SOME BANACH ALGEBRAS
WITHOUT DISCONTINUOUS DERIVATIONS

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Abstract. It is shown that the completion of $A(G)$ in either the multiplier norm or the completely bounded multiplier norm is a Banach algebra without discontinuous derivations when $G$ is either $F_2$ or $SL(2, \mathbb{R})$.

1. Introduction

In [6], it was shown that a locally compact group $G$ is amenable if and only if every derivation $D$ from $A(G)$, the Fourier algebra of $G$, into an arbitrary Banach $A(G)$-bimodule $X$ is continuous. The same result can be shown to hold for the Herz algebras $A_p(G)$, $1 < p < \infty$ [7]. Since the class of amenable groups is substantial (it includes all compact groups and all commutative groups) a large number of Banach algebras without discontinuous derivations have been identified. However, since $SL(2, \mathbb{R})$ and $F_2$, the free group on two generators, are nonamenable, $A(SL(2, \mathbb{R}))$ and $A(F_2)$ have discontinuous derivations. We will show that both $A(SL(2, \mathbb{R}))$ and $A(F_2)$ can be given natural norms in such a way that their completions will be Banach algebras without discontinuous derivations.

2. Preliminaries and notation

Let $G$ be a locally compact group. Let $A(G)$ be the Fourier algebra of $G$ as defined by P. Eymard in [4]. $A(G)$ is a Banach algebra with respect to pointwise multiplication. It is also the predual of the von Neumann algebra $VN(G)$ associated with the left-regular representation of $G$ on $L^2(G)$ and a closed ideal in $B(G)$, the Fourier Stieltjes algebra of $G$.

Let $MA(G)$ denote the space of multipliers of $A(G)$. That is, the continuous functions $\psi$ on $G$ such that $\psi u \in A(G)$ for every $u \in A(G)$. For each $\psi \in MA(G)$, $u \in A(G)$, let $m_\psi(u) = \psi u$. Denote by $\|\psi\|_m$ the operator norm of $m_\psi$. We call $\psi$ a completely bounded multiplier of $A(G)$ if $m_\psi^*$, the adjoint of $m_\psi$, is a completely bounded map on $VN(G)$ [3]. Let $\|\psi\|_{M_0}$ be the...
completely bounded norm of \( m^*_\psi \) and let \( M_0A(G) \) denote the Banach algebra of completely bounded multipliers of \( A(G) \). Then \( B(G) \subseteq M_0A(G) \subseteq MA(G) \) and \( \|u\|_M \leq \|u\|_{M_0} \leq \|u\|_{B(G)} \) for every \( u \in B(G) \).

We will denote by \( A_M(G) \) and \( A_{M_0}(G) \) the closure of \( A(G) \) in \( MA(G) \) and in \( M_0A(G) \) respectively. It is well known that \( G \) is amenable if and only if \( MA(G) = B(G) \) \([12, 13]\). In this case \( \|u\|_{B(G)} = \|u\|_M \) for every \( u \in B(G) \) and hence \( A(G) = A_M(G) = A_{M_0}(G) \).

If \( \mathcal{A} \) is a Banach algebra, then a derivation \( D \) on \( \mathcal{A} \) is a linear map from \( \mathcal{A} \) into a Banach \( \mathcal{A} \)-bimodule \( X \) such that \( D(uv) = u \cdot D(v) + D(u) \cdot v \) for every \( u, v \in \mathcal{A} \). If \( \mathcal{A} \) is commutative, we will denote the maximal ideal space of \( \mathcal{A} \) by \( \Delta(\mathcal{A}) \). We also use \( \Delta(\mathcal{A}) \) to denote the multiplicative linear functionals which correspond to the maximal ideal. Given a closed subset \( A \) of \( \Delta(\mathcal{A}) \), we denote by \( I_\mathcal{A}(A) \), the ideal \( \{u \in \mathcal{A} ; u(x) = 0 \text{ for every } x \in A\} \), where \( \mathcal{A} \) is realized as an algebra of functions on \( \Delta(\mathcal{A}) \) by means of the Gelfand transform. \( A \) is called a set of spectral synthesis for \( \mathcal{A} \) or simply an \( S \)-set if \( I_\mathcal{A}(e) = \{u \in \mathcal{A} ; \text{supp} u \text{ compact}, \text{supp} u \cap A = \emptyset\} \) is dense in \( I_\mathcal{A}(A) \).

