf{R}^n$, every $P \in \mathbf{R}^{Nn}$ and every $\phi \in W_0^{1,p}(G; \mathbf{R}^N)$,*

$$\int_G [F(P) + \nu(|D\phi|^p + |D\phi|^2)]\,dx \leq \int_G F(P + D\phi)\,dx,$$

where $\nu > 0$ is a constant.

Then, for every C^2 solution u_0 of (2) and each $x_0 \in \Omega$, there exists an $R > 0$ with $B_R(x_0) \Subset \Omega$, such that u_0 is a strong minimizer of $I(\cdot; B_R(x_0))$ on $A_R(x_0)$.

Remark. It is easy to see that (1) implies

$$|F(P)| \leq C(1 + |P|^p)$$

for some $C > 0$.

In general, even the minimizers of (1) are only partially regular (see e.g., [3, 4, 5]), i.e., there exists an open subset Ω_0 of Ω with $\mathrm{meas}(\Omega\backslash\Omega_0) = 0$, such that $u \in C^{1,\alpha}(\Omega_0; \mathbf{R}^N)$, $0 < \alpha < 1$. Therefore, for partially regular solutions of (2), we conclude that u_0 is locally stable on Ω_0, i.e., u_0 is a minimizer on $A_R(x_0)$ for some $R > 0$, $B_R(x_0) \subset\subset \Omega_0$.

Proof of the theorem. We prove the theorem by contradiction. For a fixed $x_0 \in \Omega$, if the conclusion of the theorem is not true, then there exists a sequence of positive R_j with $R_j \to 0$ as $j \to \infty$, such that $B_{R_j}(x_0) \Subset \Omega$ and u_0 is not a minimizer in $A_{R_j}(x_0)$. Hence there exists a sequence of functions $\phi_j \in W_0^{1,p}(B_{R_j}(x_0))$ such that

$$\int_{B_{R_j}(x_0)} F(Du_0 + D\phi_j)\,dx < \int_{B_{R_j}(x_0)} F(Du_0)\,dx. \tag{4}$$

Since u_0 is a solution of (2), we have, from (4),

$$\begin{aligned} 0 > \int_{B_{R_j}(x_0)} [F(Du_0 + D\phi_j) - F(Du_0)]\,dx &= \int_{B_{R_j}(x_0)} DF(Du_0)D\phi_j\,dx \\ &+ \int_{B_{R_j}(x_0)} \int_0^1 (1-t)D^2F(Du_0 + tD\phi_j)D\phi_j D\phi_j\,dt\,dx \end{aligned} \tag{5}$$

and notice that, from the divergence theorem,

$$\int_{B_{R_j}(x_0)} DF(Du_0)D\phi_j\,dx = 0,$$

so that, in (5), we have

$$(6)\qquad 0 < -\int_{B_{R_j}(x_0)}\int_0^1 (1-t)D^2F(Du_0 + tD\phi_j)D\phi_j D\phi_j\,dt\,dx.$$

On the other hand, from (2),

$$(7)\qquad \begin{aligned} &\nu\int_{B_{R_j}(x_0)} (|D\phi_j|^2 + |D\phi_j|^p)\,dx \\ &\quad\le \int_{B_{R_j}(x_0)} [F(Du_0(x_0)) + D\phi_j) - F(Du_0(x_0))]\,dx \\ &\quad= \int_{B_{R_j}(x_0)} DF(Du_0(x_0))D\phi_j\,dx \\ &\quad\quad+ \int_{B_{R_j}(x_0)}\int_0^1 (1-t)D^2F(Du_0(x_0)) + tD\phi_j)D\phi_j D\phi_j\,dt\,dx. \end{aligned}$$

