

BALL COVERING ON MANIFOLDS WITH NONNEGATIVE RICCI CURVATURE NEAR INFINITY

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ABSTRACT. Let M be a complete open Riemannian manifold with nonnegative Ricci curvature outside a compact set B . We show that the following *ball covering property* (see [LT]) is true provided that the sectional curvature has a lower bound:

For a fixed $p_0 \in M$, there exist $N > 0$ and $r_0 > 0$ such that for $r \geq r_0$, there exist $p_1, \dots, p_k \in \partial B_{p_0}(2r)$, $k \leq N$, with

$$\bigcup_{j=1}^k B_{p_j}(r) \supset \partial B_{p_0}(2r).$$

Furthermore N and r_0 depend only on the dimension, the lower bound on the sectional curvature, and the radius of the ball at p_0 that contains B .

1. INTRODUCTION

On a complete open Riemannian manifold with nonnegative Ricci curvature, the volume comparison of Bishop and Gromov [BC, GLP] is a very powerful tool. It can be used to show that the geometric growth of its ends is well controlled; namely,

(a) The volume of the geodesic ball $B_p(r)$ is bounded above by $C \cdot r^n$, where n is the dimension of the manifold and C is the volume of the unit ball in R^n .

(b) The volume of $B_p(r)$ grows at least linearly.

(c) There are a bounded number of ends (Cheeger-Gromoll's splitting theorem implies that there are at most two ends).

(d) Every end has at most linear diameter growth ([AG]).

(e) There is a *ball covering property*: for a fixed point p and any large r , there are a bounded number of points $p_1, \dots, p_k \in \partial B_p(2r)$, where the number k is independent of r , such that

$$\partial B_p(2r) \subset \bigcup_{j=1}^k B_{p_j}(r).$$

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When the global condition $\text{Ric}_M \geq 0$ is relaxed to $\text{Ric}_{M-B} \geq 0$, where B is a compact set, the relative volume comparison is weakened. However, it is still fairly easy to show that (a) and (b) remain true, even though the constant C now depends on the manifold. Moreover, the ball covering property (e), if true, would imply (c) and (d).

In their paper [LT], P. Li and L. Tam show that (e) is true if the *sectional curvature* is nonnegative outside a compact set. This ball covering property plays a crucial role in their study of positive harmonic functions on their class of manifolds. In that paper they ask whether (e) remains true if sectional curvature is replaced by Ricci curvature.

Here, we will prove that if in addition to assuming $\text{Ric}_{M-B} \geq 0$, one assumes that the sectional curvature is bounded below, then (e) is true. Furthermore, the bound can be estimated in terms of the dimension, the lower bound on the sectional curvature, and the radius of the ball $B_p(D)$ that contains the compact set B .

Theorem. *Let M^n be a complete Riemannian manifold with nonnegative Ricci curvature outside a compact set B . Assume also that the sectional curvature has a lower bound $H > -\infty$. If $B \subset B_{p_0}(D)$, then there exist $N > 0$ and $r_0 > 0$ depending only on n and $H \cdot D^2$ such that for $r \geq r_0$, we can find $p_1, \dots, p_k \in \partial B_{p_0}(2r)$, $k \leq N$, with*

$$\bigcup_{j=1}^k B_{p_j}(r) \supset \partial B_{p_0}(2r).$$

2. PRELIMINARIES

Volume comparison. Let M be an n -dimensional Riemannian manifold. Let $B_p(r)$ denote the geodesic ball of radius r at $p \in M$. Put $A_p(R, r) := B_p(R) - \overline{B_p(r)}$, $R > r > 0$. $V_p(r) := \text{vol}(B_p(r))$, $V_p(R, r) := \text{vol}(A_p(R, r))$. We use $V^H(r)$ to denote the volume of a ball of radius r in the space form of constant curvature H of the same dimension. $V^H(R, r) := V^H(R) - V^H(r)$. Then we have the well-known relative volume comparisons (see [GLP]).

