DIMENSIONS OF TOPOLOGICAL GROUPS CONTAINING THE BOUQUET OF TWO CIRCLES

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Abstract. In this paper we prove the following: If a topological group $G$ contains the bouquet $S^1 \lor S^1$, then $\dim G \geq 2$ holds. This is a counterexample to a question of Bel'nov in the one-dimensional case.

1. Introduction

In this paper we assume that all spaces are Tychonoff.

Bel'nov [1] proved that every space $X$ can be embedded into a homogeneous space $G_X$ such that $\text{ind} G_X = \text{ind} X$, $\text{Ind} G_X = \text{Ind} X$, and $\dim G_X = \dim X$ in the case when the corresponding dimension of $X$ is finite. The space $G_X$ is a free group over $X$ with continuous right translations. Bel'nov (see [5]) asked whether every space $X$ can be embedded into a topological group $G$ with $\dim G \leq \dim X$.

Shakhmatov [5] proved that if $n \neq 0, 1, 3, 7$, then the $n$-dimensional sphere $S^n$ cannot be embedded into an $n$-dimensional topological group. Thus the answer to the above question is no. However, Shakhmatov also proved that in case $\dim X = 0$ the answer to this question is yes. Hence it is natural to ask whether every space $X$ can be embedded into a topological group $G$ with $\dim G \leq \dim X$ in case $\dim X = 1$ or $3$ or $7$.

In this paper we prove that the answer to this question is no in case $\dim X = 1$. Namely, we prove that if a topological group $G$ contains the bouquet $S^1 \lor S^1$, then $\dim G \geq 2$ holds. Obviously, $\dim(S^1 \lor S^1) = 1$. Hence the bouquet $S^1 \lor S^1$ is a counterexample to a question of Bel'nov in the one-dimensional case.

2. Preliminaries and lemmas

We denote by $\mathbb{R}$, $\mathbb{Q}$, $I$, and $S^1$ the real line, the rational space, the closed unit interval $[0, 1]$, and the 1-sphere, respectively. We put $\partial I^2 = (I \times \{0, 1\}) \cup (\{0, 1\} \times I)$. We denote by $d$ the Euclidean metric on $\mathbb{R}^3$. For two continuous
mappings $f$ and $g$ of a space $X$ into $\mathbb{R}^3$ we put
\[ d(f, g) = \sup \{d(f(x), g(x)) : x \in X \}. \]
For a subset $X$ of $\mathbb{R}^3$ and for any $\varepsilon > 0$ we put
\[ U(X, \varepsilon) = \{ \alpha \in \mathbb{R}^3 : d(\alpha, X) < \varepsilon \}. \]
For a subset $X$ of $\mathbb{R}^3$ we denote by $i_X$ the inclusion mapping of $X$ into $\mathbb{R}^3$.
For a mapping $f : X \to Y$ and for a subset $A$ of $X$ we denote by $f|_A$ the restriction of $f$ to $A$. Let $N_3^1$ be the subspace of $\mathbb{R}^3$ consisting of all points which have at most one rational coordinate. The following lemma is needed in §3.

2.1. Lemma. Let $X$ be a compact metrizable space with $\dim X \leq 1$. Then for every continuous mapping $f : X \to \mathbb{R}^3$ and for any $\varepsilon > 0$ there exists an embedding $g : X \to \mathbb{R}^3$ such that $d(f, g) < \varepsilon$ and $g(X) \subset N_3^1$.

Proof. See the proof of [2, 1.11.5].

Let us set
\begin{align*}
L &= \{(x, y, 0) \in \mathbb{R}^3 : x^2 + y^2 = 9\}, \\
M &= \{(x, 0, z) \in \mathbb{R}^3 : (x-2)^2 + z^2 = 1\}, \\
C &= \{(x, y, 0) \in \mathbb{R}^3 : x^2 + y^2 = 4\}, \\
Z &= \{(0, 0, z) \in \mathbb{R}^3 : z \in \mathbb{R}\}, \\
T^2 &= \{\alpha \in \mathbb{R}^3 : d(\alpha, C) = 1\}.
\end{align*}

