REGULAR STATES AND COUNTABLE ADDITIVITY ON QUANTUM LOGICS

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Abstract. We give a counterexample of the result of Béaver and Cook concerning a generalization of the Alexandroff theorem for regular, finitely-additive states on quantum logics using states on the system of all splitting subspaces of an incomplete inner-product space. Moreover, we introduce another type of state regularity which entails countable additivity of states on logics.

1. INTRODUCTION

We recall that a quantum logic is a poset $L$ with the minimal and maximal elements 0 and 1, respectively, and with the unary operation (named orthocomplementation) $\perp : L \rightarrow L$ such that

(i) $(a^\perp)^\perp = a$, for any $a \in L$;
(ii) if $a \leq b$, then $b^\perp \leq a^\perp$;
(iii) $a \lor a^\perp = 1$, for any $a \in L$;
(iv) if $a \leq b^\perp$, then $a \lor b \in L$; and
(v) if $a \leq b$, then $b = a \lor (b \land a^\perp)$ (orthomodular property).

($\lor$ and $\land$ denote the operations sup and inf.) We note that, in view of (i)-(iv), if $a \leq b$, then $b \land a^\perp$ exists in $L$. We say that two elements $a$ and $b$ of $L$ are orthogonal, denoted $a \perp b$, if $a \leq b^\perp$. If $L$ has the property that any sequence $\{a_n\}$ of mutually orthogonal elements of $L$ has a supremum, $\lor_{n=1}^{\infty} a_n$, in $L$, then $L$ is called a $\sigma$-quantum logic.

A Boolean algebra is a poset $\mathcal{B}$ containing the minimal and maximal elements 0 and 1, respectively, such that $\mathcal{B}$ is equipped with the operation of complementation $\perp$ satisfying (i) and (ii) of the definition of quantum logic and has the additional properties

(L) $a \lor b \in \mathcal{B}$ for any $a, b \in \mathcal{B}$ (lattice property)

(so that any nonempty finite subset of $\mathcal{B}$ has supremum and infimum)

(D) $(a \lor b) \land c = (a \land c) \lor (b \land c)$ (distributive property).

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It is well known that every Boolean algebra is isomorphic to some algebra of sets.

A nonempty subset $\mathcal{B}$ of a quantum logic $L$ is a Boolean subalgebra of $L$ if (i) $0, 1 \in \mathcal{B}$; (ii) $a \in \mathcal{B}$ implies $a^\perp \in \mathcal{B}$; and (L) and (D) hold in $\mathcal{B}$.

A state (more precisely, a finitely-additive state) on a quantum logic $L$ is a mapping $m: L \to [0, 1]$ such that (i) $m(1) = 1$; (ii) $m(a \lor b) = m(a) + m(b)$ whenever $a \perp b$. A state $m$ is countably additive if $m(\bigvee_{n=1}^{\infty} a_n) = \sum_{n=1}^{\infty} m(a_n)$ whenever \{a_n\}_{n=1}^{\infty} is a sequence of mutually orthogonal elements and $\bigvee_{n=1}^{\infty} a_n \in L$. A state $m$ is completely additive if $m(\bigvee_{t \in T} a_t) = \sum_{t \in T} m(a_t)$ whenever \{a_t : t \in T\} is a system of mutually orthogonal elements of $L$ and $\bigvee_{t \in T} a_t \in L$.

In the present paper, we show that the proof of Béaver and Cook [2] on regular states contains a gap, and we present an example of a regular, finitely-additive state that is not countably additive. On the other hand, we give a new type of state regularity that will entail the countable additivity.

2. Regular states

Let $\mathcal{P}$ be a nonvoid subset of a quantum logic $L$. A state $m$ is called $\mathcal{P}$-regular (more precisely, $\mathcal{P}$-regular in the sense of Béaver and Cook), if for each $\epsilon > 0$ and each $b \in L$ there exists an $a \in \mathcal{P}$ with $a \leq b$ and $m(b \land a^\perp) < \epsilon$.

