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COMPLEX CYCLES ON REAL ALGEBRAIC MODELS OF A SMOOTH MANIFOLD

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ABSTRACT. Let M be a compact connected orientable C^∞ submanifold of $\mathbb{R}^n$ with $2\dim M + 1 \le n$. Let G be a subgroup of $H^2(M, \mathbb{Z})$ such that the quotient group $H^2(M, \mathbb{Z})$ has no torsion. Then M can be approximated in $\mathbb{R}^n$ by a nonsingular algebraic subset X such that $H^2_{\mathbb{C}\text{-alg}}(X, \mathbb{Z})$ is isomorphic to G. Here $H^2_{\mathbb{C}\text{-alg}}(X, \mathbb{Z})$ denotes the subgroup of $H^2(X, \mathbb{Z})$ generated by the cohomology classes determined by the complex algebraic hypersurfaces in a complexification of X.

INTRODUCTION

In [6, 14] we have defined a contravariant functor $H^{\text{even}}_{\mathbb{C}\text{-alg}}(\cdot, \mathbb{Z})$ from affine nonsingular real algebraic varieties to graded rings. If X is an affine nonsingular real algebraic variety, then $H^{\text{even}}_{\mathbb{C}\text{-alg}}(X, \mathbb{Z}) = \bigoplus_{k\ge0} H^{2k}_{\mathbb{C}\text{-alg}}(X, \mathbb{Z})$ is a graded subring of the graded ring $H^{\text{even}}(X, \mathbb{Z}) = \bigoplus_{k\ge0} H^{2k}(X, \mathbb{Z})$ generated, roughly speaking, by the cohomology classes determined by the complex algebraic cycles on a nonsingular complexification of X (cf. [7] and §1 for details). The functor $H^{\text{even}}_{\mathbb{C}\text{-alg}}(\cdot, \mathbb{Z})$ has played a crucial role in the study of vector bundles over real algebraic varieties [6, 9, 14] and real algebraic morphisms [11, 12]. Further applications will be discussed in our subsequent papers.

A well-known theorem of Tognoli (cf. [7, Theorem 14.1.10 or [20] or [25]) asserts that given a compact C^∞ submanifold M of $\mathbb{R}^n$ with $2\dim M+1 \le n$, one can find a C^∞ embedding $e: M \to \mathbb{R}^n$, arbitrarily close in the C^∞ topology to the inclusion mapping $M \hookrightarrow \mathbb{R}^n$ such that $X = e(M)$ is a nonsingular algebraic subvariety of $\mathbb{R}^n$ (cf. also recent result [3, 24]). The variety X is referred to as an algebraic subvariety of $\mathbb{R}^n$ approximating M. It is known that the class $\mathscr{A}(M)$ of algebraic subvarieties of $\mathbb{R}^n$ approximating M contains infinitely many varieties that are mutually birationally nonisomorphic (cf. [10, Corollary 3.3]). The study of algebro-geometric properties of the varieties in $\mathscr{A}(M)$ is important for better understanding relations between differential topology and real algebraic geometry (cf. [2, 1, 6, 12, 9]). In this paper we investigate the behavior of the groups $H^{2k}_{\mathbb{C}\text{-alg}}(X, \mathbb{Z})$ for X in $\mathscr{A}(M)$. We prove

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that for some X in $\mathscr{A}(M)$, one has $H^{2k}_{\mathbb{C}\text{-alg}}(X, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q} = H^{2k}(X, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q}$ for all $k \geq 0$ (cf. Theorem 1.1). Also, given a subgroup G of $H^2(M, \mathbb{Z})$ such that the factor group $H^2(M, \mathbb{Z})/G$ is torsionfree, and assuming M connected and orientable, we show the existence of X in $\mathscr{A}(M)$ with $H^2_{\mathbb{C}\text{-alg}}(X, \mathbb{Z})$ isomorphic to G (cf. Theorem 1.2).

For the background material on real algebraic geometry the reader may consult [7]. Unless otherwise stated, algebraic subvarieties are assumed to be Zariski closed in the ambient variety.

Main result

We first recall a definition of the functor $H^{\text{even}}_{\mathbb{C}\text{-alg}}(\cdot, \mathbb{Z})$.