Lemma 1. *If on $B_p(R)$, $\text{Ric} \geq (n - 1)H$ and $0 < r < R$, then*

$$\begin{aligned} V_p(R)/V_p(r) &\leq V^H(R)/V^H(r), \\ V_p(R, r)/V_p(R) &\leq V^H(R, r)/V^H(R), \end{aligned}$$

and

$$V_p(R, r)/V_p(r) \leq V^H(R, r)/V^H(r).$$

There is also a relative volume comparison on star-shaped sets. A star-shaped set S_p at p is a set containing p such that whenever $x \in S_p$ is not on the cut-locus of p , any point on the minimal geodesic joining p and x is also in S_p . We then have [CGT, §4, Remark 4.1].

Lemma 2. *If S_p is a star-shaped set and $\text{Ric}|_{S_p} \geq (n - 1)H$, then for $R > r > 0$ we have*

$$\text{Vol}(S_p \cap B_p(R)) / \text{Vol}(S_p \cap B_p(r)) \leq V^H(R) / V^H(r).$$

Excess functions. The excess was first introduced in [AG].

Let $(\gamma_1, \gamma_2, \gamma_3)$, where each γ_i is minimal, form a geodesic triangle in M . $\gamma_1(0) = \gamma_2(0) = p$, $\alpha = \angle(\gamma'_1(0), \gamma'_2(0))$ and $l_i := \text{length of } \gamma_i$. The excess function is defined to be $e := l_1 + l_2 - l_3$. Let h denote the altitude of the triangle with respect to the side γ_3 , i.e., $h := d(p, \gamma_3)$. The triangle inequality implies that $h \geq e/2$.

Lemma 3. *Let M be the two-dimensional space form of constant curvature -1 . Suppose in the above geodesic triangle, $l_1 = l_2 = a$, $l_3 = c$. Then the excess function e has a limit, as $a \rightarrow \infty$, if $\alpha > 0$:*

$$\lim_{a \rightarrow \infty} e = 2 \ln(\csc(\alpha/2)).$$

When $\alpha = \pi$, $e \equiv 0$ and when $\alpha = 0$, $e \equiv 2a$.

Proof. If we draw the minimal geodesic τ from p to the middle of γ_3 , then the triangle is divided into two identical right triangles. We apply the law of sines in hyperbolic geometry to each right triangle:

$$\frac{\sinh(a)}{\sinh(c/2)} = \frac{\sin(\pi/2)}{\sin(\alpha/2)}.$$

Note that as $a \rightarrow \infty$, so does c . Taking limits in the above equation yields:

$$\lim_{a \rightarrow \infty} (a - c/2) = \ln(\csc(\alpha/2)).$$

Thus

$$\lim_{a \rightarrow \infty} e = \lim_{a \rightarrow \infty} (2a - c) = 2 \ln(\csc(\alpha/2)).$$

This completes the proof of Lemma 3.

Remark 1. We observe the following properties about the excess function e that will be useful later.

(i) By the triangle inequality, if α is held fixed while the lengths of γ_1 and γ_2 are increased, then the excess e , will increase as well.

Let M_{-1} be the two-dimensional space form of constant curvature -1 and M a complete manifold with curvature bounded below by -1 .

(ii) Construct a family of geodesic triangles $(\bar{\gamma}_1, \bar{\gamma}_2, \bar{\gamma}_3)$ in M_{-1} as described in Lemma 3, with \bar{l}_1, \bar{l}_2 fixed and $\bar{\alpha}$ variable. Then as $\bar{\alpha}$ decreases, \bar{e} increases.

(iii) If $(\gamma_1, \gamma_2, \gamma_3)$ and $(\bar{\gamma}_1, \bar{\gamma}_2, \bar{\gamma}_3)$ are geodesic triangles in M and M_{-1} respectively with

$$\text{length}(\gamma_i) = \text{length}(\bar{\gamma}_i), i = 1, 2, \text{ and } \alpha = \bar{\alpha},$$

then by Toponogov's Theorem, $e \geq \bar{e}$.