One of the most important examples of a quantum logic is the system $L(H)$ of all closed subspaces of a real or complex Hilbert space $H$, which is a complete lattice and plays a considerable role in the axiomatic model of quantum mechanics (see, e.g., [15]). The famous theorem of Gleason [8] asserts that any countably-additive state $m$ on $L(H)$, $3 \leq \dim H \leq \aleph_0$, is of the following form

$$m(M) = \text{tr}(TP^M), \quad M \in L(H),$$

where $T$ is a positive operator of the trace class on $H$ and $P^M$ is the orthogonal projector from $H$ onto $M$.

More generally, let $S$ be a real or complex inner-product space with an inner product $(\cdot, \cdot)$. By a subspace of $S$ we shall understand a linear closed subspace of $S$. For any subspace $M$ of $S$, $M^\perp$ denotes the set of all $x \in S$ such that $(x, y) = 0$ for all $y \in M$. We denote by $E(S)$ the set of all subspaces $M$ of $S$ such that $M + M^\perp = S$. Then $L = E(S)$ is a quantum logic which contains any complete and, therefore, any finite-dimensional subspace. Moreover, $E(S)$ is a $\sigma$-quantum logic iff $S$ is complete [6].

Let $\mathcal{P}^\perp = \{a^\perp : a \in \mathcal{P}\}$. An element $b \in \mathcal{P}$ is called finitely coverable if, for any sequence \{a_1^\perp, a_2^\perp, \ldots\} \subseteq $\mathcal{P}^\perp$ such that $\bigvee_{k=1}^{\infty} a_k^\perp$ exists in $L$ and $b \leq \bigvee_{k=1}^{\infty} a_k^\perp$, there is an integer $n$ such that $\bigvee_{k=1}^{n} a_k^\perp$ exists in $L$ and $b \leq \bigvee_{k=1}^{n} a_k^\perp$. $\mathcal{P}$ is called finitely coverable if each element of $\mathcal{P}$ is finitely coverable.

Béaver and Cook [2] presented the following result: Let $L$ be a $\sigma$-quantum logic and $\mathcal{P} \subseteq L$ be finitely coverable such that $\mathcal{P}^\perp$ contains the join of any sequence in $\mathcal{P}^\perp$. Then any $\mathcal{P}$-regular state on $L$ is countably additive.

Unfortunately their proof is incorrect, because they used the subadditivity of a state (i.e., $m(a) \leq \sum_{i=1}^{n} m(a_i)$ if $a \leq \bigvee_{i=1}^{n} a_i$), which is invalid, in general, in quantum logics (consider, for example, a state of the form (2.1)). Also, the assumption that $L$ is a $\sigma$-quantum logic was not used in the proof. If $m$ is subadditive—for example, if $L$ is a Boolean algebra—their proof works.
Below we present an example of a quantum logic \( L \), a finitely-coverable subset \( \mathcal{P} \subset L \) such that \( \mathcal{P}^\perp \) contains the join of any sequence in \( \mathcal{P}^\perp \), and a \( \mathcal{P} \)-regular state \( m \) on \( L \) that is not countably additive. In particular, it shows that \( \mathcal{P} \)-regularity is not a sufficient condition for countable additivity.

**Counterexample 2.1.** Let \( S \) be an inner-product space. For any \( x \in S \), \( \|x\| = 1 \), the mapping \( m_x \) on \( E(S) \) defined via

\[
(2.2) \quad m_x(M) = \|x_M\|^2, \quad M \in E(S),
\]

if \( x = x_M + x_M^\perp \), where \( x_M \in M, x_M^\perp \in M^\perp \), is a state on the quantum logic \( L = E(S) \). The system \( \mathcal{P} = \mathcal{P}(S) \) of all finite-dimensional subspaces is finitely coverable. If \( S \) is a separable, incomplete inner-product space, then any \( m_x \) is a \( \mathcal{P} \)-regular state which is not countably additive.

**Proof.** If \( M \) and \( N \) are mutually orthogonal, splitting subspaces of \( S \), then \( M + N = M \cap N \in E(S) \) (see, e.g., [9, 5]). Hence, it is simple to verify that any \( m_x \) is a state of a quantum logic \( L = E(S) \).

