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HILBERT SPACES OF ANALYTIC FUNCTIONS BETWEEN THE HARDY AND THE DIRICHLET SPACE

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ABSTRACT. For a large class of Hilbert spaces of analytic functions in the unit disc lying between the Hardy and the Dirichlet space we prove that each element of the space is the quotient of two bounded functions in the same space. It follows that the multiplication operator on these spaces is cellular indecomposable and that each invariant subspace contains nontrivial bounded functions.

1. INTRODUCTION

For a positive integrable function $w \in C^2[0, 1)$, consider the space H_w of analytic functions f in the unit disc U that satisfies

$$\|f\|_w^2 = |f(0)|^2 + \int_U |f'(z)|^2 w(|z|)\, dm < \infty, \tag{1.1}$$

where m is the area measure on $\mathbb{C}$. Some simple computations with power series show that if $f(z) = \sum_{n\geq 0} a_n z^n$ is analytic in U, then

$$\|f\|_w^2 = \sum_{n\geq 0} |a_n|^2 w_n, \tag{1.2}$$

where $w_0 = 1$ and for $n \geq 1$,

$$w_n = 2\pi n^2 \int_0^1 r^{2n-1} w(r)\, dr. \tag{1.3}$$

It follows that H_w is a separable Hilbert space of analytic functions in U and that the polynomials are dense in H_w. Using the additional assumptions that w is decreasing, concave, and satisfies $\lim_{r\to 1} w(r) = 0$, locate the space H_w between the well-known Dirichlet space D obtained for $w = 1$ and the Hardy space H^2 obtained for $w(r) = 1 - r$, $r \in [0, 1)$; that is , $D \subset H_w \subseteq H^2$. The patterns for the above construction are the weighted Dirichlet spaces D_α, $0 < \alpha \leq 1$, corresponding to the choices $w_\alpha(r) = (1-r)^\alpha$, $r \in [0, 1)$, and it is the aim of this paper to prove, in the more general context considered here, a result conjectured for the spaces D_α by S. Richter and A. Shields [3]; namely, the fact that every function in H_w is the quotient of two bounded analytic

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functions in H_w. This result which is proved in §2, has some applications concerning the invariant subspaces of the multiplication operator defined on H_w by

$$(M_z f)(\zeta) = \zeta f(\zeta), \qquad \zeta \in U, \ f \in H_w. \tag{1.4}$$

Using (1.1) and (1.2) if follows easily that M_z is a bounded weighted shift on H_w. From the result mentioned above it turns out that every nontrivial invariant subspace of M_z contains a nontrivial bounded function, that each two nontrivial invariant subspaces have a nontrivial intersection and that each nontrivial invariant subspace has the codimension one property. For the usual Dirichlet space D, this was proved by S. Richter and A. Shields in [3]. These results provide positive answers to the corresponding questions for the spaces D_α [3, Conjectures 1 and 2] and are proved in §3. The method used for the proofs also implies the following cyclicity theorem for the spaces H_w, related to Question 3 in [1]. A function whose modulus is greater or equal to the modulus of a cyclic vector for M_z must also be a cyclic vector.

2. Bounded functions in H_w

We begin with a general change of variable formula that is used in order to obtain an equivalent form of the norm on H_w.

2.1. **Proposition.** *Let ϕ be a nonconstant analytic function in U and u, v be nonnegative measurable functions on $\mathbb{C}$ with respect to area measure. Then*

$$\int_U (u \circ \phi) v |\phi'|^2 \, dm = \int_{\phi(U)} u(\zeta) \left(\sum_{\phi(z)=\zeta} v(z) \right) dm(\zeta). \tag{2.1}$$

This result is actually known and was proved for $v(z) = -\log|z|$ in [4]. We include a sketch of its proof only for the sake of completeness.