Then, obviously, $T^2$ is a torus. For every $\theta$, $0 \leq \theta < 2\pi$, we put
\[ F_\theta = \{(r \cos \theta, r \sin \theta, z) \in \mathbb{R}^3 : r > 0 \text{ and } z \in \mathbb{R}\}. \]
Let $K$ be a polyhedral 1-sphere satisfying the following two conditions (*) :
\[ K \subset U(C, 1) \cap \left(\left(\left(\mathbb{R} \times \mathbb{Q}\right) \cup \left(\mathbb{Q} \times \mathbb{R}\right)\right) \times \{0\}\right) \text{ and } |F_\theta \cap K| = 1 \text{ for every } \theta, 0 \leq \theta < 2\pi. \]
We shall construct a continuous mapping $p_K : \mathbb{R}^3 - (K \cup Z) \to T^2$. For every $\theta$, $0 \leq \theta < 2\pi$, let $p_\theta : F_\theta - \{x_\theta\} \to F_\theta \cap T^2$ be the projection of $F_\theta - \{x_\theta\}$ from the point $x_\theta$ onto $F_\theta \cap T^2$, where $\{x_\theta\} = F_\theta \cap K$. Let $p_K : \mathbb{R}^3 - (K \cup Z) \to T^2$ be the mapping defined by
\[ p_K(r \cos \theta, r \sin \theta, z) = p_\theta(r \cos \theta, r \sin \theta, z) \]
for every $(r \cos \theta, r \sin \theta, z) \in \mathbb{R}^3 - (K \cup Z)$. Then $p_K$ is continuous. The proof of the following lemma is easy, so we omit the proof.

2.2. Lemma. For every $\varepsilon > 0$ there exists a polyhedral 1-sphere $K$ with property (*) such that $d(i_{U(T^2, \varepsilon)}, p_K|_{U(T^2, \varepsilon)}) < 2\varepsilon$.

Let $\lambda : I^2 \to T^2$ be the natural quotient such that
\begin{align*}
\lambda(x, 0) &= \lambda(x, 1) = (3 \cos 2\pi x, 3 \sin 2\pi x, 0),\\
\lambda(0, x) &= \lambda(1, x) = (2 + \cos 2\pi x, 0, \sin 2\pi x),
\end{align*}
and
\[ \lambda(1/2, 1/2) = (-1, 0, 0). \]
Let \( q: I^2 - \{(1/2, 1/2)\} \to \partial I^2 \) be the projection of \( I^2 - \{(1/2, 1/2)\} \) from the point \((1/2, 1/2)\) onto \( \partial I^2 \). It is obvious that \( |\lambda q \lambda^{-1}(\alpha)| = 1 \) for every \( \alpha \in \mathbb{T}^2 - \{(-1, 0, 0)\} \). Thus we can define the mapping \( \mu: \mathbb{T}^2 - \{(-1, 0, 0)\} \to \mathbb{T}^2 \) such that \( \{\mu(\alpha)\} = \lambda q \lambda^{-1}(\alpha) \). Then we have \( \mu(\mathbb{T}^2 - \{(-1, 0, 0)\}) = L \cup M \) and \( \mu \) is continuous. The proof of the following lemma is also easy, so we omit the proof.

2.3. Lemma. For every \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that

\[
d(i_{U(L \cup M, \delta) \cap \mathbb{T}^2}, \mu_{|U(L \cup M, \delta) \cap \mathbb{T}^2}, \mu_{|U(L \cup M, \delta) \cap \mathbb{T}^2}) < \varepsilon.
\]

3. The main theorem

By Lemma 2.3, we can take \( \delta > 0 \) such that

\[
d(i_{U(L \cup N, \delta) \cap \mathbb{T}^2}, \mu_{|U(L \cup M, \delta) \cap \mathbb{T}^2}) < 1/3.
\]

We fix \( \delta \) in this section.

3.1. Lemma. Let \( f: I^2 \to \mathbb{R}^3 \) be a continuous mapping such that \( d(f_{|\partial I^2}, \lambda_{|\partial I^2}) < \delta/4 \). Then \( \dim f(I^2) \geq 2 \) holds.