3. Spectral synthesis and automatic continuity of derivations

Lemma 1. Let \( G \) be a locally compact group. Then

\[
\Delta(A_M(G)) = \Delta(A_{M_0}(G)) = G.
\]

Proof. Let \( \psi \in \Delta(A_M(G)) \). Then \( \psi|_{A(G)} \), the restriction of \( \psi \) to \( A(G) \), belongs to \( \Delta(A(G)) \). By \([4, \text{p. 222}]\) there exists an \( x \in G \) such that \( \psi(u) = u(x) \) for every \( u \in A(G) \).

Let \( v \in A_M(G) \). As \( A(G) \) is dense in \( A_M(G) \), we can find \( \{u_k\} \subseteq A(G) \) such that \( \|u_k - v\|_{M} \to 0 \). However, convergence in \( A_M(G) \) implies convergence in the topology of uniform convergence on compacta and hence in the topology of pointwise convergence. Therefore \( \psi(v) = \lim_k \psi(u_k) = \lim_k u_k(x) = v(x) \). It follows that as a set \( \Delta(A_M(G)) = G \).

Now suppose that \( \{x_\alpha\}_{\alpha \in I} \) is a net in \( G \) such that \( x_\alpha \) converges to \( x \in G \) in the usual topology on \( G \). Then for each \( u \in A_M(G) \), \( u(x_\alpha) \to u(x) \). Hence \( x_\alpha \to x \) in the \( \sigma(A_M(G)^*, A_M(G)) \) topology. Conversely, if \( x_\alpha \to x \) in the \( \sigma(A_M(G)^*, A_M(G)) \) topology, then for every \( u \in A(G) \), \( u(x_\alpha) \to u(x) \). Since \( G \subseteq A(G)^* \), \( x_\alpha \to x \) in the \( \sigma(A_M(G)^*, A_M(G)) \) topology. But \( \Delta(A(G)) \) is homeomorphic to \( G \) so \( x_\alpha \to x \) in \( G \). Therefore the \( \sigma(A_M(G)^*, A_M(G)) \) topology agrees with the usual topology on \( G \).

A similar argument establishes that \( \Delta(A_{M_0}(G)) = G \). □

Proposition 1. Let \( \mathcal{A} \) be either \( A_M(G) \) or \( A_{M_0}(G) \). Then \( \{e\} \) is an \( S \)-set of \( \mathcal{A} \).

Proof. Let \( v \in \mathcal{A} \) with \( v \in I_\mathcal{A}(\{e\}) \). Then there exists \( u_k \in A(G) \) with \( \|u_k - v\|_\mathcal{A} \to 0 \). Furthermore since \( u_k(e) \to v(e) = 0 \), we can assume that \( |u_k(e)| < 1/2k \). We can also find \( w_k \in A(G) \) with \( \|w_k\|_{A(G)} = w_k(e) = u_k(e) \). Let \( v_k = u_k - w_k \). Then \( v_k(e) = 0 \). Since \( v_k \in I_{A(G)}(\{e\}) \) and \( \{e\} \) is an \( S \)-set for \( A(G) \) \([4, \text{p. 229}]\), there exists \( z_k \in A(G) \) with \( \|z_k - v_k\|_{A(G)} \leq 1/2k \).
supp $z_k$ is compact and $\text{supp } z_k \cap \{e\} = \emptyset$. However

$$\|z_k - v\|_{A'} \leq \|z_k - u_k\|_{A'} + \|v_k - v\|_{A'}$$

$$\leq \|z_k - u_k\|_{A(G)} + \|u_k - v\|_{A} + \|w_k\|_{A(G)}$$

$$\leq \|u_k - v\|_{A'} + 1/k.$$

Hence $\|z_k - v\|_{A'} \to 0$. Therefore $I_{A'}^0(\{e\})$ is dense in $I_A(\{e\})$ and $\{e\}$ is an $S$-set. □

**Proposition 2.** Let $\mathcal{A}$ be either $A_M(G)$ or $A_{M_0}(G)$. If $\mathcal{A}$ has a bounded approximate identity, then so does $I_{\mathcal{A}}^0(\{e\})$.

