SOME BANACH ALGEBRAS
WITHOUT DISCONTINUOUS DERIVATIONS

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Abstract. It is shown that the completion of $A(G)$ in either the multiplier norm or the completely bounded multiplier norm is a Banach algebra without discontinuous derivations when $G$ is either $F_2$ or $SL(2, \mathbb{R})$.

1. Introduction

In [6], it was shown that a locally compact group $G$ is amenable if and only if every derivation $D$ from $A(G)$, the Fourier algebra of $G$, into an arbitrary Banach $A(G)$-bimodule $X$ is continuous. The same result can be shown to hold for the Herz algebras $A_p(G), 1 < p < \infty$ [7]. Since the class of amenable groups is substantial (it includes all compact groups and all commutative groups) a large number of Banach algebras without discontinuous derivations have been identified. However, since $SL(2, \mathbb{R})$ and $F_2$, the free group on two generators, are nonamenable, $A(SL(2, \mathbb{R}))$ and $A(F_2)$ have discontinuous derivations. We will show that both $A(SL(2, \mathbb{R}))$ and $A(F_2)$ can be given natural norms in such a way that their completions will be Banach algebras without discontinuous derivations.

2. Preliminaries and notation

Let $G$ be a locally compact group. Let $A(G)$ be the Fourier algebra of $G$ as defined by P. Eymard in [4]. $A(G)$ is a Banach algebra with respect to pointwise multiplication. It is also the predual of the von Neumann algebra $VN(G)$ associated with the left-regular representation of $G$ on $L^2(G)$ and a closed ideal in $B(G)$, the Fourier Stieltjes algebra of $G$.

Let $MA(G)$ denote the space of multipliers of $A(G)$. That is, the continuous functions $\psi$ on $G$ such that $\psi u \in A(G)$ for every $u \in A(G)$. For each $\psi \in MA(G), u \in A(G)$, let $m_\psi(u) = \psi u$. Denote by $\|\psi\|_m$ the operator norm of $m_\psi$. We call $\psi$ a completely bounded multiplier of $A(G)$ if $m_\psi^*$, the adjoint of $m_\psi$, is a completely bounded map on $VN(G)$ [3]. Let $\|\psi\|_{M_0}$ be the

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completely bounded norm of $m^*$ and let $M_0A(G)$ denote the Banach algebra of completely bounded multipliers of $A(G)$. Then $B(G) \subset M_0A(G) \subset MA(G)$ and $\|u\|_M \leq \|u\|_{M_0} \leq \|u\|_{B(G)}$ for every $u \in B(G)$.

We will denote by $A_M(G)$ and $A_{M_0}(G)$ the closure of $A(G)$ in $MA(G)$ and in $M_0A(G)$ respectively. It is well known that $G$ is amenable if and only if $MA(G) = B(G)$ [12, 13]. In this case $\|u\|_{B(G)} = \|u\|_M$ for every $u \in B(G)$ and hence $A(G) = A_M(G) = A_{M_0}(G)$.

If $\mathscr{A}$ is a Banach algebra, then a derivation $D$ on $\mathscr{A}$ is a linear map from $\mathscr{A}$ into a Banach $\mathscr{A}$-bimodule $X$ such that $D(uv) = u \cdot D(v) + D(u) \cdot v$ for every $u, v \in \mathscr{A}$. If $\mathscr{A}$ is commutative, we will denote the maximal ideal space of $\mathscr{A}$ by $\Delta(\mathscr{A})$. We also use $\Delta(\mathscr{A})$ to denote the multiplicative linear functionals which correspond to the maximal ideal. Given a closed subset $A$ of $\Delta(\mathscr{A})$, we denote by $I_{\mathscr{A}}(A)$, the ideal $\{u \in \mathscr{A}; u(x) = 0 \text{ for every } x \in A\}$, where $\mathscr{A}$ is realized as an algebra of functions on $\Delta(\mathscr{A})$ by means of the Gelfand transform. $A$ is called a set of spectral synthesis for $\mathscr{A}$ or simply an $S$-set if $I_{\mathscr{A}}^0(A) = \{u \in \mathscr{A}; \text{ supp } u \text{ is compact, supp } u \cap A = \emptyset\}$ is dense in $I_{\mathscr{A}}(A)$.

