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REMARK ON COMMUTATIVE APPROXIMATE IDENTITIES ON HOMOGENEOUS GROUPS

JACEK DZIUBAŃSKI

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ABSTRACT. We give, using the functional calculus of Hulanicki [4], a construction of a commutative approximate identity on every homogeneous group.

In [2] Folland and Stein asked, whether on every homogeneous group $\mathcal{N}$ there is a function ϕ in the Schwartz class $\mathcal{S}(\mathcal{N})$ with the following properties: $\int_{\mathcal{N}} \phi(x)\,dx = 1$, $\phi_t * \phi_s = \phi_s * \phi_t$ where $\phi_t(x) = t^{-Q}\phi(\delta_{t^{-1}}x)$, and Q is the homogeneous dimension of $\mathcal{N}$. The family $\{\phi_t\}$ is then called a commutative approximate identity and is used for characterizing Hardy spaces $H^p(\mathcal{N})$, cf. [2]. Folland and Stein [2] produced a commutative approximate identity in the case when $\mathcal{N}$ is graded. Glowacki showed in [3] that for every homogeneous group the densities of the stable semigroup of symmetric measures generated by the functional

$$\langle P, f\rangle = \lim_{\varepsilon\to 0}\int_{|x|>\varepsilon} \frac{f(0)-f(x)}{|x|^{Q+1}}\Omega(x)\,dx,$$

(where Ω is a nonzero nonnegative smooth away from the origin homogeneous of degree 0 symmetric function and $|\cdot|$ is a smooth homogeneous norm on $\mathcal{N}$) belong to $\mathcal{R}(\mathcal{N})$, cf. [2, p. 253]. The construction of a commutative approximate identity from such a semigroup of measures was presented in [2, pp. 258–260]. The purpose of this note is to give (using the functional calculus of Hulanicki [4]) an alternative construction of a commutative approximate identity from the distribution P. We prove that on every homogeneous group $\mathcal{N}$ the following theorem holds.

Theorem 1. *If $F \in C_c(-1, 1)$, $F = 1$ in a neighborhood of 0, then there is a function $\phi \in \mathcal{S}(\mathcal{N})$ such that $\int_0^\infty F(\lambda)\,dE_P(\lambda)f = f * \phi$, where E_P is the spectral resolution of the operator $f \mapsto Pf = f * P$.*

Corollary 1. *Since $\int_{\mathcal{N}} \phi(x)\,dx = F(0)$ and $\int_0^\infty F(t\lambda)\,dE_P(\lambda)f = f * \phi_t$, the family $\{\phi_t\}$ forms a commutative approximate identity.*

Let $U = \{x \in \mathcal{N} : |x| \le 1\}$ and $\tau(x) = \inf\{n : x \in U^n\}$. For every $\alpha \ge 0$ the function $w_\alpha(x) = (1 + \tau(x))^\alpha$ is submultiplicative. Moreover, there are

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constants c, C, a, $b > 0$ such that $a < 1$, $2 < b$, and

$$c\tau(x)^a \le |x| \le C\tau(x)^b \quad \text{for } |x| > 1, \tag{1}$$

cf. [4, Lemma 1.1]. Denote by M_α the *algebra of Borel measures μ on $\mathcal{N}$ such that $\int \omega_\alpha(x)\, d|\mu|(x) < \infty$. If A is a selfadjoint operator on $L^2(\mathcal{N})$, E_A is its spectral resolution, and m is a bounded function on $\mathbf{R}$, then denote by $m(A)$ the operator $\int_{\mathbf{R}} m(\lambda)\, dE_A(\lambda)$. If $Af = f * \psi$ then $m(\psi)$ is the abbreviation for $m(A)$. The following theorem has been proved in [4]

Theorem 2. *Suppose that* $\psi = \psi^* \in M_\alpha \cap L^2(\mathcal{N})$, $\alpha > \beta + \frac{b}{2}Q + 2$, $m \in C_c^k(\mathbf{R})$ *with* $k > 3(\beta + \frac{b}{2}Q + 3)$, $m(0) = 0$, *then there is a measure* ν *in* M_β *such that* $m(\psi)f = f * \nu$.

The theorem below, which has been actually proved in [1], plays here the fundamental role.

Theorem 3. *For every natural* $N > Q$, *let* $T_t^{(N)}$ *be the semigroup of operators on* $L^2(\mathcal{N})$ *generated by* P^N. *Then* $T_t^{(N)} f = f * q_t^{(N)}$, *where* $q_t^{(N)} \in C^\infty(\mathcal{N})$ *and* $\sup_{x \in \mathcal{N}} |\partial q_t^{(N)}(x)|(1 + |x|)^{N+Q} \le C(t, \partial, N) < \infty$ *for every left-invariant differential operator* ∂ *on* $\mathcal{N}$.

Proposition 1. *If* $m \in C_c^\infty(-1, 1)$, *then for every natural* $N > (\beta + \frac{b}{2}Q + 2)/a$ *there is a unique function* ϕ *such that* $m(P^N) = f * \phi$. *Moreover, for every left-invariant differential operator* ∂ *on* $\mathcal{N}$, $\sup_{x \in \mathcal{N}} |\partial\phi(x)| w_\beta(x) < \infty$.

Proof. Let $n(\lambda) = m(-\log\lambda)/\lambda$. Then $n \in C_c^\infty(e^{-1}, e)$. Moreover, $m(P^N)f = T_1^{(N)} n(q_1^{(N)})f$. By Theorems 2, 3 and (1), $n(q_1^{(N)})f = f * \nu$ holds with $\nu \in M_\beta$. Using Theorem 3 one obtains that $\phi = \nu * q_1^{(N)}$ satisfies required conditions.

Proof of Theorem 1. In virtue of (1) it suffices to show that there is a C^∞ function ϕ such that for every $\beta > 0$ and every left-invariant differential operator ∂, one has $\int_0^\infty F(\lambda)\, dE_P(\lambda) f = f * \phi$ and $\sup_{x \in \mathcal{N}} |\partial\phi(x)| w_\beta(x) < \infty$. But for every natural N

$$\int_0^\infty F(\lambda)\, dE_P(\lambda) f = \int_0^\infty F_N(\lambda)\, dE_{P^N}(\lambda) f, \tag{2}$$

where $F_N(\lambda) = F(|\lambda|^{1/N})$. By the definition of F, $F_N \in C_c^\infty(-1, 1)$. Using (2) and Proposition 1 with sufficiently large N, the required estimate follows.

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Mathematical Institute, Wrocław University, pl. Grunwaldzki 2/4, 50–384 Wrocław, Poland