Let X be an affine nonsingular real algebraic variety. Let $\varphi\colon X \to \mathbb{R}P^n$ be an algebraic embedding of X in the real projective space $\mathbb{R}P^n$, that is, $\varphi(X)$ is a Zariski locally closed algebraic subvariety of $\mathbb{R}P^n$ and φ induces a biregular isomorphism from X onto $\varphi(X)$. Consider $\mathbb{R}P^n$ as a subset of the complex projective space $\mathbb{C}P^n$, and let V be the Zariski (complex) closure of $\varphi(X)$ in $\mathbb{C}P^n$. Let U be a Zariski (complex) neighborhood of $\varphi(X)$ in the set of nonsingular points of V. Given a complex algebraic subvariety W of U, we let $[W]$ denote its fundamental class in the Borel-Moore homology group $H^{\text{BM}}_*(W, \mathbb{Z})$ of W (cf. [13] or [16, Chapter 19]). Let a_W be the element of the cohomology group $H^*(U, \mathbb{Z})$ Poincaré dual to the image of $[W]$ under the homomorphism $H^{\text{BM}}_*(W, \mathbb{Z}) \to H^{\text{BM}}_*(U, \mathbb{Z})$ induced by the inclusion mapping $W \hookrightarrow U$. Denote by $H^{2k}_{\text{alg}}(U, \mathbb{Z})$ the subgroup of $H^{2k}(U, \mathbb{Z})$ generated by all elements of the form a_w, where W runs through the set of complex algebraic subvarieties of U of codimension k. It is known that $H^{\text{even}}_{\text{alg}}(U, \mathbb{Z}) = \bigoplus_{k\geq 0} H^{2k}_{\text{alg}}(U, \mathbb{Z})$ is a graded subring of $H^{\text{even}}(U, \mathbb{Z})$ (cf. [13] or [16]). Define $\varphi_U\colon X \to U$ by $\varphi_U(x) = \varphi(x)$ for x in X and set

$$H^{2k}_{\mathbb{C}\text{-alg}}(X, \mathbb{Z}) = \varphi^*_U(H^{2k}_{\text{alg}}(U, \mathbb{Z})),$$
$$H^{\text{even}}_{\mathbb{C}\text{-alg}}(X, \mathbb{Z}) = \bigoplus_{k\geq 0} H^{2k}_{\mathbb{C}\text{-alg}}(X, \mathbb{Z}).$$

One shows that the graded ring $H^{\text{even}}_{\mathbb{C}\text{-alg}}(X, \mathbb{Z})$ does not depend on the choice of the embedding φ and the neighborhood U (cf. [6, §3]).

The group $H^2_{\mathbb{C}\text{-alg}}(X, \mathbb{Z})$ has a simple, purely algebraic, interpretation. Namely, let $\mathscr{R}(X, \mathbb{C})$ denote the ring of regular functions from X to $\mathbb{C}$ (cf. [7, Chapter 12]). Then $H^2_{\mathbb{C}\text{-alg}}(X, \mathbb{Z})$ is canonically isomorphic to the Picard group $\text{Pic}(\mathscr{R}(X, \mathbb{C}))$ of $\mathscr{R}(X, \mathbb{C})$ (cf. [6, Remark 5.4]).

If $f\colon X \to Y$ is a regular mapping between affine nonsingular real algebraic varieties, then the induced homomorphism $f^*\colon H^*(Y, \mathbb{Z}) \to H^*(X, \mathbb{Z})$ maps $H^{\text{even}}_{\mathbb{C}\text{-alg}}(Y, \mathbb{Z})$ into $H^{\text{even}}_{\mathbb{C}\text{-alg}}(X, \mathbb{Z})$ (cf. [6, §3]).

We identify $H^*(X, \mathbb{Z}) \otimes \mathbb{Q}$ with $H^*(X, \mathbb{Q})$ and consider $H^{\text{even}}_{\mathbb{C}\text{-alg}}(X, \mathbb{Q}) = H^{\text{even}}_{\mathbb{C}\text{-alg}}(X, \mathbb{Z}) \otimes \mathbb{Q}$ as a subring of $H^{\text{even}}(X, \mathbb{Q})$.

Now we can state our results.

Theorem 1.1. *Let M be a compact C^∞ submanifold of $\mathbb{R}^n$ with $2\dim M + 1 \leq n$. Then there exists a C^∞ embedding $e\colon M \to \mathbb{R}^n$, arbitrarily close in the C^∞ topology to the inclusion mapping $M \hookrightarrow \mathbb{R}^n$, such that $X = e(M)$ is a* *nonsingular algebraic subvariety of* $\mathbb{R}^n$ *and the following conditions are satisfied*:

(i) $H^{\text{even}}_{\mathbb{C}\text{-alg}}(X, \mathbb{Q}) = H^{\text{even}}(X, \mathbb{Q})$,

(ii) $H^2_{\mathbb{C}\text{-alg}}(X, \mathbb{Z}) = H^2(X, \mathbb{Z})$.