**Proof.** Let $\{u_\alpha\}_{\alpha \in I}$ be a bounded approximate identity in $\mathcal{A}$. Let $\mathcal{F}(\{e\}) = \{K \subset G; K \text{ is compact, } K \cap \{e\} = \emptyset\}$. For every $K \in \mathcal{F}(\{e\})$, there exists $v_K \in B(G)$ such that $\|v_K\|_{\mathcal{A}} \leq \|v_K\|_{B(G)} = 1$, $u_K(e) = 1$, and $v_K(x) = 0$ for every $x \in K$. Define $u_{K, \alpha} \in \mathcal{A}$ by $u_{K, \alpha} = u_\alpha - v_K u_\alpha$. Then $u_{K, \alpha}(e) = 0$, $\|u_{K, \alpha}\|_{\mathcal{A}} \leq 2\|u_\alpha\|_{\mathcal{A}}$, and $u_{K, \alpha} v = u_\alpha v$ for every $v \in \mathcal{A}$ with $\text{supp } v \subset K$.

Order $K \times I$ by $(K_1, \alpha_1) \leq (K_2, \alpha_2)$ if and only if $K_1 \subseteq K_2$ and $\alpha_1 \leq \alpha_2$. If $v \in \mathcal{A}$ and $\text{supp } v \in \mathcal{F}(\{e\})$, then $v = \lim_{K, \alpha} u_{K, \alpha} v$. By Proposition 1, such $v$'s are dense in $I_{\mathcal{A}}^0(\{e\})$. As $\{u_{K, \alpha}\}_{K \times I}$ is bounded, $\lim_{K, \alpha} u_{K, \alpha} w = w$ for every $w \in I_{\mathcal{A}}^0(\{e\})$. □

The proof of Proposition 2 is a modification of the proof of [7, Proposition 3.2]. The case where $G$ is amenable is due to A. Lau [11, Corollary 4.11].

**Corollary 1.** Let $\mathcal{A}$ be either $A_M(G)$ or $A_{M_0}(G)$. If $\mathcal{A}$ has a bounded approximate identity, then $\mathcal{A}$ satisfies Ditkin’s condition. Furthermore, if $A$ has a closed subset of $G$ and the boundary of $A$ contains no nontrivial perfect set, then $A$ is an $S$-set. In particular, every finite subset of $G$ is an $S$-set for $\mathcal{A}$.

**Proof.** Let $x \in G$ and $u \in \mathcal{A}$ be such that $u(x) = 0$. From the proof of Proposition 2 (translate is $x \neq e$), we see there exists a sequence $\{v_n\} \subset \mathcal{A}$ for which each $v_n$ vanishes in a neighborhood $V_n$ of $\{x\}$ and $\lim_{n} \|v_n u - u\|_{\mathcal{A}} = 0$.

Assume that $G$ is not compact. Let $u \in \mathcal{A}$, $u \neq 0$. Let $\varepsilon > 0$. If $\{v_\alpha\}_{\alpha \in I}$ is a bounded approximate identity in $\mathcal{A}$, then there exists $u_\alpha$ such that $\|u_\alpha - u\|_{\mathcal{A}} < \varepsilon/2$. We can find $v \in A(G)$ with $\text{supp } v$ compact such that $\|u_\alpha - v\|_{\mathcal{A}} \leq \varepsilon/2\|u\|_{\mathcal{A}}$. Then $\|vu - u\|_{\mathcal{A}} \leq \|vu - u_\alpha u\|_{\mathcal{A}} + \|u_\alpha u - u\|_{\mathcal{A}} \leq \varepsilon$.