3. Spectral synthesis and automatic continuity of derivations

Lemma 1. Let $G$ be a locally compact group. Then

$$\Delta(A_M(G)) = \Delta(A_{M_0}(G)) = G.$$  

Proof. Let $\psi \in \Delta(A_M(G))$. Then $\psi|_{A(G)}$, the restriction of $\psi$ to $A(G)$, belongs to $\Delta(A(G))$. By [4, p. 222] there exists an $x \in G$ such that $\psi(u) = u(x)$ for every $u \in A(G)$.

Let $v \in A_M(G)$. As $A(G)$ is dense in $A_M(G)$, we can find $\{u_k\} \subset A(G)$ such that $\|u_k - v\|_M \to 0$. However, convergence in $A_M(G)$ implies convergence in the topology of uniform convergence on compacta and hence in the topology of pointwise convergence. Therefore $\psi(v) = \lim_{k} \psi(u_k) = \lim_{k} u_k(x) = v(x)$. It follows that as a set $\Delta(A_M(G)) = G$.

Now suppose that $\{x_\alpha\}_{\alpha \in I}$ is a net in $G$ such that $x_\alpha$ converges to $x \in G$ in the usual topology on $G$. Then for each $u \in A_M(G)$, $u(x_\alpha) \to u(x)$. Hence $x_\alpha \to x$ in the $\sigma(A_M(G)^*, A_M(G))$ topology. Conversely, if $x_\alpha \to x$ in the $\sigma(A_M(G)^*, A_M(G))$ topology, then for every $u \in A(G)$, $u(x_\alpha) \to u(x)$. Since $G \subseteq A(G)^*$, $x_\alpha \to x$ in the $\sigma(A_M(G)^*, A_M(G))$ topology. But $\Delta(A(G))$ is homeomorphic to $G$ so $x_\alpha \to x$ in $G$. Therefore the $\sigma(A_M(G)^*, A_M(G))$ topology agrees with the usual topology on $G$.

A similar argument establishes that $\Delta(A_{M_0}(G)) = G$. □

Proposition 1. Let $\mathscr{A}$ be either $A_M(G)$ or $A_{M_0}(G)$. Then $\{e\}$ is an $S$-set of $\mathscr{A}$.

Proof. Let $v \in \mathscr{A}$ with $v \in I_{\mathscr{A}}(\{e\})$. Then there exists $u_k \in A(G)$ with $\|u_k - v\|_{\mathscr{A}} \to 0$. Furthermore since $u_k(e) \to v(e) = 0$, we can assume that $|u_k(e)| < 1/2k$. We can also find $w_k \in A(G)$ with $\|w_k\|_{A(G)} = w_k(e) = u_k(e)$. Let $v_k = u_k - w_k$. Then $v_k(e) = 0$. Since $v_k \in I_{A(G)}(\{e\})$ and $\{e\}$ is an $S$-set for $A(G)$ [4, p. 229], there exists $z_k \in A(G)$ with $\|z_k - v_k\|_{A(G)} \leq 1/2k$.
supp \( z_k \) is compact and \( \text{supp } z_k \cap \{ e \} = \emptyset \). However

\[
\| z_k - v \|_{\mathcal{A}} \leq \| z_k - v \|_{\mathcal{A}} + \| v \|_{\mathcal{A}} \\
\leq \| z_k \|_{A(G)} + \| u_k - v \|_{\mathcal{A}} + \| w_k \|_{A(G)} \\
\leq \| u_k - v \|_{\mathcal{A}} + 1/k.
\]

Hence \( \| z_k - v \|_{\mathcal{A}} \to 0 \). Therefore \( I^0_{\mathcal{A}}(\{ e \}) \) is dense in \( I_{\mathcal{A}}(\{ e \}) \) and \( \{ e \} \) is an \( S \)-set. □

**Proposition 2.** Let \( \mathcal{A} \) be either \( AM(G) \) or \( AM_0(G) \). If \( \mathcal{A} \) has a bounded approximate identity, then so does \( I_{\mathcal{A}}(\{ e \}) \).