3. PROOF OF THE THEOREM

In this section we prove the theorem stated in §1.

We assume that all geodesics are parametrized by arclength.

Proof of the theorem. If $H \geq 0$, i.e., $K_M \geq 0$, the theorem is already known. So we assume that $H < 0$. If we multiply the metric on M by $\sqrt{-H}$, then for the new metric, $K \geq -1$, and $B \subset B_{p_0}(\sqrt{-H} \cdot D)$. This normalization does not affect the validity of the ball covering property. Thus we put $D_1 = \sqrt{-H} \cdot D$ and work with the rescaled metric.

Take an $\alpha > 0$ small enough so that in Lemma 3 we have $\lim_{a \rightarrow \infty} e \geq 4D_1$. Then there exists $\alpha_0 > 0$, such that

$$(1) \quad e(\alpha, a) > 3D_1, \quad a \geq \alpha_0.$$

Note that α is bounded below in terms of n and D_1 .

We now divide the manifold M into *cones* as follows: first divide the unit sphere in the tangent space at p_0 into m closed domains $\{U_1, \dots, U_m\}$ so that in each domain U_j , we have

$$\angle(v, w) \leq \alpha, \quad \forall v, w \in U_j.$$

Clearly $m = m(n, D_1)$ is bounded above in terms of n and D_1 . Define the cone K_j by:

$$x \in K_j \quad \text{if and only if there is a minimal geodesic } \gamma \text{ from } p_0 \\ \text{to } x \text{ with } \gamma'(0) \in U_j.$$

Now let $r \geq \max\{a_0, D_1\}$. On $\partial B_{p_0}(2r)$, take a maximal set of points $\{p_1, \dots, p_k\}$ such that $\text{dist}(p_i, p_j) \geq r$, $i \neq j$. Then

$$(2) \quad \bigcup B_{p_i}(r) \supset \partial B_{p_0}(2r);$$

$$(3) \quad B_{p_i}(r/2) \cap B_{p_j}(r/2) = \emptyset, \quad i \neq j.$$

To estimate the bound on k , we divide the balls $B_{p_i}(r/2)$ into m families as follows: for each ball $B_{p_i}(r/2)$, look at $\text{Vol}(B_{p_i}(r/2) \cap K_j)$, $j = 1, \dots, m$. Fix a j_i such that $\text{Vol}(B_{p_i}(r/2) \cap K_{j_i})$ is maximal. Then

$$(4) \quad \text{Vol}(B_{p_i}(r/2) \cap K_{j_i}) \geq \frac{1}{m} \text{Vol}(B_{p_i}(r/2)).$$

We denote

$$B_{p_i}(r/2) \cap K_{j_i} = B_{p_i}^{L, j_i},$$

or simply $B_{p_i}^L$, and place the ball $B_{p_i}(r/2)$ in the j_i th family.

Now we estimate the number of balls in a cone K_j . Suppose B_p^L has the smallest volume among all $B_{p_i}^{L, j}$ in this cone. Since p may not be in K_j , we need the following

Lemma 4. *Assume that $\text{Ric}|_{B_p(R)} \geq 0$. There exists a δ with $0 < \delta < 1$, depending only on n and m , such that whenever a subset $W \subset B_p(R)$ has*

$$\text{Vol}(W) \geq 1/m \cdot \text{Vol}(B_p(R)),$$

then there exists $q \in W$ such that $\text{dist}(q, p) \leq \delta R$. Hence $B_q((1 - \delta)R) \subset B_p(R)$.

Proof. If for any $q \in W$, $\text{dist}(p, q) > \delta R$, then $W \subset A_p(R, \delta R)$. By Lemma 1,

$$V(R, \delta R)/V_p(R) \leq \frac{R^n - \delta^n R^n}{R^n} = 1 - \delta^n.$$

So $1 - \delta^n \geq 1/m$, i.e., $\delta \leq (1 - 1/m)^{1/n}$. Thus we can take

$$(5) \quad \delta = (1 - 1/(2m))^{1/n}.$$

This finishes the proof of Lemma 4.