Hence $\mathcal{A}$ satisfies Ditkin’s condition [9, p. 49]. The remaining statements follow immediately from Ditkin’s theorem [9, p. 497]. □

Corollary 1 is simply [7, Lemma 5.2, Proposition 5.3] when $G$ is assumed to be amenable.

**Proposition 3.** Let $\mathcal{A}$ be either $A_M(G)$ or $A_{M_0}(G)$. If $\mathcal{A}$ has a bounded approximate identity and $I$ is a closed cofinite ideal in $\mathcal{A}$ (i.e., $\text{dim } \mathcal{A}/I < \infty$), then $I = I(A)$ for some finite set $A = \{x_1, \ldots, x_n\}$ where $n$ is the codimension of $I$.

**Proof.** Let $A = Z(I) = \{x \in G; u(x) = 0 \text{ for every } u \in I\}$. Let $n = \text{codim } I$.

Assume that $A$ contains $n+1$ distinct elements $\{x_1, \ldots, x_{n+1}\}$. We can find a compact neighborhood $V_k$ of each $x_k$ such that $V_j \cap V_k = \emptyset$ if $j \neq k$. We can also find $u_k \in A(G)$ such that $\text{supp } u_k \subseteq V_k$ and $u_k(x_k) = 1$ for $1 \leq k \leq n + 1$. But if $\psi: \mathcal{A} \to \mathcal{A}/I$ is the canonical homomorphism, then $\{\psi(u_1), \ldots, \psi(u_{n+1})\}$ is a linearly independent subset, which is impossible if
codim\(I = n\). Therefore \(A\) has at most \(n\) elements. Since \(A\) is finite, it is an \(S\)-set by Corollary 1. Therefore \(I = I(A)\).

Let \(A = \{x_1, \ldots, x_k\}\). Let \(u_k \in A(G)\) be such that \(u_k(x_k) = 1\) and \(u_k(x_j) = 0\) if \(j \neq k\). Let \(u \in A\). Then

\[
u = \sum_{i=1}^{k} u(x_i)u_i + \left(u - \sum_{i=1}^{k} u(x_i)u_i\right).
\]

Since

\[
u - \sum_{i=1}^{k} u(x_i)u_i \in I(A), \quad k \geq n.
\]

Hence \(k = n\). \(\Box\)

**Proposition 4.** Let \(A\) be either \(A_M(G)\) or \(A_{M_0}(G)\). Assume that \(A\) has a bounded approximate identity. Let \(I\) be a closed cofinite ideal of \(A\). Then \(I\) has a bounded approximate identity. In particular \(I^2 = \{\sum_{i=1}^{n} u_i v_i, \; u_i, v_i \in I\} = I\).

**Proof.** By Proposition 3, \(I = I(\{x_1, \ldots, x_n\})\) for some finite subset of \(G\).

Let \(A, B\) be two closed subsets of \(G\). Assume that \(I_A(\{e\})\) and \(I_B(\{x\})\) have bounded approximate identities \(\{u_i\}_{i \in T}, \{v_j\}_{j \in J}\) respectively. Then it is easy to see that \(\{u_i v_j\}_{i \times j}\) is a bounded approximate identity for \(I(A \cup B)\).

Since \(A\) has a bounded approximate identity, Proposition 2 implies that \(I_A(\{e\})\) has a bounded approximate identity. By translating, we see that \(I_A(\{x\})\) has a bounded approximate identity for every \(x \in G\). A simple induction argument shows that \(I\) also has a bounded approximate identity. Then \(I^2 = I\) follows from Cohen’s factorization theorem [9, p. 268]. \(\Box\)

**Theorem 1.** Let \(A\) be either \(A_M(G)\) or \(A_{M_0}(G)\). If \(A\) has a bounded approximate identity, then \(I\) is a cofinite ideal of \(A\) if and only if \(I = I(A)\) for some finite subset \(A\) of \(G\).

**Proof.** Proposition 4 shows that every closed cofinite ideal of \(A\) is idempotent. By [2, Theorem 2.3] every cofinite ideal of \(A\) must be closed and is therefore of the form \(I(A)\) for some finite subset \(A\) of \(G\) by Proposition 3.