Proof. Suppose that \( \dim f(I^2) \leq 1 \). By Lemma 2.1, we can take an embedding \( g: f(I^2) \to \mathbb{R}^3 \) such that \( d(i_{f(I^2)}, g) < \delta/4 \) and \( g \circ f(I^2) \subset N^3_\delta \). By Lemma 2.2, we can take a polyhedral 1-sphere \( K \) with property (\( * \)) such that \( d(i_{f(I^2), \partial I^2}, P_{K|U(I^2, \delta/2)}) < \delta \). Note that \( g \circ f(I^2) \cap (K \cup \mathbb{Z}) = \emptyset \), because \( g \circ f(I^2) \subset N^3_\delta \). Since \( K \subset ((\mathbb{R} \times \mathbb{Q}) \cup (\mathbb{Q} \times \mathbb{R})) \times \{0\} \) and \( g \circ f(I^2) \cap \{(-3, 0, 0)\} = \emptyset \), we have \( (-1, 0, 0) \notin p_K \circ g \circ f(I^2) \). Let \( \nu = \mu \circ p_K \circ g \circ f \). Then \( \nu \) is a continuous mapping of \( I^2 \) into \( L \cup M \) and \( d(\nu_{|\partial I^2}, \lambda_{|\partial I^2}) < 1/3 \). We take a lifting \( \bar{\nu}: I^2 \to \mathbb{R}^2 \) of \( \nu \). Let us set \( A_i = I \times \{i\} \) for \( i = 0, 1 \), and \( B_i = \{i\} \times I \) for \( i = 0, 1 \). Since \( d(\nu_{|\partial I^2}, \lambda_{|\partial I^2}) < 1/3 \), we may assume that

\[
\bar{\nu}(A_i) \subset V(A_i, 1/3) \quad \text{for } i = 0, 1,
\]

and

\[
\bar{\nu}(B_i) \subset V(B_i, 1/3) \quad \text{for } i = 0, 1,
\]

where \( V(N, \varepsilon) \) is the \( \varepsilon \)-neighborhood of \( N \) in \( \mathbb{R}^2 \). Let us set \( D = \mathbb{R} \times \{1/2\} \) and \( E = \{1/2\} \times \mathbb{R} \). Then \( D \) is a partition in \( \mathbb{R}^2 \) between \( \bar{\nu}(A_0) \) and \( \bar{\nu}(A_1) \), and \( E \) is a partition in \( \mathbb{R}^2 \) between \( \bar{\nu}(B_0) \) and \( \bar{\nu}(B_1) \). Thus \( \bar{\nu}^{-1}(D) \) is a partition in \( I^2 \) between \( A_0 \) and \( A_1 \), and \( \bar{\nu}^{-1}(E) \) is a partition in \( I^2 \) between \( B_0 \) and \( B_1 \). Thus we have \( \bar{\nu}^{-1}(D) \cap \bar{\nu}^{-1}(E) \neq \emptyset \). On the other hand, since \( \nu(I^2) \subset L \cup M \), we have \( \bar{\nu}(I^2) \subset p^{-1}(L \cup M) \), where \( p: \mathbb{R}^2 \to \mathbb{T}^2 \) is the covering mapping. Thus we have \( D \cap E \neq \emptyset \). This implies that \( \bar{\nu}^{-1}(D) \cap \bar{\nu}^{-1}(E) = \emptyset \). This is a contradiction. Hence we have \( \dim f(I^2) \geq 2 \). Lemma 3.1 has been proved.

Recall that the bouquet \( S^1 \vee S^1 \) is the one point union of \( S^1 \) and \( S^1 \) with the common point.

We are now in a position to establish our main theorem.

3.2. Theorem. If a topological group \( G \) contains the bouquet \( S^1 \vee S^1 \), then \( \dim G \geq 2 \) holds.

Proof. Suppose that a topological group \( G \) with \( \dim G \leq 1 \) contains the bouquet \( S^1 \vee S^1 \). Since \( L \cup M \) is homeomorphic to \( S^1 \vee S^1 \) and since \( G \) is
homogeneous, we may assume that \( e = (3, 0, 0) \in L \cup M \subset G \), where \( e \) is the neutral element of \( G \). Let \( \varphi_1 : I \to G \) and \( \varphi_2 : I \to G \) be the mappings defined by

\[
\varphi_1(x) = (3 \cos 2\pi x, 3 \sin 2\pi x, 0)
\]

and

\[
\varphi_2(x) = (2 + \cos 2\pi x, 0, \sin 2\pi x)
\]

for every \( x \in I \). Let \( \varphi : I^2 \to G \) be the mapping defined by \( \varphi(x, y) = \varphi_1(x) \cdot \varphi_2(y) \) for every \( (x, y) \in I^2 \), where \( \cdot \) is the group composition of \( G \). Then \( \varphi(I^2) \) is compact and metrizable. Since \( \varphi_1(0) = \varphi_1(1) = \varphi_2(0) = \varphi_2(1) = (3, 0, 0) = e \), we have \( \varphi|_{\partial I^2} = \lambda_{\partial I^2} \). Let \( f : \varphi(I^2) \to \mathbb{R}^3 \) be a continuous mapping such that \( f|_{\varphi(I^2)} = i_{LU} \). Since \( \dim \varphi(I^2) \leq 1 \), we have \( \dim \varphi(I^2) \leq 1 \). Thus, by Lemma 2.1, we can take an embedding \( g : \varphi(I^2) \to \mathbb{R}^3 \) such that \( d(f, g) < \delta/4 \). Then we have \( d(g \circ \varphi|_{\partial I^2}, \lambda_{\partial I^2}) < \delta/4 \). Thus, by Lemma 3.1, we have \( \dim \varphi(I^2) = \dim g \circ \varphi(I^2) \geq 2 \). This is a contradiction. Theorem 3.2 has been proved.

4. Remarks and questions

4.1 Remark. Vopenka ([7] or see [3, 18–11]) constructed a compact space \( X \) such that \( \dim X = n < \omega \) and \( \text{ind} X = \infty \). Shakhmatov [4] proved that the above space \( X \) cannot be embedded into a finite-dimensional, locally pseudo-compact topological group. However, it is unknown whether the above space \( X \) can be embedded into a topological group with the same dimension of \( X \).

4.2. Remark. In the proof of Theorem 3.2 we get a contradiction to assume that \( \dim \varphi(I^2) \leq 1 \). Because \( \varphi(I^2) \) is compact and metrizable, if \( \text{ind} G \leq 1 \) or \( \text{Ind} G \leq 1 \), then we have \( \dim \varphi(I^2) \leq 1 \). Hence the bouquet \( S^1 \vee S^1 \) cannot be embedded into a topological group \( G \) such that \( \text{ind} G \leq 1 \) or \( \text{Ind} G \leq 1 \) or \( \dim G \leq 1 \).

It is well known that every separable metrizable space \( X \) with \( \dim X = n \) can be embedded into the \((2n + 1)\)-dimensional cube \( I^{2n+1} \). Obviously, \( I^{2n+1} \) can be embedded into the \((2n + 1)\)-dimensional torus \( T^{2n+1} \). Hence every separable metrizable space \( X \) with \( \dim X = n \) can be embedded into a \((2n + 1)\)-dimensional compact metrizable topological group. It is easy to see that every \( n \)-dimensional, locally separable metrizable space can be embedded into a \((2n + 1)\)-dimensional metrizable topological group.

4.3. Question. Does there exist a mapping \( \varphi : \omega \to \omega \) such that every metrizable space \( X \) with \( \dim X = n \) can be embedded into a (metrizable) topological group \( G \) with \( \dim G \leq \varphi(n) \)?

Shakhmatov also gave some similar problems in [6].

It is obvious that the bouquet \( S^1 \vee S^1 \) can be embedded into the 2-dimensional torus \( T^2 \). However, it is well known that if \( n \geq 8 \), then the complete graph \( K_n \) cannot be embedded into \( T^2 \). Since \( K_8 \) contains \( S^1 \vee S^1 \), \( K_n \) cannot be embedded into a topological group \( G \) with \( \dim G \leq 1 \) for each \( n \geq 8 \). Obviously, \( K_n \) can be embedded into the 3-dimensional torus \( T^3 \).
4.4. **Question.** Let $n \geq 8$. Can the complete graph $K_n$ be embedded into a topological group $G$ with $\dim G \leq 2$?

A space $X$ is **atriodic** if $X$ does not contain the complete bipartite graph $K_{1,3}$. Obviously, the bouquet $S^1 \vee S^1$ is not atriodic.

4.5. **Question.** Can the complete bipartite graph $K_{1,3}$ be embedded into a topological group $G$ with $\dim G = 1$?

If a graph $X$ is atriodic, then every component of $X$ is homeomorphic to $S^1$ or $I$ or the singleton. Thus an atriodic graph can be embedded into $S^1 \times \mathbb{Z}_p$, where $p$ is the number of components of $X$ and $\mathbb{Z}_p$ is a group consisting of $p$ points with discrete topology. Hence if the answer to the above question is negative, then a graph $X$ can be embedded into a topological group $G$ with $\dim G = 1$ if and only if $X$ is atriodic.

*Added in proof.* H. Kato and J. Kulesza, independently, constructed an $n$-dimensional space $X$ such that every topological group $G$ containing $X$ has $\dim G \geq n + 1$. They also gave a negative answer to Question 4.5.

**References**

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