Now we continue the proof of the theorem. By the above lemma and (4), there is a point $q \in B_p^L$ such that

$$(6) \quad B_q((1 - \delta)r/2) \subset B_p(r/2).$$

We construct a star-shaped set U_q at q as follows: $y \in U_q$ if and only if there is a point x belonging to either $B_q((1 - \delta)r/2)$ or one of $B_{p_i}^{L,j}(r/2)$ in the cone K_j and there is a minimal geodesic γ connecting q and x which passes y . Note that

(a) This geodesic γ will not pass through $B_{p_0}(D_1)$ by the construction of the cone. To see this, note that if x is in one of $B_{p_i}^{L,j}(r/2)$, then both x and q are in K_j . The claim follows from (1) and the remarks following Lemma 3; if x is in $B_q((1 - \delta)r/2)$ then the claim is immediate by the triangle inequality.

(b) The length of γ is not bigger than $r/2 + 2r + 2r + r/2 = 5r$.

(c) $B_q((1 - \delta)r/2) \subset U_q \subset M - B_{p_0}(D_1)$.

Hence we can apply Lemma 2. We have

$$(7) \quad \text{Vol}(U_q)/V_q((1 - \delta)r/2) \leq V^0(5r)/V^0((1 - \delta)r/2) = 10^n \cdot (1 - \delta)^{-n}.$$

On the other hand let N_j be the number of balls in the j th family. By (4) and (6),

$$(8) \quad \text{Vol}(U_q)/V_q((1 - \delta)r/2) \geq \sum_{B_{p_i}^L \in j\text{th family}} \text{Vol}(B_{p_i}^L)/V_p(r/2) \geq N_j/m.$$

So (7) and (8) imply $N_j \leq m \cdot 10^n(1 - \delta)^{-n}$. Adding up the contributions from all m families using (5), we have

$$(9) \quad k \leq 10^n m^2 [1 - (1 - 1/(2m))^{1/n}]^{-n}.$$

Since m depends only on n and D_1 so does the right-hand side of (9). The theorem is proved.

Remark 2. From the above proof, we see that we can replace $\partial B_{p_0}(2r)$ by any subset S satisfying $\sup_{x \in S} \text{dist}(p_0, x) \leq C_0 \text{dist}(p_0, S)$; hence we have the following:

Suppose $\text{Ric}_{M-B} \geq 0$, $K_M \geq H > -\infty$, and $B \subset B_{p_0}(D)$. Then $\forall \mu > 0$, $\forall C_0 > 0$, $\exists N > 0$, and $r_0 > 0$ depending only on n , μ , $H \cdot D^2$, and C_0 , such that if S is a subset satisfying

$$\sup_{x \in S} \text{dist}(p_0, x) \leq C_0 \text{dist}(p_0, S),$$

$$r = \text{dist}(p_0, S) > r_0,$$

we can find $p_1, \dots, p_k \in S$, $k \leq N$, with

$$\bigcup_{j=1}^k B_{p_j}(\mu \cdot r) \supset S.$$

This more general form will be useful in the next section.

4. APPLICATIONS

Corollary 1. *Under the same assumption as in the theorem, the number of ends is finite and bounded by the same constant N in the theorem.*

Proof. If not, take r large enough so that in $M - B_{p_0}(r)$, there are $n > N$ separated ends. It is apparent that each such end E requires at least one ball of radius r to cover $E \cap \partial B_{p_0}(2r)$. This contradicts the theorem.

There are several different definitions of the concept of “diameter growth.” (Very roughly speaking, the diameter growth measures the intrinsic diameter of $\partial B_p(r)$ as a function of r .) As far as the question of linearity of diameter growth is concerned, they are all equivalent. See for example, [AG] and [S].