This theorem can be easily derived from [6] and is recorded here for the sake of completeness, since our more interesting and more precise result below concerns only $H^2_{\mathbb{C}\text{-alg}}$.

Theorem 1.2. *Let* M *be a compact connected orientable* C^∞ *submanifold of* $\mathbb{R}^n$ *with* $2\dim M + 1 \le n$. *Let* G *be a subgroup of* $H^2(M, \mathbb{Z})$ *such that the quotient group* $H^2(M, \mathbb{Z})/G$ *is torsionfree. Then there exists a* C^∞ *embedding* $e\colon M \to \mathbb{R}^n$, *arbitrarily close in the* C^∞ *topology to the inclusion mapping* $M \hookrightarrow \mathbb{R}^n$, *such that* $X = e(M)$ *is a nonsingular algebraic subvariety of* $\mathbb{R}^n$ *and* $h^*(G) = H_{\mathbb{C}\text{-alg}}(X, \mathbb{Z})$ *where* $h\colon X \to M$ *is the* C^∞ *diffeomorphism defined by* $h(e(m)) = m$ *for* m *in* M.

As an application we immediately obtain the following

Corollary 1.3. *Let* M *be a compact connected orientable* C^∞ *submanifold of* $\mathbb{R}^n$ *with* $2\dim M + 1 \le n$ *and the cohomology group* $H^2(M, \mathbb{Z})$ *torsionfree. Then there exists a* C^∞ *embedding* $e\colon M \to \mathbb{R}^n$, *arbitrarily close in the* C^∞ *topology to the inclusion mapping* $M \hookrightarrow \mathbb{R}^n$, *such that* $X = e(M)$ *is a nonsingular algebraic subvariety of* $\mathbb{R}^n$ *and the ring* $\mathscr{R}(X, \mathbb{C})$ *is a unique factorization domain.*

Proof. By Theorem 1.2 there exists a C^∞ embedding $e\colon M \to \mathbb{R}^n$, arbitrarily close in the C^∞ topology to the inclusion mapping $M \hookrightarrow \mathbb{R}^n$, such that $X = e(M)$ is a nonsingular algebraic subvariety of $\mathbb{R}^n$ and $H^2_{\mathbb{C}\text{-alg}}(X, \mathbb{Z}) = 0$. Since $H^2_{\mathbb{C}\text{-alg}}(X, \mathbb{Z})$ is isomorphic to $\operatorname{Pic}(\mathscr{R}(X, \mathbb{C}))$, we obtain $\operatorname{Pic}(\mathscr{R}(X, \mathbb{C})) = 0$. It follows that the ring $\mathscr{R}(X, \mathbb{C})$ is a unique factorization domain. □

Remark 1.4. In Corollary 1.3 the assumption that $H^2(M, \mathbb{Z})$ is torsionfree cannot be dropped. Indeed, if M is a compact, connected, nonorientable surface of odd genus, then $H^2_{\mathbb{C}\text{-alg}}(X, \mathbb{Z}) = H^2(X, \mathbb{Z}) \cong \mathbb{Z}/2$ for each affine nonsingular real algebraic variety X homeomorphic to M (cf. [8, Theorem 1.7(iii); 9, Proposition 1.2]). □

2. Proof of Theorem 1.1

Let X be an affine real algebraic variety. Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. An algebraic $\mathbb{F}$-vector bundle over X is said to be *strongly algebraic* if it is algebraically isomorphic to a subbundle of a product vector bundle $X \times \mathbb{F}^n$ for some n (cf. [7, Chapter 12] for an exposition of the theory of strongly algebraic vector bundles). A topological $\mathbb{F}$-vector bundle over X is said to admit an *algebraic structure* if it is topologically isomorphic to a strongly algebraic vector bundle over X (cf. [6]). Denote by $K_{\mathbb{F}}(X)$ the Grothendieck group of topological $\mathbb{F}$-vector bundles over X (cf. [19]) and by $K_{\mathbb{F}\text{-alg}}(X)$ the subgroup of $K_{\mathbb{F}}(X)$ generated by the classes of $\mathbb{F}$-vector bundles admitting an algebraic structure.