Still from the divergence theorem, we have

$$\int_{B_{R_j}(x_0)} DF(Du_0(x_0))D\phi_j\,dx = 0,$$

so that adding (6) to (7) gives

$$(8)\qquad \begin{aligned} &\nu\int_{B_{R_j}(x_0)} (|D\phi_j|^p + |D\phi_j|^2)\,dx \\ &\quad< \int_{B_{R_j}(x_0)}\int_0^1 (1-t)[D^2F(Du_0(x_0) + tD\phi_j) \\ &\quad\quad - D^2F(Du_0(x) + tD\phi_j)]D\phi_j D\phi_j\,dt\,dx \\ &\quad\le \int_{B_{R_j}(x_0)} C[1 + |Du_0(x)|^{p-2-\alpha} + |Du_0(x_0)|^{p-2-\alpha} \\ &\quad\quad + |D\phi_j|^{p-2-\alpha}]|Du_0(x) - Du_0(x_0)|^\alpha |D\phi_j|^2\,dx \\ &\quad\le C(\|Du_0\|_C)\int_{B_{R_j}(x_0)} |Du_0(x_0) - Du_0(x)|^\alpha(1 + |D\phi_j|^{p-2})|D\phi_j|^2\,dx \\ &\quad\le C\max_{x\in B_{R_j}}\{|Du_0(x_0) - Du_0(x)|^\alpha\}\int_{B_{R_j}(x_0)} (|D\phi_j|^p + |D\phi_j|^2)\,dx. \end{aligned}$$

Dividing both sides of (8) by $\int_{B_{R_j}(x_0)}(|D\phi_j|^p + |D\phi_j|^2)\,dx$ we obtain

$$(9)\qquad \nu < C\max_{x\in B_{R_j}(x_0)}\{|Du_0(x_0) - Du_0(x)|^\alpha\}.$$

Passing to the limit $j\to\infty$, we have $R_j \to 0$ and

$$\max_{x\in B_{R_j}(x_0)} |\{Du_0(x_0) - Du_0(x)|\} \to 0$$

so that

$$\nu \leq \lim_{j\to\infty} C \max_{x\in B_{R_j}(x_0)} \{|Du_0(x_0) - Du_0(x)|^\alpha\} = 0.$$

This contradicts to the assumption $\nu > 0$.

Remark. This result could be easily extended to the case where F depends explicitly on x, say, $F = F(x, P)$. It seems to be unknown whether general weak equilibria of (1) in $W^{1,p}(\Omega: \mathbf{R}^N)$ are local minima for variations of sufficiently small support near a Lebesgue point of Du.

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References

1. J. M. Ball, *Convexity conditions and existence theorems in nonlinear elasticity*, Arch. Rational Mech. Anal. **63** (1977), 337–403.
2. L. Cesari, *Optimisation-theory and applications*, Springer-Verlag, Berlin, 1983.
3. L. C. Evans, *Quasiconvexity and partial regularity in the calculus of variations*, Arch. Rational Mech. Anal. **95** (1986), 227–252.
4. N. Fusco and J. Hutchinson, $C^{1,\alpha}$ *partial regularity of functions minimizing quasiconvex integrals*, Manuscripta Math. **54** (1985), 121–143.
5. M. Giaquinta and G. Modica, *Partial regularity of minimizers of quasiconvex integrals*, Ann. Inst. H. Poincare Anal. Non Lineaire **3** (1986), 185–208.
6. F. John, *Uniqueness of non-linear elastic equilibrium for prescribed boundary displacements and sufficiently small strains*, Comm. Pure Appl. Math. **25** (1972), 617–634.
7. R. J. Knops and C. A. Stuart, *Quasiconvexity and uniqueness of equilibrium solutions in nonlinear elasticity*, Arch. Rational Mech. Anal. **86** (1984), 233–249.
8. H. Rund, *The Hamilton-Jacobi theory in the calculus of variations*, Van-Nostrand, London, 1966.
9. J. Sivaloganathan, *The generalised Hamilton-Jacobi inequality and the stability of equilibria in nonlinear elasticity*, preprint, 1988.
10. K. Zhang, *Polyconvexity and stability of equilibria in nonlinear elasticity*, Quart. J. Mech. Appl. Math. **43** (1990), 215–221.

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