REMARKS ON QUASICONEVEXITY AND STABILITY OF EQUILIBRIA FOR VARIATIONAL INTEGRALS

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Abstract. Let $F : \mathbb{R}^{nN} \rightarrow \mathbb{R}$ be a uniformly strictly quasiconvex function (see [3, 4]) of class $C^{2+a}$, $(0 < a < 1)$, and be of polynomial growth. Then every smooth solution of the Euler-Lagrangian equation of the multiple integral $I(u; \Omega) = \int_{\Omega} F(Du(x)) \, dx$ is a minimum of $I$ for variations of sufficiently small supports contained in $\Omega$.

This note establishes the stability of solutions to the equilibrium equations for variational integrals under the constitutive assumption of uniformly strictly quasiconvexity. We show that, for a class of quasiconvex integrands, all equilibria are strong local minimizers of sufficiently small support. This work could be compared with that of Sivaloganathan [9] and Zhang [10], where the similar problems are studied under the constitutive assumption of polyconvexity.

Let $\Omega \subset \mathbb{R}^n$ be bounded and open. To any given map $u : \Omega \rightarrow \mathbb{R}^N$, we associate an energy

$$I(u ; \Omega) = \int_{\Omega} F(Du(x)) \, dx.$$  

It is well known that any smooth minimizer of $I$ satisfies the corresponding Euler-Lagrange equations:

$$\frac{\partial}{\partial x^\alpha} \left[ \frac{\partial F}{\partial P^\alpha_i}(Du(x)) \right] = 0, \quad \text{for all } x \in \Omega, \ i = 1, 2, \ldots, N.$$  

We define the set of admissible maps

$$A_R(x_0) = \{ u \in W^{1,p}(B_R(x_0); \mathbb{R}^N) : u|_{\partial \Omega} = u_0|_{\partial \Omega}, \ B_R(x_0) \subset \subset \Omega \}$$  

and consider the question of whether a given solution $u_0$ of (2) is a strong local minimizer of $I$, in the sense that $u_0$ minimizes $I$ in $A_R(x_0)$ for some $x_0 \in \Omega$ and for some $R > 0$ (where we use $B_R(x_0)$, to denote the ball in $\mathbb{R}^n$ centered at $x_0$ with radius $R > 0$). We study this question in the case where the integrand $F$ is uniformly strictly quasiconvex and of class $C^{2+\alpha}$.

This problem has been studied by many authors (see, e.g., Cesari [2], Rund [8] in the case that $F$ is strictly convex and may depend on $x, u$ as well, Sivaloganathan [9] for polyconvexity case, and the references therein). However, in

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the present case nonuniqueness phenomenon of solutions may occur (see Ball [1], John [6], Knops and Stuart [7]). It is interesting to study the behaviour of other solutions of (2) than the minimizers of (1).

Throughout this paper, summation convention is applied. For a function \( F: \mathbb{R}^{Nn} \rightarrow \mathbb{R} \), denote by \( DF(P) \), \( D^2F(P) \) the first and the second order derivatives of \( F \) while \( \| \cdot \|_C \) denotes the supremum norm on the space of continuous functions on some \( \overline{\Omega} \). We use various \( C \) to denote positive constants independent of the variables.

**Theorem.** Suppose \( F: \mathbb{R}^{Nn} \rightarrow \mathbb{R} \) is of class \( C^{2,\alpha}_{\text{loc}} \) with \( 0 < \alpha \leq 1 \) and satisfies

1. for some \( p \geq 2 + \alpha \),
   \[
   |D^2F(P + Q) - D^2F(P)| \leq C(1 + |P|^{p-2-\alpha} + |Q|^{p-2-\alpha})|Q|^\alpha;
   \]
2. (uniformly strictly quasiconvexity (see Evans [3], Giaquinta and Modica [5], Fusco and Hutchinson [4])). For every open bounded set \( G \subset \mathbb{R}^n \), every \( P \in \mathbb{R}^{Nn} \) and every \( \phi \in W_0^{1,p}(G; \mathbb{R}^N) \),
   \[
   \int_G [F(P) + \nu(|D\phi|^p + |D\phi|^2)] \, dx \leq \int_G F(P + D\phi) \, dx,
   \]
   where \( \nu > 0 \) is a constant.

Then, for every \( C^2 \) solution \( u_0 \) of (2) and each \( x_0 \in \Omega \), there exists an \( R > 0 \) with \( B_R(x_0) \subseteq \Omega \), such that \( u_0 \) is a strong minimizer of \( I(\cdot; B_R(x_0)) \) on \( A_R(x_0) \).

**Remark.** It is easy to see that (1) implies

\[
|F(P)| \leq C(1 + |P|^p)
\]

for some \( C > 0 \).

In general, even the minimizers of (1) are only partially regular (see e.g., [3, 4, 5]), i.e., there exists an open subset \( \Omega_0 \) of \( \Omega \) with \( \text{meas}(\Omega \setminus \Omega_0) = 0 \), such that \( u \in C^{1,\alpha}(\Omega_0; \mathbb{R}^N) \), \( 0 < \alpha < 1 \). Therefore, for partially regular solutions of (2), we conclude that \( u_0 \) is locally stable on \( \Omega_0 \), i.e., \( u_0 \) is a minimizer on \( A_{R}(x_0) \) for some \( R > 0 \), \( B_R(x_0) \subseteq \Omega_0 \).

**Proof of the theorem.** We prove the theorem by contradiction. For a fixed \( x_0 \in \Omega \), if the conclusion of the theorem is not true, then there exists a sequence of positive \( R_j \) with \( R_j \rightarrow 0 \) as \( j \rightarrow \infty \), such that \( B_{R_j}(x_0) \subseteq \Omega \) and \( u_0 \) is not a minimizer in \( A_{R_j}(x_0) \). Hence there exists a sequence of functions \( \phi_j \in W_0^{1,p}(B_{R_j}(x_0)) \) such that

\[
\int_{B_{R_j}(x_0)} F(Du_0 + D\phi_j) \, dx < \int_{B_{R_j}(x_0)} F(Du_0) \, dx.
\]

Since \( u_0 \) is a solution of (2), we have, from (4),

\[
0 > \int_{B_{R_j}(x_0)} [F(Du_0 + D\phi_j) - F(Du_0)] \, dx = \int_{B_{R_j}(x_0)} DF(Du_0)D\phi_j \, dx
\]

\[
+ \int_{B_{R_j}(x_0)} \int_0^1 (1 - t)D^2F(Du_0 + tD\phi_j)D\phi_j D\phi_j \, dt \, dx
\]
and notice that, from the divergence theorem,
\[ \int_{B_{R_j}(x_0)} DF(Du_0)D\phi_j \, dx = 0, \]
so that, in (5), we have
\[ 0 < -\int_{B_{R_j}(x_0)} \int_0^1 (1 - t)D^2F(Du_0 + tD\phi_j)D\phi_jD\phi_j \, dt \, dx. \]

On the other hand, from (2),
\[ \nu \int_{B_{R_j}(x_0)} (|D\phi_j|^2 + |D\phi_j|^p) \, dx \]
\[ \leq \int_{B_{R_j}(x_0)} [F(Du_0(x_0)) + D\phi_j) - F(Du_0(x_0))] \, dx \]
\[ = \int_{B_{R_j}(x_0)} DF(Du_0(x_0))D\phi_j \, dx \]
\[ + \int_{B_{R_j}(x_0)} \int_0^1 (1 - t)D^2F(Du_0(x_0)) + tD\phi_j)D\phi_jD\phi_j \, dt \, dx. \]

Still from the divergence theorem, we have
\[ \int_{B_{R_j}(x_0)} DF(Du_0(x_0))D\phi_j \, dx = 0, \]
so that adding (6) to (7) gives
\[ \nu \int_{B_{R_j}(x_0)} (|D\phi_j|^p + |D\phi_j|^2) \, dx \]
\[ \leq \int_{B_{R_j}(x_0)} \int_0^1 (1 - t)[D^2F(Du_0(x_0) + tD\phi_j) \]
\[ - D^2F(Du_0(x) + tD\phi_j))]D\phi_jD\phi_j \, dt \, dx \]
\[ \leq \int_{B_{R_j}(x_0)} C[1 + |Du_0(x)|^{p-2} + |Du_0(x_0)|^{p-2} \]
\[ + |D\phi_j|^{p-2}]|Du_0(x) - Du_0(x_0)|^\alpha|D\phi_j|^2 \, dx \]
\[ \leq C(||Du_0||C) \int_{B_{R_j}(x_0)} |Du_0(x_0) - Du_0(x)|^\alpha(1 + |D\phi_j|^{p-2})|D\phi_j|^2 \, dx \]
\[ \leq C \max_{x \in B_{R_j}(x_0)} \{|Du_0(x_0) - Du_0(x)|^\alpha \} \int_{B_{R_j}(x_0)} (|D\phi_j|^p + |D\phi_j|^2) \, dx. \]

Dividing both sides of (8) by \( \int_{B_{R_j}(x_0)} (|D\phi_j|^p + |D\phi_j|^2) \, dx \) we obtain
\[ \nu < C \max_{x \in B_{R_j}(x_0)} \{|Du_0(x_0) - Du_0(x)|^\alpha \}. \]

Passing to the limit \( j \to \infty \), we have \( R_j \to 0 \) and
\[ \max_{x \in B_{R_j}(x_0)} |\{Du_0(x_0) - Du_0(x)| \to 0 \]
so that
\[ \nu \leq \lim_{j \to \infty} C \max_{x \in B_j(x_0)} \{|Du_0(x_0) - Du_0(x)|^p\} = 0. \]

This contradicts to the assumption \( \nu > 0 \).

**Remark.** This result could be easily extended to the case where \( F \) depends explicitly on \( x \), say, \( F = F(x, P) \). It seems to be unknown whether general weak equilibria of (1) in \( W^{1,p}(\Omega; \mathbb{R}^N) \) are local minima for variations of sufficiently small support near a Lebesgue point of \( Du \).

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**REFERENCES**


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