Proof of Theorem 1.1. By [4, Theorem 4.2] there exists a C^∞ embedding $e\colon M \to \mathbb{R}^n$, arbitrarily close in the C^∞ topology to the inclusion mapping $M \hookrightarrow \mathbb{R}^n$, such that $X = e(M)$ is a nonsingular algebraic subvariety of $\mathbb{R}^n$ and

$K_{\mathbb{F}\text{-alg}}(X) = K_{\mathbb{F}}(X)$. (Strictly speaking, the above statement is proved in [4] for $\mathbb{F} = \mathbb{R}$. However, in order to obtian a proof for $\mathbb{F} = \mathbb{C}$ only a straightforward modification is required. Below we use this result with $\mathbb{F} = \mathbb{C}$.) It is known that the isomorphism

$$\mathrm{ch}\colon K_{\mathbb{C}}(X) \otimes \mathbb{Q} \to H^{\mathrm{even}}(X, \mathbb{Q})$$

induced by the Chern character (cf. [21]) maps $K_{\mathbb{C}\text{-alg}}(X) \otimes \mathbb{Q}$ onto $H^{\mathrm{even}}_{\mathbb{C}\text{-alg}}(X, \mathbb{Q})$ (cf. [6, Proposition 5.9]). Hence (i) follows at once.

Since the homomorphism $c_1\colon K_{\mathbb{C}}(X) \to H^2(X, \mathbb{Z})$ determined by the first Chern class is surjective (cf. [19, Chapter 16, Theorem 3.4]) and maps $K_{\mathbb{C}\text{-alg}}(X)$ on to $H^2_{\mathbb{C}\text{-alg}}(X, \mathbb{Z})$ (cf. [6, Theorem 5.2 and Remark 5.4]), we also immediately obtain (ii). $\square$

3. Proof of Theorem 1.2

We first need a few auxiliary results.

Let, as before, $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Denote by $G_{n,p}(\mathbb{F})$ the Grassmann variety of p-dimensional vector subspaces of $\mathbb{F}^n$. Recall that $G_{n,p}(\mathbb{F})$ is an affine nonsingular real algebraic variety [7, Theorem 3.4.4, Proposition 3.4.8], and the universal vector bundle $\gamma_{n,p}(\mathbb{F})$ over $G_{n,p}(\mathbb{F})$ is strongly algebraic [7, Theorem 12.1.7, p. 280].

Let X be an affine nonsingular real algebraic variety. An algebraic subvariety Y of X is said to be *quasi-regular* if for each point x in X the ideal of real analytic function-germs $(X, x) \to \mathbb{R}$ vanishing on Y is generated by regular functions on X vanishing on Y. One can show that a union of finitely many nonsingular algebraic subvarieties of X is quasi-regular [23, p. 75].

Lemma 3.1. *Let X be a compact affine nonsingular real algebraic variety, and let Y be a quasi-regular algebraic subvariety of X. Let $f\colon X \to G_{n,p}(\mathbb{F})$ be a C^∞ mapping whose restriction $f|Y$ to Y is a regular mapping. If the pullback vector bundle $f^*\gamma_{n,p}(\mathbb{F})$ admits an algebraic structure, then there exists a regular mapping $g\colon X \to G_{n,p}(\mathbb{F})$, arbitrarily close in the C^0 topology to f, such that $g|Y = f|Y$.*

Proof. Let $\gamma = \gamma_{n,p}(\mathbb{F})$, and let ξ be a strongly algebraic $\mathbb{F}$-vector bundle over X which is topologically, hence also C^∞, isomorphic to $f^*\gamma$. Take a C^∞ section $u\colon X \to \mathrm{Hom}(\xi, f^*\gamma)$ of $\mathrm{Hom}(\xi, f^*\gamma)$ such that $u(x)\colon \xi_x \to (f^*\gamma)_x$ is an isomorphism of fibres for all x in X. Since $f|Y$ is a regular mapping, it follows that $\mathrm{Hom}(\xi, f^*\gamma)|Y$ is a strongly algebraic $\mathbb{F}$-vector bundle over Y (cf. [7, Proposition 12.1.8]). By [7, Theorem 12.3.1] $u|Y$ can be approximated in the C^0 topology by regular sections $v\colon Y \to \mathrm{Hom}(\xi, f^*\gamma)|Y$. If v is sufficiently close to $u|Y$, then it extends to a C^∞ section, say w, of $\mathrm{Hom}(\xi, f^*\gamma)$ such that w is close in the C^0 topology to u. In particular we may assume that $w(x)\colon \xi_x \to (f^*\gamma)_x$ is an isomorphism for all x in X.