Proof. There exists a sequence of pairwise disjoint sets $R_n \subset U$, $n \geq 1$, of the form $R_n = \{z \in U ; a_n < |z| < b_n, \ \alpha_n < \arg z < \beta_n\}$ such that $\phi_{|R_n}$ is injective and $m(U \setminus \bigcup_{n\geq 1} R_n) = 0$. By the usual change of variable formula, we have

$$\int_U (u \circ \phi) v |\phi'|^2 \, dm = \sum_{n\geq 1} \int_{\phi(R_n)} u[v \circ (\phi_{|R_n})^{-1}] \, dm. \tag{2.2}$$

Let χ_n be the characteristic function of $\phi(R_n)$ and ϕ_n be a measurable extension of $(\phi_{|R_n})^{-1}$. We have

$$\sum_{n\geq 1} \int_{\phi(R_n)} u(v \circ \phi_n) \, dm = \int_{\mathbb{C}} u \left[\sum_{n\geq 1} \chi_n (v \circ \phi_n) \right] dm. \tag{2.3}$$

If Γ is the complement of $\bigcup_{n\geq 1} R_n$, then $m(\phi(\Gamma)) = 0$ and for each $\zeta \in \phi(U) \setminus \phi(\Gamma)$

$$\sum_{n\geq 1} \chi_n(\zeta) v \circ \phi_n(\zeta) = \sum_{\phi(z)=\zeta} v(z), \tag{2.4}$$

i.e. the above sums coincide a.e. on $\phi(U)$.

2.2. **Corollary.** *If $f \in H_w$ is nonconstant, then*

$$\|f\|_w^2 = |f(0)|^2 + \int_{f(U)} \left(\sum_{f(z)=\zeta} w(|z|) \right) dm(\zeta) . \tag{2.5}$$

In what follows, for a nonconstant analytic function f in U, $\zeta \in f(U)$ and u a nonnegative measurable function on $[0, 1)$, we denote

$$N_{u,f}(\zeta) = \sum_{f(z)=\zeta} u(|z|) . \tag{2.6}$$

In the special case when $u(r) = u_0(r) = \log 1/r$, $r \in [0, 1)$, (2.6) gives the usual Nevanlinna counting function of f and we denote $N_{u_0, f} = N_f$

2.3. **Lemma.** *Let $f \in H_w$ be nonconstant and for z, $\lambda \in U$ let $\varphi_z(\lambda) = (z+\lambda) \times (1 + \overline{z}\lambda)^{-1}$. Then for every $\zeta \in f(U)$*

$$N_{w,f}(\zeta) = -\frac{1}{2\pi} \int_U \Delta\tilde{w}(z) N_{f\circ\varphi_z}(\zeta)\, dm(z) , \tag{2.7}$$

where $\tilde{w}$ is defined on U by $\tilde{w}(z) = w(|z|)$ and Δ denotes the Laplace operator.

Proof. Since $\lim_{|z|\to 1} \tilde{w}(z) = 0$, by Green's theorem, for every $\lambda \in U$ we have

$$\begin{aligned} w(|\lambda|) = \tilde{w}(\lambda) &= \frac{1}{2\pi} \int_U \Delta\tilde{w}(z) \log \left| \frac{z-\lambda}{1-\overline{\lambda}z} \right| dm(z) \\ &= \frac{1}{2\pi} \int_U \Delta\tilde{w}(z) \log |\varphi_z^{-1}(\lambda)|\, dm(z) . \end{aligned} \tag{2.8}$$

Moreover, $\Delta\tilde{w}(z) = (1/|z|)w'(|z|) + w''(|z|) \le 0$ because w is decreasing and concave. Then by (2.8) and the monotone convergence theorem, we obtain

$$\begin{aligned} N_{w,f}(\zeta) &= \frac{1}{2\pi} \sum_{f(\lambda)=\zeta} \int_U \Delta\tilde{w}(z) \log |\varphi_z^{-1}(\lambda)|\, dm(z) \\ &= -\frac{1}{2\pi} \int_U \Delta\tilde{w} \left(\sum_{f(\lambda)=\zeta} -\log |\varphi_z^{-1}(\lambda)| \right) dm(z) . \end{aligned} \tag{2.9}$$

Finally for $\zeta \in f(U)$ and z, $\lambda \in U$, we have that $f(\lambda) = \zeta$ if and only if $f \circ \varphi_z(\varphi_z^{-1}(\lambda)) = \zeta$, which proves (2.7).