Corollary 2. *Under the same assumption as in the theorem, the diameter growth of each end is at most linear.*

Proof. Apply Remark 2 to S , a curve connecting two points on a distance sphere. We omit the details.

An immediate consequence of the theorem in the study of harmonic functions is the following:

Proposition 1. *Assume that $\text{Ric}_{M-B} \geq 0$, where $B \subset B_{p_0}(D)$ is a compact subset of M , and the sectional curvature of M is bounded below. Let E denote an end of M , that is, E is an unbounded component of $M - B_{p_0}(r_0)$. Assume that $f(x)$ is a positive harmonic function defined on E . Then either $\lim_{x \rightarrow \infty} f(x) = \infty$ or $\lim_{x \rightarrow \infty} f(x) = a$ for some constant a , provided that one of the following conditions on E holds:*

- (a) *There is an $R_0 > 0$, such that $\pi_1(E - B_{p_0}(R_0)) = 0$.*
- (b) *$\forall r > 0$, $\exists R > r$, such that the unbounded component $E - B_{p_0}(R)$ has a connected boundary.*
- (c) *There is a sequence of connected $(n-1)$ -dimensional compact submanifolds $S_j \subset E$, such that (i) S_j and S_{j+1} bound a connected compact region D_j for $j = 1, 2, 3, \dots$; (ii) for some large r , $\bigcup D_j \supset E - B_p(r)$; (iii) $\text{dist}(p_0, S_j) \rightarrow \infty$; and (iv) for some constant C*

$$\max_{x \in S_j} \text{dist}(p_0, x) / \min_{x \in S_j} \text{dist}(p_0, x) \leq C.$$

Proof. We follow the same line of arguments as in the proof of Theorems 3.2 and 3.3 in [LT].

Since (a) implies (b) (see [AG, Proposition 4.3 and its proof]) which in turn implies (c), let us assume that the end E satisfies condition (c). The proposition is a consequence of the following two lemmas.

Lemma 5 (Harnack inequality; see [Y, LY], also see [LT, Lemma 3.2]). *There is a constant C_1 , such that if $f(x)$ is a positive harmonic function on E and $\text{Ric}_{B_q(2r)} \geq 0$, where $B_q(2r) \subset E$, then*

$$|\nabla f(x)|/f(x) \leq C_1/r$$

for any $x \in B_q(r)$.

Lemma 6. *There is a constant $C_2 > 0$ independent of j such that for any harmonic function $f(x)$ defined on E and any large j , we have*

$$\max_{x \in S_j} f(x) \leq C_2 \min_{x \in S_j} f(x).$$

Proof. Choose $y, z \in S_j$ with

$$f(y) = \max_{x \in S_j} f, \quad f(z) = \min_{x \in S_j} f.$$

Let γ be a curve in S_j connecting y and z . Denote $r_j = \text{dist}(p_0, S_j)$. Without loss of generality, we assume that $r_j > 2r_0$. Remark 2 implies that there are a bounded number of points $y = y_1, \dots, y_k = z, k \leq C_3$, on the curve γ such that

$$\gamma \subset \bigcup_i B_{y_i}(1/4 \cdot r_j).$$

Now Harnack's inequality implies that

$$f(y)/f(z) \leq C_1^{(2k-1)}.$$

This proves the lemma.

Now, Proposition 1 follows from the above lemma. The argument is as in the proof of Theorem 3.3 of [LT]. Namely, suppose that $f(x)$ is not bounded on E . We are going to show that

$$\lim_{x \rightarrow \infty} f(x) = \infty.$$

Since $f(x)$ is not bounded, there is a sequence of points $x_j, \lim \text{dist}(x_j, p_0) = \infty$, such that $\lim f(x_j) = \infty$. Suppose $x_j \in D_{i_j}$, then

$$\max_{x \in D_{i_j}} f(x) \geq f(x_j).$$

By the maximum principle the maximum is either on S_{i_j} , in which case we denote $\sum_j = S_{i_j}$, or on S_{i_j+1} , in which case we denote $\sum_j = S_{i_j+1}$. We write U_j for the domain bounded by \sum_j and \sum_{j+1} . Then we have

$$\min_{x \in \sum_j} f(x) \geq 1/C_2 \max_{x \in \sum_j} f(x) \geq 1/C_2 f(x_j).$$

Thus by the maximum principle $\min_{x \in U_j} f(x) \geq 1/C_2 \min\{f(x_j), f(x_{j+1})\}$. Therefore $\lim_{x \rightarrow \infty} f(x) = \infty$.