Consider $f^*\gamma$ as a $C^\infty\mathbb{F}$-vector subbundle of the product vector bundle $\varepsilon = X \times \mathbb{F}^n$, and let $i\colon f^*\gamma \to \varepsilon$ be the inclusion homomorphism. By [24] there exists a regular section $s\colon X \to \mathrm{Hom}(\xi, \varepsilon)$, arbitrarily close in the C^∞ topology to $i \circ w$, such that $s|Y = i \circ w|Y$ (in particular, we may assume that $s(x)\colon \xi_x \to \varepsilon_x$ is a monomorphism for all x in X). Define $g\colon X \to G_{n,p}(\mathbb{F})$ by $g(x) = \rho(s(x)(\xi_x))$ for x in X, where $\rho\colon X \times \mathbb{F}^n \to \mathbb{F}^n$ is the standard projection. By construction, g is a regular mapping (cf. [7, Proposition 3.4.9]) close in the C^0 topology to f and $g|Y = f|Y$. □

Lemma 3.2. *Let M be a compact C^∞ submanifold of $\mathbb{R}^q$. Let $Y_1, \ldots, Y_k$ be compact C^∞ submanifolds of M that are in general position (considered as submanifolds of M). Let $f: M \to G_{n,p}(\mathbb{F})$ be a C^∞ mapping. Assume that Y_i is a nonsingular algebraic subvariety of $\mathbb{R}^q$ and the pullback vector bundle $(f|Y_i)^*\gamma_{n,p}(\mathbb{F})$ over Y_i, $i = 1, \ldots, k$, admits an algebraic structure. Then there exists a C^∞ mapping $g: M \to G_{n,p}(\mathbb{F})$, arbitrarily close in the C^0 topology to f, such that $g|Y$ is a regular mapping where $Y = Y_1 \cup \cdots \cup Y_k$.*

Proof. Set $Y^0 = \varnothing$ and $Y^l = Y_1 \cup \cdots \cup Y_l$ for $l = 1, \ldots, k$. Let us fix l, $0 \le l \le k-1$, and suppose that one can find a C^∞ mapping $g^l: M \to G_{n,p}$, arbitrarily close in the C^0 topology to f, such that $g^l|Y^l$ is a regular mapping. We claim that there exists a C^∞ mapping $g^{l+1}: M \to G_{n,p}(\mathbb{F})$, arbitrarily close in the C^0 topology to f, such that $g^{l+1}|Y^{l+1}$ is a regular mapping.

Indeed, observe that $Y_1 \cap Y_{l+1}, \ldots, Y_l \cap Y_{l+1}$ are nonsingular subvarieties of Y_{l+1} that are in general position (considered as C^∞ submanifolds to Y_{l+1}). In particular, by Lemma 3.1, there exists a regular mapping $\varphi: Y_{l+1} \to G_{n,p}(\mathbb{F})$, arbitrarily close in the C^0 topology to $g^l|Y_{l+1}$, such that $\varphi|Z = g^l|Z$ where $Z = \bigcup_{i=1}^{l}(Y_i \cap Y_{l+1})$. Let $h: Y^{l+1} \to G_{n,p}(\mathbb{F})$ be a mapping defined by $h|Y^l = g^l|Y^l$, $h|Y_{l+1} = \varphi$. Since, obviously, h is continuous, it follows from [4, Lemma 3], that h is regular. It suffices to define g^{l+1} to be a C^∞ extension to M of h that is close in the C^0 topology to g^l. Thus the claim is proved.

Now Lemma 3.2 follows from the claim by induction. □

Lemma 3.3. *Let S be a compact orientable C^∞ surface in $\mathbb{R}^n$. Assume that S is connected if $n = 5$. Then there exists a C^∞ embedding $f: S \to \mathbb{R}^n$, arbitrarily close in the C^∞ topology to the inclusion mapping $S \hookrightarrow \mathbb{R}^n$, such that $Y = f(S)$ is a nonsingular algebraic subvariety of $\mathbb{R}^n$ with $H^2_{\mathbb{C}\text{-alg}}(Y, \mathbb{Z}) = 0$.*

Proof. Let $c = \operatorname{codim} S$. We claim that there exist compact C^∞ hypersurfaces $H_1, \ldots, H_c$ of $\mathbb{R}^n$ that are in general position at each point of S and satisfy $S = H_1 \cap \cdots \cap H_c$.