**Proof.** Let \( \{ u_\alpha \}_{\alpha \in I} \) be a bounded approximate identity in \( \mathcal{A} \). Let \( \mathcal{F}(\{ e \}) = \{ K \subset G ; K \text{ is compact, } K \cap \{ e \} = \emptyset \} \). For every \( K \in \mathcal{F}(\{ e \}) \), there exists \( v_K \in B(G) \) such that \( \| v_K \|_{\mathcal{A}} \leq \| v_K \|_{B(G)} = 1 \), \( u_K(e) = 1 \), and \( v_K(x) = 0 \) for every \( x \in K \). Define \( w_{K, \alpha} \in \mathcal{A} \) by \( w_{K, \alpha} = u_\alpha - v_K u_\alpha \). Then \( w_{K, \alpha}(e) = 0 \), \( \| w_{K, \alpha} \|_{\mathcal{A}} \leq 2 \| u_\alpha \|_{\mathcal{A}} \), and \( w_{K, \alpha}v = u_\alpha v \) for every \( v \in \mathcal{A} \) with \( \text{supp } v \subseteq K \).

Order \( K \times I \) by \( (K_1, \alpha_1) \leq (K_2, \alpha_2) \) if and only if \( K_1 \subseteq K_2 \) and \( \alpha_1 \leq \alpha_2 \). If \( v \in \mathcal{A} \) and \( \text{supp } v \subseteq \mathcal{F}(\{ e \}) \), then \( v = \lim_{K, \alpha} w_{K, \alpha} v \). By Proposition 1, such \( v \)'s are dense in \( I_{\mathcal{A}}(\{ e \}) \). As \( \{ w_{K, \alpha} \}_{K \times I} \) is bounded, \( \lim_{K, \alpha} w_{K, \alpha} w = w \) for every \( w \in I_{\mathcal{A}}(\{ e \}) \). □

The proof of Proposition 2 is a modification of the proof of [7, Proposition 3.2]. The case where \( G \) is amenable is due to A. Lau [11, Corollary 4.11].

**Corollary 1.** Let \( \mathcal{A} \) be either \( AM(G) \) or \( AM_0(G) \). If \( \mathcal{A} \) has a bounded approximate identity, then \( \mathcal{A} \) satisfies Ditkin's condition. Furthermore, if \( A \) has a closed subset of \( G \) and the boundary of \( A \) contains no nontrivial perfect set, then \( A \) is an \( S \)-set. In particular, every finite subset of \( G \) is an \( S \)-set for \( \mathcal{A} \).

**Proof.** Let \( x \in G \) and \( u \in \mathcal{A} \) be such that \( u(x) = 0 \). From the proof of Proposition 2 (translate is \( x \neq e \)), we see there exists a sequence \( \{ v_n \} \subset \mathcal{A} \) for which each \( v_n \) vanishes in a neighborhood \( V_n \) of \( \{ x \} \) and \( \lim_n \| v_n u - u \|_{\mathcal{A}} = 0 \).

Assume that \( G \) is not compact. Let \( u \in \mathcal{A} \), \( u \neq 0 \). Let \( \epsilon > 0 \). If \( \{ u_\alpha \}_{\alpha \in I} \) is a bounded approximate identity in \( \mathcal{A} \), then there exists \( u_\alpha \), such that \( \| u_\alpha u - u \|_{\mathcal{A}} < \epsilon/2 \). We can find \( v \in A(G) \) with \( \text{supp } v \) compact such that \( \| u_\alpha u - v \|_{\mathcal{A}} \leq \epsilon/2 \| v \|_{\mathcal{A}} \). Then \( \| vu - u \|_{\mathcal{A}} \leq \| vu - u_\alpha u \|_{\mathcal{A}} + \| u_\alpha u - u \|_{\mathcal{A}} \leq \epsilon \).

Hence \( \mathcal{A} \) satisfies Ditkin's condition [9, p. 49]. The remaining statements follow immediately from Ditkin's theorem [9, p. 497]. □

Corollary 1 is simply [7, Lemma 5.2, Proposition 5.3] when \( G \) is assumed to be amenable.

**Proposition 3.** Let \( \mathcal{A} \) be either \( AM(G) \) or \( AM_0(G) \). If \( \mathcal{A} \) has a bounded approximate identity and \( I \) is a closed cofinite ideal in \( \mathcal{A} \) (i.e., \( \dim \mathcal{A} / I < \infty \)), then \( I = I(A) \) for some finite set \( A = \{ x_1, \ldots, x_n \} \) where \( n \) is the codimension of \( I \).

**Proof.** Let \( A = Z(I) = \{ x \in G ; u(x) = 0 \text{ for every } u \in I \} \). Let \( n = \text{codim } I \).

Assume that \( A \) contains \( n + 1 \) distinct elements \( \{ x_1, \ldots, x_{n+1} \} \). We can find a compact neighborhood \( V_k \) of each \( x_k \) such that \( V_j \cap V_k = \emptyset \) if \( j \neq k \). We can also find \( u_k \in A(G) \) such that \( \text{supp } u_k \subseteq V_k \) and \( u_k(x_j) = 1 \) for \( 1 \leq k \leq n + 1 \). But if \( \psi : \mathcal{A} \to \mathcal{A} / I \) is the canonical homomorphism, then \( \{ \psi(u_1), \ldots, \psi(u_{n+1}) \} \) is a linearly independent subset, which is impossible if
codim \( I = n \). Therefore \( A \) has at most \( n \) elements. Since \( A \) is finite, it is an \( S \)-set by Corollary 1. Therefore \( I = I(A) \).

Let \( A = \{x_1, \ldots, x_k\} \). Let \( u_k \in A(G) \) be such that \( u_k(x_k) = 1 \) and \( u_k(x_j) = 0 \) if \( j \neq k \). Let \( u \in \mathcal{A} \). Then

\[
u = \sum_{i=1}^{k} u(x_i)u_i + \left(u - \sum_{i=1}^{k} u(x_i)u_i\right).
\]

Since

\[
u - \sum_{i=1}^{k} u(x_i)u_i \in I(A), \quad k \geq n.
\]

Hence \( k = n \). \( \square \)

**Proposition 4.** Let \( \mathcal{A} \) be either \( A_M(G) \) or \( A_{M_0}(G) \). Assume that \( \mathcal{A} \) has a bounded approximate identity. Let \( I \) be a closed cofinite ideal of \( \mathcal{A} \). Then \( I \) has a bounded approximate identity. In particular \( I^2 = \{\sum_{i=1}^{n} u_i v_i, \ u_i, v_i \in I\} = I \).

**Proof.** By Proposition 3, \( I = I(\{x_1, \ldots, x_n\}) \) for some finite subset of \( G \).

Let \( A, B \) be two closed subsets of \( G \). Assume that \( I_{\mathcal{A}}(A) \) and \( I_{\mathcal{A}}(B) \) have bounded approximate identities \( \{u_i\}_{i \in I}, \{v_j\}_{j \in J} \) respectively. Then it is easy to see that \( \{u_i v_j\}_{i \in I, j \in J} \) is a bounded approximate identity for \( I(A \cup B) \).

Since \( \mathcal{A} \) has a bounded approximate identity, Proposition 2 implies that \( I_{\mathcal{A}}(\{e\}) \) has a bounded approximate identity. By translating, we see that \( I_{\mathcal{A}}(\{x\}) \) has a bounded approximate identity for every \( x \in G \). A simple induction argument shows that \( I \) also has a bounded approximate identity. Then \( I^2 = I \) follows from Cohen's factorization theorem [9, p. 268]. \( \square \)

**Theorem 1.** Let \( \mathcal{A} \) be either \( A_M(G) \) or \( A_{M_0}(G) \). If \( \mathcal{A} \) has a bounded approximate identity, then \( I \) is a cofinite ideal of \( \mathcal{A} \) if and only if \( I = I(A) \) for some finite subset \( A \) of \( G \).

**Proof.** Proposition 4 shows that every closed cofinite ideal of \( \mathcal{A} \) is idempotent. By [2, Theorem 2.3] every cofinite ideal of \( \mathcal{A} \) must be closed and is therefore of the form \( I(A) \) for some finite subset \( A \) of \( G \) by Proposition 3.