Now we show that \( \mathcal{P} = \mathcal{P}(S) \) is finitely coverable. In fact, any \( M \in E(S) \) is finitely coverable by \( \mathcal{P} \). To see this, let \( M \in E(S) \) and let \( \{M_1^\perp, M_2^\perp, \ldots \} \) be a sequence in \( \mathcal{P} \) such that \( M \subseteq \bigvee_{k=1}^\infty M_k^\perp \) (the latter join belongs always to \( E(S) \)). Each \( M_k^\perp \) is a subspace of finite codimension. If the integer \( n_k, k = 1, 2, \ldots \), is the codimension of \( M_1^\perp \cap \cdots \cap M_k^\perp \in E(S) \), then \( n_1 \geq n_2 \geq \cdots \geq 0 \). Thus, for some \( i \), \( n_i = n_{i+1} = \cdots \) and \( M \leq \bigvee_{k=1}^i M_k^\perp \).

Suppose that \( S \) is separable and incomplete. It is straightforward to show that \( m_x \) is of the form

\[
(2.3) \quad m_x(M) = \|P_M x\|^2, \quad M \in E(S),
\]

where \( P_M \) is the orthoprojector from the completion \( \overline{S} \) of \( S \) onto the completion \( \overline{M} \) of \( M \). The separability of \( S \) entails the existence of an orthonormal basis (ONB) \( \{x_n\} \) in any splitting subspace \( M \) of \( S \). It is simple to show that \( \{x_n\} \) is an ONB in \( \overline{M} \), too. Therefore, \( m_x(M) = \|P_M x\|^2 = \| \sum_{n} P_{x_n} x \|^2 = \sum_{n} \|P_{x_n} x\|^2 \), where \( P_u \) is an orthogonal projection onto one-dimensional subspace spanned by a non-zero vector \( u \in S \).

Given \( \varepsilon > 0 \), we can find a finite-dimensional subspace \( N = \text{sp}(x_1, \ldots, x_n) \subset M \), where \( \text{sp} \) denotes the span over \( x_1, \ldots, x_n \) such that \( m_x(M \cap N^\perp) < \varepsilon \).

Since \( S \) is incomplete and separable, there is a maximal orthonormal set \( \{x_i\}_{i=1}^\infty \) in \( S \) that is not a basis (see, for example [11]). Consequently, there is a \( z \in S \) such that \( 1 = \|z\|^2 \neq \sum_{i=1}^\infty \|z, x_i\|^2 \). Therefore, \( m_z(S) = \|z\|^2 \neq \sum_{i=1}^\infty m_z(P_{x_i}) \), although \( \bigvee_{i=1}^\infty P_{x_i} = S \), and \( m_z \) is not countably additive. Actually, in this case \( E(S) \) does not possess any countably-additive states, as a consequence of the result of [6] saying that \( S \) is complete iff \( E(S) \) has at least one countably-additive state (completely additive for general \( S \)). Therefore, any of the states \( m_x \) is a \( \mathcal{P} \)-regular state but not countably additive. Q.E.D.

On the other hand, we show below that on a very important quantum logic, \( L(H) \) of a Hilbert space \( H \), the assertion of Béaver and Cook is correct, even when a finitely-additive state is not subadditive.
Theorem 2.2. A finitely-additive state $m$ on $L(H)$, $\dim H \geq 3$, is of the form
(2.1) iff $m$ is $\mathcal{P}(S)$-regular (in the sense of Béaver and Cook).

Proof. Suppose that $m$ is of the form (2.1). Then $m$ is completely additive. Let $M$ be an arbitrary element of $L(H)$ and an ONB $\{f_i\}$ in $M$. Due to the complete additivity of $m$, there is a sequence $\{f_n\} \subset \{f_i\}$ such that
$m(M) = \sum_{n=1}^{\infty} m(P_{f_n})$. For $\epsilon > 0$, there exists an integer $k$ sufficiently large for which $N = \text{sp}(f_1, \ldots, f_k)$ has the property $m(M \cap N^\perp) = m(M) - m(N) = \sum_{n=k+1}^{\infty} m(P_{f_n}) < \epsilon$.

Now let $m$ be a $\mathcal{P}(H)$-regular (in the sense of Béaver and Cook), finitely-additive state on $L(H)$. Due to the result by Aarnes [1, Proposition 2, p. 609], any finitely-additive state on $L(H)$ is uniquely decomposed into the sum $m = m_1 + m_2$, where $m_1$ is a completely-additive state and $m_2$ is a finitely-additive state that vanishes on finite-dimensional subspaces of $H$. Due to the Maeda theorem [12] or Aarnes [1], $m_1$ is of the form (2.1) for some positive operator of trace class on $H$. The $\mathcal{P}(H)$-regularity of $m$ and $m_1$ entails $m_2 = 0$. Thus, if $m$ is a $\mathcal{P}(H)$-regular state on $L(H)$, then $m = m_1$ and $m$ is of the form (2.1). Q.E.D.

A system of states, $\mathcal{M}$, of a quantum logic $L$ is full if the following condition is satisfied: If $m(a) \leq m(b)$ for all $m \in \mathcal{M}$, then $a \leq b$.

Example 2.3. There is a lattice $\sigma$-quantum logic $L$ with a full system of countably-additive states on $L$, a subquantum logic $L_0$ of $L$ that is not a lattice, and a finitely-coverable system $\mathcal{P} \subseteq L_0$ such that the restriction of any $m \in \mathcal{M}$ onto $L_0$ is a $\mathcal{P}$-regular state which is not countably additive.

Proof. Let $S$ be an incomplete, separable inner-product space. Put $L = L(\overline{S})$, and for any $M \in E(S)$ define $\varphi(M) = \overline{M} \in L$. Then

(i) if $M \neq N$, then $\varphi(M) \neq \varphi(N)$;
(ii) $\varphi(M \vee N) = \varphi(M) \vee \varphi(N)$ if $M \perp N$; and
(iii) $\varphi(M^\perp) = \overline{\varphi(M)} = \{x \in \overline{S} : (x, y) = 0 \text{ for each } y \in \varphi(M)\}$.

If we put $L_0 = \{\varphi(M) : M \in E(S)\}$, then $L_0$ is a subquantum logic of $L$ that is isomorphic to $E(S)$. The system $\mathcal{M} = \{w_x : x \in S, \|x\| = 1\}$, where $w_x$ is a mapping on $L$ defined in a manner analogous to $m_x$ in (2.2), is a full system of countably-additive states on $L$. Indeed, let $w_x(M) \leq w_x(N)$, $x \in S$, $\|x\| = 1$, then we have $w_x(M) = (P^M x, x)$, $M \in L(\overline{S})$, where $P^M$ is the orthoprojector from $\overline{S}$ onto $\overline{M}$. Therefore, for all vectors $x$ from $S$ we have $(P^M x, x) \leq (P^N x, x)$, so that $(P^M x, x) \leq (P^N x, x)$ for all $x \in \overline{S}$; i.e., $M \subseteq N$.

A finitely-coverable system $\mathcal{P}$ is defined as $\mathcal{P} = \{\varphi(M) : M \in E(S), \dim M < \infty\}$. Following the lines of Counterexample 2.1 and noting that $w_x(\varphi(M)) = m_x(M)$ for each $M \in E(S)$, we see that $w_x|L_0$ is $\mathcal{P}$-regular but is not countable additive. Q.E.D.

3. ALEXANDROFF'S THEOREM ON QUANTUM LOGICS

Now we introduce another type of state regularity that will imply countable additivity. Let $\mathcal{P}$ be a nonvoid subset of a quantum logic $L$. We say that a state $m$ is $\mathcal{P}$-regular if, for every sequence $\{q_n\}_{n=1}^{\infty}$ of mutually orthogonal
elements of $L$ such that $q = \bigvee_{n=1}^{\infty} q_n$ exists in $L$, there is a block $\mathcal{B} \subseteq L$ (i.e., a maximal Boolean subalgebra of $L$) such that for each $\varepsilon > 0$ and every $r \in \{q, q_{1}^{\perp}, q_{2}^{\perp}, \ldots\}$, there exists a $p \in \mathcal{B} \cap \mathcal{P}$ with $p \leq r$ and $m(r \wedge p^{\perp}) < \varepsilon$.

It is evident that any $\mathcal{P}$-regular state is a $\mathcal{P}$-regular state in the sense of Béaver and Cook. The converse assertion is not true, in general, as we shall see below.

If $L$ is a Boolean algebra, then both notions coincide.

**Theorem 3.1.** Let $L$ be a quantum logic, $\mathcal{P} \subseteq L$ be finitely coverable, and $\mathcal{P}^{\perp}$ contain the join of any sequence in $\mathcal{P}^{\perp}$. Then any $\mathcal{P}$-regular state $m$ on $L$ is countably additive.

**Proof.** The proof is similar to that in [2]. Let $\{q_n\}_{n=1}^{\infty}$ be an orthogonal sequence in $L$ with $q = \bigvee_{n=1}^{\infty} q_n$ in $L$. Then $q \geq \bigvee_{i=1}^{n} q_i$ for all $n \geq 1$ and $m(q) \geq \sum_{i=1}^{n} m(q_i)$, so that $m(q) \geq \sum_{i=1}^{\infty} m(q_i)$. For $\{q_n\}_{n=1}^{\infty}$, there is a block $\mathcal{B}$ of $L$ such that, for any $\varepsilon > 0$, there is a sequence $\{p_1, p_2, \ldots\} \subseteq \mathcal{B} \cap \mathcal{P}$ with $p_n \leq q_n^{\perp}$ and $m(q_n^{\perp} \wedge p_n^{\perp}) < \varepsilon/2^n$. Also, there exists a $p \in \mathcal{B} \cap \mathcal{P}$ with $p \leq q$ and $m(q \wedge p^{\perp}) < \varepsilon$. Thus, $p^{\perp} \geq q^{\perp} = \bigwedge_{i=1}^{\infty} q_i^{\perp} \geq \bigwedge_{i=1}^{\infty} p_i$, so $p \leq \bigvee_{i=1}^{k} p_i^{\perp}$. Since $\mathcal{P}$ is finitely coverable, there exists an integer $k$ such that $p \leq \bigvee_{i=1}^{k} p_i^{\perp}$. Also, we have $m(p_n^{\perp}) = m(q_n) + m(q_n^{\perp} \wedge p_n^{\perp})$, and this implies $m(p_n^{\perp}) - \varepsilon/2^n \leq m(q_n)$. Similarly, $m(p) + \varepsilon \geq m(q)$. Using the subadditivity of $m$ in Boolean subalgebras, we conclude that

$$\sum_{n=1}^{\infty} m(q_n) \geq \sum_{n=1}^{\infty} m(p_n^{\perp}) - \varepsilon \geq \sum_{n=1}^{k} m(p_n^{\perp}) - \varepsilon \geq m(p) - \varepsilon \geq m(q) - 2\varepsilon.$$  

Therefore, $m(q) = \sum_{n=1}^{\infty} m(q_n)$. Q.E.D.

Let $L = L(H)$ be the quantum logic of a Hilbert space $H$, and $\mathcal{P} = \mathcal{P}(H)$ be the system of all finite-dimensional subspaces of $H$. Then any countably-additive state $m$ on $L$ of the form $m(M) = \text{tr}(TP^M)$, $M \in L(H)$, where $T$ is a positive operator of trace equal to 1, is $\mathcal{P}$-regular, and $m$, $\mathcal{P}$, $L$ satisfy the conditions of Theorem 3.1. Moreover, in view of Theorem 2.2, the $\mathcal{P}(H)$-regularity and the $\mathcal{P}(H)$-regularity (in the sense of Béaver and Cook) coincide on $L(H)$ if $\dim H \geq 3$. On the other hand, we recall that any countably additive state on $L(H)$ is of the form (2.1) iff the dimension of $H$ is a nonmeasurable cardinal $\neq 2$ ([4, 7]).