If $c = 1$, the claim is obvious. Let us assume then that $c \ge 2$, that is, $n \ge 4$. Identify $\mathbb{R}^n$, via the stereographic projection, with $S^n \backslash \{a\}$, where S^n is the n-dimensional unit sphere and $a = (0, \ldots, 0, 1)$. Given a positive integer m with $m < n$, we also identify S^m with $\{(x_0, \ldots, x_n) \in S^n | x_{m+1} = \cdots = x_n = 0\}$. Observe that one can find a C^∞ diffeomorphism $h: S^n \to S^n$ such that $h(S)$ is contained in $S^4 \subset S^n$. (This is a standard result for $n \ge 6$ [18] and it follows from [13, p. 43] for $n = 5$; to apply [13], we use the connectedness of S.) Replacing S by $h(S)$, we may assume that S is contained in S^4. This reduces the proof of the claim to the case $c = 2$. One easily sees that the normal vector bundle of S in S^4 is trivial. It follows that there exist a C^∞ mapping $g: S^4 \to S^2$ and a regular value b of g such that $S = g^{-1}(b)$. Let η be an orientable C^∞ real vector bundle over S^2, and let u be a C^∞ section of η such that $\operatorname{rank} \eta = 2$, u is transverse to the zero section of η, and the set of zeros $u^{-1}(0)$ of u is equal to $\{b\}$. Then the pullback section g^*u of the pullback vector bundle $g^*\eta$ is transverse to the zero section of $g^*\eta$ and $(g^*u)^{-1}(0) = S$. Since $H^2(S^4, \mathbb{Z}) = 0$, the vector bundle $g^*\eta$ is trivial (cf. [19, Chapter 16, Theorem 3.4]), and hence the claim follows.

Now the conclusion of Lemma 3.3 follows at once from the claim and [6, Theorem 7.1, Remark 5.4]. □

We still need a few preliminary remarks.

If $\mathbb{C}P^q$ is considered as a real algebraic variety, then $H^2_{\mathbb{C}\text{-alg}}(\mathbb{C}P^q, \mathbb{Z}) = H^2(\mathbb{C}P^q, \mathbb{Z})$. Indeed, the universal complex bundle γ_q over $\mathbb{C}P^q$ is a strongly algebraic complex line bundle and hence its first Chern class $c_1(\gamma_q)$ is in $H^2_{\mathbb{C}\text{-alg}}(\mathbb{C}P^q, \mathbb{Z})$ (cf. [6, Theorem 5.3]). Now our claim is obvious since $c_1(\gamma_q)$ generates $H^2(\mathbb{C}P^q, \mathbb{Z})$. It easily follows that $H^2_{\mathbb{C}\text{-alg}}(P, \mathbb{Z}) = H^2(P, \mathbb{Z})$ where $P = \mathbb{C}P^{q(1)} \times \cdots \times \mathbb{C}P^{q(k)}$. Furthermore, every homology class in $H_l(P, \mathbb{Z}/2)$, $l \geq 0$, can be represented by a nonsingular real algebraic subvariety of P. This last fact implies that every unoriented bordism class of P is represented by a regular mapping $f: X \to P$ where X is a compact affine nonsingular real algebraic variety [2, 20].

Proof of Theorem 1.2. Let $m = \dim M$. If $m \leq 1$, then $H^2(M, \mathbb{Z}) = 0$ and Theorem 1.2 is just a very special case of Tognoli's theorem (cf. Introduction), while for $m = 2$ it follows from Lemma 3.3. Thus we assume below that $m \geq 3$.

Since the group $H^2(M, \mathbb{Z})/G$ is torsionfree, there exists a torsionfree subgroup H of $H^2(M, \mathbb{Z})$ such that $H^2(M, \mathbb{Z}) = G \oplus H$. The bilinear mapping

$$H^2(M, \mathbb{Q}) \times H^{m-2}(M, \mathbb{Q}) \to H^m(M, \mathbb{Q}), \qquad (u, v) \to u \cup v,$$

where $\cup$ stands for the cup product, is a dual pairing (cf. [15, Proposition 8.13]), and hence one can find elements $v_1, \ldots, v_k$ in $H^{m-2}(M, \mathbb{Z})$ with the property that for a given element a in $H^2(M, \mathbb{Z})$ there is $a \cup v_i = 0$ for $i = 1, \ldots, k$ if and only if a is in G. Let α_i be the homology class in $H_2(M, \mathbb{Z})$ Poincaré dual to v_i. By Thom's theorem [22, Theorem II.26], α_i can be represented by a compact oriented C^∞ surface S_i in M. Clearly, we may assume that the surfaces S_i are in general position.