From the fact that H_w is contained in H^2, it follows that each function $f \in H_w$ has nontangential limits $f(e^{i\theta})$ a.e. on $[0, 2\pi]$ and that its boundary function is in $L^2[0, 2\pi]$. For $z \in U$, let $P_z(\theta) = \mathrm{Re}(e^{i\theta} + z)/(e^{i\theta} - z)$, be the Poisson kernel. We prove

2.4. **Proposition.** *Let $f \in H_w$. Then*

$$\|f\|_w^2 = |f(0)|^2 - \frac{1}{4} \int_U \Delta\tilde{w}(z) \left[\frac{1}{2\pi} \int_0^{2\pi} P_z(\theta) |f(e^{i\theta})|^2\, d\theta - |f(z)|^2 \right] dm(z) . \tag{2.10}$$

Proof. For a nonconstant $f \in H_w$, we have

$$\begin{aligned}\|f\|_w^2 - |f(0)|^2 &= \int_{f(U)} N_{w,f}\,dm \\ &= \int_{f(U)} \left(-\frac{1}{2\pi}\int_U \Delta\tilde{w}(z) N_{f\circ\varphi_z}(\zeta)\,dm(z)\right) dm(\zeta) \\ &= -\frac{1}{4}\int_U \Delta\tilde{w}(z)\left(\frac{2}{\pi}\int_{f(U)} N_{f\circ\varphi_z}(\zeta)\,dm(\zeta)\right) dm(z),\end{aligned} \tag{2.11}$$

by Corollary 2.2, Lemma 2.3, and Fubini's theorem. By (2.1) and the Littlewood-Paley formula

$$\begin{aligned}\frac{2}{\pi}\int_{f(U)} N_{f\circ\varphi_z}\,dm &= \frac{2}{\pi}\int_U |(f\circ\varphi_z)'(\lambda)|^2 \log\frac{1}{|\lambda|}\,dm \\ &= \|f\circ\varphi_z\|_{H^2}^2 - |f\circ\varphi_z(0)|^2.\end{aligned} \tag{2.12}$$

Obviously $f \circ \varphi_z(0) = f(z)$ and by elementary computations with harmonic measures for the unit disc, we obtain

$$\|f\circ\varphi_z\|_{H^2}^2 = \frac{1}{2\pi}\int_0^{2\pi} |f\circ\varphi_z(e^{i\theta})|^2\,d\theta = \frac{1}{2\pi}\int_0^{2\pi} P_z(\theta)|f(e^{i\theta})|^2\,d\theta, \tag{2.13}$$

and the proof is complete.

2.5. **Corollary.** *Let $f \in H_w$ and F be its outer factor. Then $F \in H_w$ and*

$$\|F\|_w^2 - |F(0)|^2 \le \|f\|_w^2 - |f(0)|^2. \tag{2.14}$$

Proof. For every $z \in U$, we have $|f(z)| \le |F(z)|$ and $|f(e^{i\theta})| = |F(e^{i\theta})|$ a.e. on $[0, 2\pi]$. Thus

$$\frac{1}{2\pi}\int_0^{2\pi} P_z(\theta)|F(e^{i\theta})|^2\,d\theta - |F(z)|^2 \le \frac{1}{2\pi}\int_0^{2\pi} P_z(\theta)|f(e^{i\theta})|^2\,d\theta - |f(z)|^2, \tag{2.15}$$

and the result follows by Proposition 2.4.

For a function f in the Nevanlinna class N, $f \not\equiv 0$, we denote by ϕ_f the outer function satisfying $|\phi_f(e^{i\theta})| = \min\{1, 1/|f(e^{i\theta})|\}$ a.e. on $[0, 2\pi]$; that is,

$$\phi_f(z) = \exp\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}\log\min\{1, 1/|f(e^{i\theta}|\}\,d\theta. \tag{2.16}$$

Our main result is

2.6. **Theorem.** *Let $f \in H_w$, $f \not\equiv 0$. Then ϕ_f, $f\phi_f$ are in H_w and satisfy*

$$\|\phi_f\|_w^2 - |\phi_f(0)|^2 \le \|f\|_w^2 - |f(0)|^2 \quad \textit{and} \quad \|f\phi_f\|_w \le \|f\|_w. \tag{2.17}$$

The proof uses the following inequalities:

2.7. **Lemma.** *Let (X, μ) be a probability space and $f \in L^1(\mu)$ such that $f > 0$ μ-a.e. on X and $\log f \in L^1(\mu)$. Let*

$$E(f) = \int_X f\,d\mu - \exp\int_X \log f\,d\mu. \tag{2.18}$$

Then

$$E(\min\{1, f\}) \le E(f), \tag{2.19}$$

and

$$E(\max\{1, f\}) \le E(f). \tag{2.20}$$

Proof. