REMARK ON COMMUTATIVE APPROXIMATE IDENTITIES ON HOMOGENEOUS GROUPS

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(Communicated by J. Marshall Ash)

Abstract. We give, using the functional calculus of Hulanicki [4], a construction of a commutative approximate identity on every homogeneous group.

In [2] Folland and Stein asked, whether on every homogeneous group \( N \) there is a function \( \phi \) in the Schwartz class \( \mathcal{S}(N) \) with the following properties: \( \int_N \phi(x) \, dx = 1 \), \( \phi_s \ast \phi_s = \phi_s \ast \phi_t \), where \( \phi_t(x) = t^{-Q} \phi(t^{-1}x) \), and \( Q \) is the homogeneous dimension of \( N \). The family \( \{\phi_t\} \) is then called a commutative approximate identity and is used for characterizing Hardy spaces \( H^p(N) \), cf. [2]. Folland and Stein [2] produced a commutative approximate identity in the case when \( N \) is graded. Glowacki showed in [3] that for every homogeneous group the densities of the stable semigroup of symmetric measures generated by the functional

\[
\langle P, f \rangle = \lim_{\varepsilon \to 0} \int_{|x| > \varepsilon} \frac{f(0) - f(x)}{|x|^{Q+1}} \Omega(x) \, dx,
\]

(where \( \Omega \) is a nonzero nonnegative smooth away from the origin homogeneous of degree 0 symmetric function and \( |\cdot| \) is a smooth homogeneous norm on \( N \)) belong to \( \mathcal{R}(N) \), cf. [2, p. 253]. The construction of a commutative approximate identity from such a semigroup of measures was presented in [2, pp. 258–260]. The purpose of this note is to give (using the functional calculus of Hulanicki [4]) an alternative construction of a commutative approximate identity from the distribution \( P \). We prove that on every homogeneous group \( N \) the following theorem holds.

**Theorem 1.** If \( f \in C_c(-1, 1) \), \( f = 1 \) in a neighborhood of 0, then there is a function \( \phi \in \mathcal{S}(N) \) such that \( \int_{-\infty}^{\infty} F(\lambda) \, dE_P(\lambda) f = f \ast \phi \), where \( E_P \) is the spectral resolution of the operator \( f \mapsto P f = f \ast P \).

**Corollary 1.** Since \( \int_N \phi(x) \, dx = F(0) \) and \( \int_{-\infty}^{\infty} F(t\lambda) \, dE_P(\lambda) f = f \ast \phi_t \), the family \( \{\phi_t\} \) forms a commutative approximate identity.

Let \( U = \{x \in N : |x| \leq 1\} \) and \( \tau(x) = \inf\{n : x \in U^n\} \). For every \( \alpha \geq 0 \) the function \( w_\alpha(x) = (1 + \tau(x))^\alpha \) is submultiplicative. Moreover, there are

Received by the editors August 28, 1990.

1991 Mathematics Subject Classification. Primary 43A80.

Key words and phrases. Homogeneous groups.
constants $c, C, a, b > 0$ such that $a < 1, 2 < b$, and

\begin{equation}
ct(x)^a \leq |x| \leq C\tau(x)^b \quad \text{for } |x| > 1,
\end{equation}

cf. [4, Lemma 1.1]. Denote by $M_\alpha$ the algebra of Borel measures $\mu$ on $\mathcal{N}$ such that $\int \omega_\alpha(x) d|\mu|(x) < \infty$. If $A$ is a selfadjoint operator on $L^2(\mathcal{N})$, $E_A$ is its spectral resolution, and $m$ is a bounded function on $\mathbb{R}$, then denote by $m(A)$ the operator $\int_{\mathbb{R}} m(\lambda) dE_A(\lambda)$. If $Af = f * \psi$ then $m(\psi)$ is the abbreviation for $m(A)$. The following theorem has been proved in [4]

**Theorem 2.** Suppose that $\psi = \psi^* \in M_\alpha \cap L^2(\mathcal{N})$, $\alpha > \beta + \frac{k}{2} Q + 2$, $m \in C^k_c(\mathbb{R})$ with $k > 3(\beta + \frac{k}{2} Q + 3)$, $m(0) = 0$, then there is a measure $\nu$ in $M_\beta$ such that $m(\psi) = f * \nu$.

The theorem below, which has been actually proved in [1], plays here the fundamental role.

**Theorem 3.** For every natural $N > Q$, let $T_i^{(N)}$ be the semigroup of operators on $L^2(\mathcal{N})$ generated by $P_i$. Then $T_i^{(N)} f = f * q_i^{(N)}$, where $q_i^{(N)} \in C^\infty(\mathcal{N})$ and $\sup_{x \in \mathcal{N}} |\partial q_i^{(N)}(x)|(1 + |x|)^{N+Q} \leq C(t, \theta, N) < \infty$ for every left-invariant differential operator $\partial$ on $\mathcal{N}$.

**Proposition 1.** If $m \in C^\infty_c(-1, 1)$, then for every natural $N > (\beta + \frac{k}{2} Q + 2)/a$ there is a unique function $\phi$ such that $m(P_i^n) = f * \phi$. Moreover, for every left-invariant differential operator $\partial$ on $\mathcal{N}$, $\sup_{x \in \mathcal{N}} |\partial \phi(x)| w_\theta(x) < \infty$.

**Proof.** Let $n(\lambda) = m(- \log \lambda)/\lambda$. Then $n \in C^\infty_c(e^{-1}, e)$. Moreover, $m(P_i^n) f = T_i^{(N)} n(q_i^{(N)}) f$. By Theorems 2, 3 and (1), $m(q_i^{(N)}) f = f * \nu$ holds with $\nu \in M_\beta$. Using Theorem 3 one obtains that $\phi = \nu * q_1^{(N)}$ satisfies required conditions.

**Proof of Theorem 1.** In virtue of (1) it suffices to show that there is a $C^\infty$ function $\phi$ such that for every $\beta > 0$ and every left-invariant differential operator $\partial$, one has $\int_0^\infty F(\lambda) dE_\beta(\lambda) f = \int_0^\infty F_N(\lambda) dE_\beta(\lambda) f$, where $F_N(\lambda) = F(|\lambda|^{1/N})$. By the definition of $F$, $F_N \in C^\infty_c(-1, 1)$. Using (2) and Proposition 1 with sufficiently large $N$, the required estimate follows.

**References**