Note that if $r_i: H^2(M, \mathbb{Z}) \to H^2(S_i, \mathbb{Z})$ is the homomorphism induced by the inclusion mapping $S_i \hookrightarrow M$, then given an element w in $H^2(M, \mathbb{Z})$, we have

(1) $$r_i(w) = 0 \quad \text{for } i = 1, \ldots, k \text{ if and only if } w \in G.$$

Moreover, by Lemma 3.3, there exists a C^∞ embedding $f_i: S_i \to \mathbb{R}^n$, arbitrarily close in the C^∞ topology to the inclusion mapping $S_i \hookrightarrow \mathbb{R}^n$, such that $f_i(S_i)$ is a nonsingular algebraic subvariety of $\mathbb{R}^n$ and $H^2_{\mathbb{C}\text{-alg}}(f_i(S_i), \mathbb{Z}) = 0$. Now one can find a C^∞ embedding $F: M \to \mathbb{R}^n$, close in the C^∞ topology to the inclusion mapping $M \hookrightarrow \mathbb{R}^n$, such that $F|S_i = f_i$ for $i = 1, \ldots, k$ (cf. [2, Lemma 2.9]). Hence replacing possibly M by $F(M)$, we may assume that S_i is a nonsingular algebraic subvariety of $\mathbb{R}^n$ and

(2) $$H^2_{\mathbb{C}\text{-alg}}(S_i, \mathbb{Z}) = 0 \quad \text{for } i = 1, \ldots, k.$$

Let $a_1, \ldots, a_l$ be generators of G. For each $j = 1, \ldots, l$, one can find a positive integer $q(j)$ and a C^∞ mapping $\varphi_j: M \to \mathbb{C}P^{q(j)}$ such that $\varphi_j^*(z_j) = a_j$ where z_j is a generator of $H^2(\mathbb{C}P^{q(j)}, \mathbb{Z})$. Setting $P = \mathbb{C}P^{q(1)} \times \cdots \times \mathbb{C}P^{q(l)}$ and $\varphi = (\varphi_1, \ldots, \varphi_l)$, we get

(3) $$\varphi^*(H^2(P, \mathbb{Z})) = G.$$

Note that if γ^j is the universal complex line bundle over $\mathbb{C}P^{q(j)}$, then in virtue of (1) and (3), we have $c_1((\varphi_j|S_i)^*\gamma^j) = 0$ in $H^2(S_i, \mathbb{Z})$, and hence the complex line bundle $(\varphi_j|S_i)^*\gamma^j$ over S_i is topologically trivial (cf. [19, Chapter 16, Theorem 3.4]). This implies (cf. Lemma 3.2) that there exists a C^∞ mapping $\sigma: M \to P$, arbitrarily close in the C^0 topology to φ, such that $\sigma|S$ is a regular mapping where $S = S_1 \cup \cdots \cup S_k$ (in particular, we may assume that σ is homotopic to φ). Hence replacing possibly φ by σ, we may assume that $\varphi|S$ is a regular mapping.

Let $g: M \to G_{n,n-m}(\mathbb{R})$ be the C^∞ mapping defined by

$$g(x) = \text{the orthogonal complement of } T_xM \text{ in } \mathbb{R}^n$$

for x in M, where T_xM is the tangent space to M at x. Note that $\nu_i = (g^*\gamma_{n,n-m}(\mathbb{R}))|S_i$ is the restriction to S_i of the normal vector bundle of M in $\mathbb{R}^n$. Since M is orientable, the vector bundle ν_i is also orientable and hence admits an algebraic structure (cf. [7, Corollary 12.5.4]). By Lemma 3.2 $g|S$ can be approximated in the C^0 topology by regular mappings.

Now the properties of φ, S_i, and P established above and the fact that each unoriented bordism class of P is represented by a regular mapping (cf. the paragraph preceding the proof of Theorem 1.2), allow us to apply the approximation theorem [4, Theorem 3] (cf. also [2, Propositon 2.8; 23]). It follows that one can find a C^∞ embedding $e: M \to \mathbb{R}^n$, arbitrarily close in the C^∞ topology to the inclusion mapping $M \hookrightarrow \mathbb{R}^n$, such that $X = e(M)$ is a nonsingular algebraic subvariety of $\mathbb{R}^n$, $e(x) = x$ for x in S, and there exists a regular mapping $\psi: X \to P$ with $\psi \circ e$ close in the C^∞ topology to φ (in particular, we may assume that $\psi \circ e$ is homotopic to φ). Let $h: X \to M$ be the C^∞ diffeomorphism defined by $h(e(m)) = m$ for m in M. By construction $h^*(G) = \psi^*(H^2(P, \mathbb{Z}))$. Since ψ is a regular mapping and $H^2_{\mathbb{C}\text{-alg}}(P, \mathbb{Z}) = H^2(P, \mathbb{Z})$, we obtain that $h^*(G)$ is contained in $H^2_{\mathbb{C}\text{-alg}}(X, \mathbb{Z})$. Moreover, if $\rho_i: H^2(X, \mathbb{Z}) \to H^2(S_i, \mathbb{Z})$ is the homomorphism induced by the inclusion mapping, then given an element w in $H^2(X, \mathbb{Z})$ we have

$$\rho_i(w) = 0 \quad \text{for } i = 1, \ldots, k \text{ if and only if } w \in h^*(G)$$

