ON THE MEAN CURVATURE ESTIMATES
FOR BOUNDED SUBMANIFOLDS

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Abstract. A Liouville-type theorem is proved for strongly subharmonic functions on complete riemannian manifolds of bounded curvature. We use this to give a simple proof of a theorem of Jorge, Koutroufiotis and Xavier, which gives an estimate for the exterior size of a submanifold in terms of the sup of the length of its mean curvature.

We give a short proof of the following theorem of Jorge and Xavier [3].

Theorem 1. Let $M$ and $\bar{M}$ be riemannian manifolds and let $f : M \to \bar{M}$ be an isometric immersion. Suppose that $M$ is complete with inf scalar curvature $> -\infty$. Let $S := \sup$ of the sectional curvature of $\bar{M}$, $H := \sup$ of the length of mean curvature vector of $f$, and $\lambda$ be such that there exists some closed normal ball $B_{\lambda^{-}}$ of radius $\lambda$ in $\bar{M}$ containing $f(M)$. Then,

$$
\lambda \geq \begin{cases} 
\frac{1}{2} \tan^{-1}(\sqrt{\delta}/H) & \text{if } \delta < 0, \\
1/H & \text{if } \delta = 0, \\
\min\{\frac{1}{2}\tan^{-1}(\sqrt{\delta}/H), \pi/(2\sqrt{\delta})\} & \text{if } \delta > 0.
\end{cases}
$$

Theorem 1 follows from

Theorem 2. Let $M$ be a complete riemannian manifold with bounded sectional curvature and $\theta$ a positive constant. Then, every $C^2$ solution to the inequality $\Delta u \geq \theta$ is unbounded.

Proof. First, we claim that given any $r > 0$, there exists $\alpha > 0$ such that for any $p \in M$, we can construct a $C^2$ function $\tilde{v}_{p,r} := M \to \mathbb{R}_+$ such that $\tilde{v}_{p,r}(p) = 1$, $\tilde{v}_{p,r}$ vanishes outside $B_r(p)$, and $|\text{Hess}_x \tilde{v}_{p,r}| \leq \alpha$ for all $x \in M$. That such a $\tilde{v}_{p,r}$ exists independent of the injectivity radius of $M$ follows by smoothing a bump function by convolution in the tangent space using the techniques of Theorem 1.8 in [1]. Set $v_{p,r} := \theta \tilde{v}_{p,r}/(\alpha\sqrt{d})$, where $d := \dim M$. Then, $|\text{Hess} \ v_{p,r}| \leq \theta/\sqrt{d}$.

Now, assume that $u$ is bounded. Let $n := \sup u$. Then for $r > 0$ fixed, there exists a point $q \in M$ such that $n - u(q) < \theta/(\alpha\sqrt{d})$ where $\alpha$ is as in the

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construction above. Then,

\[ \Delta(u + v_{q,r}) \geq \Delta u - |\Delta v_{q,r}| \geq \Delta u - \frac{|\text{Hess } v_{q,r}|}{\sqrt{d}} \geq \theta - \theta = 0, \]

so the function \( u + v_{q,r} \) is subharmonic on \( M \). On the other hand, \( u(q) + v_{q,r}(q) > n - \theta/(\alpha \sqrt{d}) + \theta \delta_{q,r}(q)/(\alpha \sqrt{d}) = n \) while \( u + v_{q,r} \) coincides with \( u \) outside \( B_r(q) \). Therefore, \( u + v_{q,r} \) must attain some maximum point in the set \( B_r(q) \), contradicting the maximum principle.

**Proof of Theorem 1.** By the assumption on the scalar curvature and the Gauss equation, the sectional curvature of \( M \) is bounded (cf. [2, p. 722]). Assume that there exists a point \( o \in M \) such that \( f(M) \subset B_1(o) \). If \( \delta > 0 \), assume also that \( \lambda < \pi/(2\sqrt{\delta}) \). Then, it is well known (cf. [2, 3]) that the function \( \varphi : M \to \mathbb{R}^+ \) defined by \( \varphi(x) := \text{dist}_M(x, o)^2 \) satisfies \( \Delta \varphi(x) \geq d(\varepsilon - H\lambda) \) where

\[
\varepsilon := \begin{cases} 
\lambda \sqrt{-\delta} \coth(\lambda \sqrt{-\delta}) & \text{if } \delta < 0, \\
1 & \text{if } \delta = 0, \\
\lambda \sqrt{\delta} \cot(\lambda \sqrt{\delta}) & \text{if } \delta > 0.
\end{cases}
\]

Hence substituting, if \( \lambda \) is less than the prescribed constant, \( \Delta \varphi \geq \theta := \sqrt{d} \varepsilon \). By Theorem 1, \( \varphi \) is unbounded, which is absurd.

**References**


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