Suppose that $f(x)$ is a bounded E . Let

$$A = \lim_{r \rightarrow \infty} \left(\inf_{E-B_{p_0}(r)} f \right); \quad 0 \leq A < \infty.$$

For any $\varepsilon > 0$, there exists a sequence of points $x_j, \lim \text{dist}(x_j, p_0) = \infty$, such that $A - \varepsilon < f(x_j) < A + \varepsilon$. Suppose $x_j \in D_{i_j}$. By the maximum principle $\min_{D_{i_j}} f$ is either on S_{i_j} , in which case we denote $\sum_j = S_{i_j}$, or on S_{i_j+1} , in which case we denote $\sum_j = S_{i_j+1}$. We write U_j for the domain bounded by \sum_j and \sum_{j+1} . Also, there is a j_0 , such that

$$\inf_{\bigcup_{i>j} U_i} f > A - \varepsilon$$

for any $j \geq j_0$.

Let $g = f - (A - \varepsilon)$, which is a positive harmonic function on $\bigcup_{i>j_0} U_i$. Lemma 6 implies that for sufficiently large j ,

$$\max_{x \in \sum_j} g(x) \leq C_2 \min_{x \in \sum_j} g(x).$$

Hence

$$\max_{x \in \sum_j} g(x) \leq C_2 \{ \min_{x \in \sum_j} f - (A - \varepsilon) \} \leq 2C_2 \varepsilon,$$

$$\max_{x \in \sum_j} f \leq (2C_2 - 1)\varepsilon + A.$$

Since ε is arbitrary, by the maximum principle again we have

$$\lim_{j \rightarrow \infty} \left(\sup_{\bigcup_{i>j} U_i} f \right) \leq A.$$

This shows that $\lim_{x \rightarrow \infty} f = A$. The proposition is proved.

We intend to further investigate the structure of the space of positive harmonic functions elsewhere.

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Added in proof. The original version of this paper was submitted for publication in May 1990.

At the AMS Summer Institute on Differential Geometry held at U.C.L.A., July 1990, P. Li announced that he and L. Tam had proved that $\text{Ric}_{M-B} \leq 0$ implies that M has finitely many ends (without assuming a lower bound on the sectional curvature), and the number of ends depends only on n and $H\phi^2$, where ϕ is the diameter of B and $\text{Ric}_B \geq (n-1)H$. Their approach is analytic in nature. At about the same time, M. Cai independently proved this same result by purely geometric means.

Cai's argument suggested to us that our original approach could be slightly modified by using the triangle inequality in place of Toponogov's Theorem, thus eliminating the requirement of an arbitrary lower bound on the sectional curvature; i.e. we can establish the ball covering property assuming only $\text{Ric}_{M-B} \geq 0$, and the covering number depends only on n and HD^2 , where H is as above and D as in the theorem. (Ball covering is not treated in either of the aforementioned works.)

In fact, it is only necessary to modify the definitions of the subsets U_j and of the corresponding cones K_j . We can define $\{U_1, \dots, U_m\}$ as subsets of $\partial B_{p_0}(2D_1)$ such that $d_M(U_j) \leq 2D_1$, while K_j is the star-shaped set consisting of all minimal geodesics emanating from p_0 that intersect U_j . Then the triangle inequality easily implies that any minimal geodesic connecting two points in K_j which are sufficiently far away from p_0 will not intersect B . The rest of the construction remains unchanged. Details will be given elsewhere.

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