**Example 3.2.** Let $S$ be an incomplete, separable inner-product space. For any unit vector $x \in \overline{S}$ we define a mapping $m_x: E(S) \to [0, 1]$ via $m_x(M) = \|P^M x\|^2$, $M \in E(S)$, and let $\mathcal{P}$ be the system of all finite-dimensional subspaces of $S$. Then $m_x$ is $\mathcal{P}$-regular in the sense of Béaver and Cook and not $\mathcal{P}$-regular. This follows from the result in [6] saying that $S$ is complete iff $E(S)$ possesses at least one countably additive state.

**Corollary 3.3.** Let $S$ be of a countable orthogonal dimension (i.e., the cardinality of any maximal orthonormal system in $S$ is countable). $S$ is complete iff $E(S)$ possesses at least one $\mathcal{P}$-regular state, where $\mathcal{P}$ is the system of all finite-dimensional subspaces of $S$.

**Proof.** This follows from Theorem 3.1 and the result in [6]. Q.E.D.
We note that according to [3, pp. 21, 38], the range of the observable corresponding to the momentum operator is a block in \( L = L(H) \) that does not contain nonzero finite-dimensional subspaces. Therefore, not every block in \( L(H) \) may be used for an approximation of a \( P(H) \)-regular state.

4. Regularity on \( \sigma \)-classes

Now we exhibit the problem of the countable additivity of regular states on a special type of \( \sigma \)-quantum logics that are called \( \sigma \)-classes, and we present two results in this direction.

Let \( X \) be a nonempty set. A \( \sigma \)-class \( L \) of subsets of \( X \) is a collection of subsets of \( X \) that satisfy the following:

(i) \( X \in L \);
(ii) if \( E \in L \), then \( E^c : X - E \in L \);
(iii) if \( A_i \in L, i \geq 1 \), are mutually disjoint, then \( \bigcup_{i=1}^{\infty} A_i \in L \).

The set \( L \) may be regarded as a partially ordered set, where the partial ordering is defined by the set-theoretical inclusion, and \( A^\perp = A^c \). It is easy to check that \( L \) is a \( \sigma \)-quantum logic, where sup and inf, \( F \lor G \) and \( F \land G \), respectively, are defined in the usual way relative to \( L \). Note, however, that \( F \lor G \) (\( F \land G \)) need not equal \( F \cup G \) (\( F \cap G \)) even if the former exist in \( L \); they are equal if the latter are in \( L \).

In the following two examples, we show that the subadditivity does not hold, in general, relative neither to \( \cup \) nor to \( \lor \).

Example 4.1 [10, p. 71]. Let \( X = [0,6] \) and \( L = \{ \emptyset, X, A, B, C, A^c, B^c, C^c \} \), where \( A = [0,4], B = [2,5], C = [0,1] \cup [2,3] \cup [5,6] \). Now we define the state \( m \) on \( L \) as follows:

\[
\begin{align*}
m(\emptyset) &= 0, & m(A) = m(B) = m(C) &= 1/4, \\
m(A^c) = m(B^c) = m(C^c) &= 3/4, & m(X) &= 1.
\end{align*}
\]

Then \( 1 = m(X) = m(A \cup B \cup C) > 3/4 = m(A) + m(B) + m(C) \), so that \( m \) is not subadditive.

Example 4.2. Let \( X = \{ 1, 2, 3, 4 \}, L = \{ \emptyset, X, \{ 1, 2 \}, \{ 3, 4 \}, \{ 1, 3 \}, \{ 2, 4 \} \}, m(\emptyset) = 0, m(X) = 1, m(\{ 1, 2 \}) = m(\{ 3, 4 \}) = 1/2, m(\{ 1, 3 \}) = 1/3, m(\{ 2, 4 \}) = 2/3. \) Then

\[
1 = m(X) = m(\{ 1, 2 \} \lor \{ 1, 3 \}) > m(\{ 1, 2 \}) + m(\{ 1, 3 \}) = 5/6.
\]

On the other hand, any \( m \) satisfies the condition of subadditivity (with respect to the union \( \lor \)) for two sets \( A, B \in L \) if \( A \lor B \in L \) [10, p. 71].