If \(I = I(A)\) for a finite subset \(A\) of \(G\), then the proof of Proposition 3 shows that \(I\) is cofinite. \(\Box\)

If \(G\) is amenable, then Proposition 3 and Theorem 1 follow from [7, Corollary 5.6, Theorem 5.8].

**Theorem 2.** Let \(A\) be either \(A_M(G)\) or \(A_{M_0}(G)\). If \(A\) has a bounded approximate identity, then every homomorphism from \(A\) with finite-dimensional range is continuous.

**Proof.** The statement follows immediately from Theorem 1, Proposition 4, and [2, Theorem 2.3]. \(\Box\)

**Lemma 2.** Let \(A\) be either \(A_M(G)\) or \(A_{M_0}(G)\). Assume that \(A\) has a bounded approximate identity. Let \(I\) be a closed ideal in \(A\) with infinite codimension. Then there exist sequences \(\{u_n\}, \{v_n\}\) in \(A(G)\) such that \(u_n v_1 \cdots v_{n-1} \notin I\) but \(u_n v_1 \cdots v_n \in I\) for \(n \geq 2\).

**Proof.** If \(I\) has infinite codimension, then \(A = Z(I) = \{x \in G, \; u(x) = 0 \; \forall u \in I\}\) must be infinite by Theorem 1.
From the proof of [6, Lemma 2], we see that we can find sequences \( \{u_n\} \), \( \{v_n\} \subseteq A(G) \) such that \( u_nv_1 \cdots v_{n-1} \notin I_{A(G)}(A) \) while \( u_nv_1 \cdots v_n = 0 \). Since \( I_{A(G)}(A) \subseteq I(A) \), the result follows. 

**Theorem 3.** Let \( \mathcal{A} \) be either \( A_M(G) \) or \( A_{M_0}(G) \). Assume that \( \mathcal{A} \) has a bounded approximate identity. Then every derivation \( D \) of \( \mathcal{A} \) into a Banach \( \mathcal{A} \)-bimodule \( X \) is continuous.

**Proof.** This follows immediately from Proposition 4, Lemma 2, and [10, Theorem 2]. 

For the algebra \( A(G) \), the automatic continuity properties listed in Theorems 2, 3 are characteristic of amenable groups. We see that if \( A_M(G) \) or \( A_{M_0}(G) \) has a bounded approximate identity, then it possesses many of the properties of the Fourier algebra of an amenable group.

In [5] it was shown that \( A(F) \) has an approximate identity which is necessarily unbounded in \( A(G) \) but is bounded in \( \| \cdot \|_M \). Hence for the prototypical nonamenable group \( F_2 \), \( A_M(F_2) \) has a bounded approximate identity.

In [3] it was shown that the bounded approximate identity in \( A_M(F_2) \) is also bounded in \( \| \cdot \|_M \). Moreover, if \( G = SL(2, \mathbb{R}) \), \( G = SO(n, 1) \), or \( G \) is any closed subgroup of any of these groups, then \( A(G) \) has an approximate identity bounded in \( \| \cdot \|_M \). Therefore we have:

**Theorem 4.** Let \( G \) be \( SL(2, \mathbb{R}) \), \( SO(n, 1) \), or \( F_n \) for \( n = 2, 3, \ldots \). Let \( \mathcal{A} \) be either \( A_M(G) \) or \( A_{M_0}(G) \). Then every homomorphism for \( A \) with finite-dimensional range is continuous and every derivation from \( \mathcal{A} \) into a Banach \( \mathcal{A} \)-bimodule is continuous.

**Proof.** This follows immediately from Theorem 2, Theorem 3, and [3, Theorem 3.7]. 

De Canniere and Haagerup had speculated about the existence of a bounded approximate identity in \( A_M(G) \) and \( A_{M_0}(G) \) for any locally compact group [3]. This would have provided a large new class of Banach spaces without discontinuous derivations, however, it has recently been shown that this is not the case (see [1]).

It would be of interest to know whether the above automatic continuity properties are characteristic of those groups for which either \( A_M(G) \) or \( A_{M_0}(G) \) has a bounded approximate identity.

**Bibliography**

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