If \( I = I(A) \) for a finite subset \( A \) of \( G \), then the proof of Proposition 3 shows that \( I \) is cofinite. \( \square \)

If \( G \) is amenable, then Proposition 3 and Theorem 1 follow from [7, Corollary 5.6, Theorem 5.8].

**Theorem 2.** Let \( \mathcal{A} \) be either \( A_M(G) \) or \( A_{M_0}(G) \). If \( \mathcal{A} \) has a bounded approximate identity, then every homomorphism from \( \mathcal{A} \) with finite-dimensional range is continuous.

**Proof.** The statement follows immediately from Theorem 1, Proposition 4, and [2, Theorem 2.3]. \( \square \)

**Lemma 2.** Let \( \mathcal{A} \) be either \( A_M(G) \) or \( A_{M_0}(G) \). Assume that \( \mathcal{A} \) has a bounded approximate identity. Let \( I \) be a closed ideal in \( \mathcal{A} \) with infinite codimension. Then there exist sequences \( \{u_n\}, \{v_n\} \) in \( A(G) \) such that \( u_n v_1 \cdots v_{n-1} \notin I \) but \( u_n v_1 \cdots v_n \in I \) for \( n \geq 2 \).

**Proof.** If \( I \) has infinite codimension, then \( A = Z(I) = \{x \in G, u(x) = 0 \ \forall u \in I\} \) must be infinite by Theorem 1.
From the proof of [6, Lemma 2], we see that we can find sequences \( \{u_n\} \), \( \{v_n\} \subseteq A(G) \) such that \( u_n v_1 \cdots v_{n-1} \notin I_{A(G)}(A) \) while \( u_n v_1 \cdots v_n = 0 \). Since \( I_{A(G)}(A) \subseteq I_A(A) \), the result follows. \( \square \)

**Theorem 3.** Let \( \mathcal{A} \) be either \( A_M(G) \) or \( A_{M_0}(G) \). Assume that \( \mathcal{A} \) has a bounded approximate identity. Then every derivation \( D \) of \( \mathcal{A} \) into a Banach \( \mathcal{A} \)-bimodule \( X \) is continuous.

**Proof.** This follows immediately from Proposition 4, Lemma 2, and [10, Theorem 3]. \( \square \)

For the algebra \( A(G) \), the automatic continuity properties listed in Theorems 2, 3 are characteristic of amenable groups. We see that if \( A_M(G) \) or \( A_{M_0}(G) \) has a bounded approximate identity, then it possesses many of the properties of the Fourier algebra of an amenable group.

In [5] it was shown that \( A(F_2) \) has an approximate identity which is necessarily unbounded in \( A(G) \) but is bounded in \( \| \cdot \|_M \). Hence for the prototypical nonamenable group \( F_2 \), \( A_M(F_2) \) has a bounded approximate identity.

In [3] it was shown that the bounded approximate identity in \( A_M(F_2) \) is also bounded in \( \| \cdot \|_M \). Moreover, if \( G = SL(2, \mathbb{R}) \), \( G = SO(n, 1) \), or \( G \) is any closed subgroup of any of these groups, then \( A(G) \) has an approximate identity bounded in \( \| \cdot \|_M \). Therefore we have:

**Theorem 4.** Let \( G \) be \( SL(2, \mathbb{R}) \), \( SO(n, 1) \), or \( F_n \) for \( n = 2, 3, \ldots \). Let \( \mathcal{A} \) be either \( A_M(g) \) or \( A_{M_0}(G) \). Then every homomorphism for \( A \) with finite-dimensional range is continuous and every derivation from \( \mathcal{A} \) into a Banach \( \mathcal{A} \)-bimodule is continuous.

**Proof.** This follows immediately from Theorem 2, Theorem 3, and [3, Theorem 3.7]. \( \square \)

De Canniere and Haagerup had speculated about the existence of a bounded approximate identity in \( A_M(G) \) and \( A_{M_0}(G) \) for any locally compact group [3]. This would have provided a large new class of Banach spaces without discontinuous derivations, however, it has recently been shown that this is not the case (see [1]).

It would be of interest to know whether the above automatic continuity properties are characteristic of those groups for which either \( A_M(G) \) or \( A_{M_0}(G) \) has a bounded approximate identity.

**Bibliography**


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