Now we show that the proof of the result of Béaver and Cook works in the case of \( \sigma \)-classes in the "almost" original formulation. Namely, the following is true:

Theorem 4.3. Let \( L \) be a \( \sigma \)-class of subsets of a set \( X \neq \emptyset \). Let \( \mathcal{P} \subseteq L \) be finitely coverable (with respect to \( \lor \)) and contain the union of any sequence in it. Then any \( \mathcal{P} \)-regular state \( m \) (in the sense of Béaver and Cook) on \( L \) is countably additive.

Proof. We show that \( m \) is \( \mathcal{P} \)-regular. Since \( \mathcal{P} \) is closed with respect to any union of elements from \( \mathcal{P} \), we conclude by [13] that there is a Boolean subalgebra \( \mathcal{B} \subseteq L \) containing \( \mathcal{P} \). Without loss of generality, we may assume
that $\mathcal{H}$ is a block of question which is necessary for the validity of Theorem 3.1. Q.E.D.

A different approach to that of Alexandroff for a criterion of $\sigma$-additivity of a set function defined on a $\sigma$-algebra of subsets of a set $X$ is that of E. Marczewski [14]. In this case, no topology on $X$ is supposed. We show that such an approach may be applied to $\sigma$-classes.

A collection $\mathcal{H}$ of subsets of a set $X \neq \emptyset$ is said to be compact [14] if for any sequence $\{K_n\}_{n=1}^\infty$ of elements of $\mathcal{H}$ we have $K_1 \cap K_2 \cap \cdots \cap K_n \neq \emptyset$ for all $n \geq 1$, imply $\bigcap_{n=1}^\infty K_n \neq \emptyset$. Let $L$ be a $\sigma$-class of subsets of a set $X$ and $m$ be a state on $L$. We say that $m$ is compact (with respect to $\mathcal{H}$), provided that for any $E \in L$ and any $\varepsilon > 0$ there exist a $K \in \mathcal{H}$ and an $F \in L$, such that $E \supseteq K \supseteq F$ and $m(E \cap F^c) < \varepsilon$.

Denote $\bar{L} = \{E \cap F^c : E \in L, F \in L, E \supseteq F \text{ and there is a } K \in \mathcal{H} \text{ such that } E \supseteq K \supseteq F\}.$

**Theorem 4.4.** Let $L$ be a $\sigma$-class of subsets of a set $X$. Let $\mathcal{H} \subseteq 2^X$ be compact, $m$ a compact state on $L$, and $L$ contain every finite union of elements from $\bar{L}$. Then $m$ is countably additive.

**Proof.** It suffices to show that $\lim_n m(E_n) = 0$ for each decreasing sequence $\{E_n\}$ in $L$ such that $\bigcap_{n=1}^\infty E_n = \emptyset$. Let $\varepsilon > 0$, and for any $n \geq 1$, choose a $K_n \in \mathcal{H}$ such that $E_n \supseteq K_n \supseteq F_n$, where $F_n \in L$ and $m(E_n \cap F_n^c) < \varepsilon/2^n$. Evidently $\emptyset = \bigcap_{n=1}^\infty E_n \supseteq \bigcap_{n=1}^\infty K_n \supseteq \bigcap_{n=1}^\infty F_n$. Therefore, there is an integer $n_0$ such that, for $n > n_0$, we have $\bigcap_{i=1}^n K_i = \emptyset$; hence $\bigcap_{i=1}^n F_i = \emptyset$.

As in the proof of Theorem 4.3, we conclude that $\bar{L}$ is contained in a Boolean subalgebra of $L$, so that $m$ is subadditive on it. Therefore, for any $n > n_0$, we have

$$m(E_n) = m \left( \bigcap_{i=1}^n E_i \right) = m \left( \bigcap_{i=1}^n E_i - \bigcap_{i=1}^n F_i \right) \leq m \left( \bigcup_{i=1}^n (E_i \cap F_i^c) \right) < \varepsilon.$$ 

Thus, $\lim_n m(E_n) = 0$, which entails the $\sigma$-additivity of $M$ on $L$. Q.E.D.

**Remark.** Observe that in Theorem 4.4, $\mathcal{H}$ need not be contained in $L$.

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**References**