(cf. (1)). If w is in $H^2_{\mathbb{C}\text{-alg}}(X, \mathbb{Z})$, then by (2), $\rho_i(w) = 0$ for $i = 1, \ldots, k$, and hence w belongs to $h^*(G)$.

Thus, finally, $h^*(G) = H^2_{\mathbb{C}\text{-alg}}(X, \mathbb{Z})$, which finishes the proof. □

References

1. S. Akbulut and H. King, *A relative Nash theorem*, Trans. Amer. Math. Soc. **267** (1981), 465–481.
2. ——, *The topology of real algebraic sets with isolated singularities*, Ann. of Math. (2) **113** (1981), 425–426.
3. ——, *On approximating submanifolds by algebraic sets*, preprint, 1989.
4. R. Benedetti and A. Tognoli, *Approximation theorems in real algebraic geometry*, Algebra é Geometria, Boll. Un. Mat. Ital. Suppl. **2** (1980), 209–228.
5. ——, *On real algebraic vector bundles*, Bull. Sci. Math. (2) **104** (1980), 89–112.
6. J. Bochnak, M. Buchner, and W. Kucharz, *Vector bundles over real algebraic varieties*, *K*-Theory J. **3** (1990), 271–298.
7. J. Bochnak, M. Coste, and M.-F. Roy, *Géométrie algébrique réelle*, Ergeb. Math. Grenzgeb. (3), vol. 12, Springer-Verlag, New York and Berlin, 1987.

8. J. Bochnak and W. Kucharz, *Algebraic approximation of mappings into spheres*, Michigan Math. J. **34** (1987), 119–125.

9. ——, *Algebraic cycles and approximation theorems in real algebraic grometry*, preprint, 1989.

10. ——, *Algebraic models of smooth manifolds*, Invent. Math. **97** (1989), 585–611.

11. ——, *On real algebraic morphisms into even-dimensional spheres*, Ann. of Math. (2) **128** (1988), 415–433.

12. ——, *On vector bundles and real algebraic morphisms*, Real Analytic and Algebraic Geometry, Trento 1988, Lecture Notes in Math., vol. 1420, Springer-Verlag, Berlin and New York, 1990, pp. 65–71.

13. A Borel and A. Haefliger, *La classe d'homologie fondamentale d'un espace analytique*, Bull. Soc. Math. France **89** (1961), 461–513.

14. M. Buchner and W. Kucharz, *Algebraic vector bundles over real algebraic varieties*, Bull. Amer. Math. Soc. **17** (1987), 279–282.

15. A. Dold, *Lectures on algebraic topology*, Grundlehren Math. Wiss., vol. 200, Springer, Berlin and New York 1980.

16. W. Fulton, *Intersection theory*, Ergeb. Math. Grenzgeb. (3), vol. 2, Springer-Verlag, New York and Berlin, 1984.

17. A. Haefliger, *Plongement différentiables de variétés dans variétés*, Comment. Math. Helv. **36** (1961), 47–82.

18. M. Hirsch, *Differential topology*, Graduate Texts in Math., vol. 33, Springer, Berlin and New York 1976.

19. M. Husemoller, *Fibre bundles*, 2nd ed., Springer, Berlin and New York 1975.

20. N. Ivanov, *Approximation of smooth manifolds by real algebraic sets*, Russian Math. Surveys **37** (1982), 1–59.

21. M. Karoubi, *K-Theory, an introduction*, Springer, Berlin and New York 1978.

22. R. Thom, *Quelques propriétés globales des variétés differentiables*, Comment. Math. Helv. **28** (1954), 17–86.

23. A. Tognoli, *Algebraic approximation of manifolds and spaces*, Séminaire Bourbaki 32e année, 1979/1980, no. 548, Lecture Notes in Math., vol. 842, Springer, Berlin, Heidelberg and New York, 1981, pp. 73–94.

24. ——, *Any compact differential submanifold of* $\mathbb{R}^n$ *has an algebraic approximation in* $\mathbb{R}^n$, Topology **27** (1988), 205–210.

25. ——, *Su una congettura di Nash*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **27** (1973), 167–185.

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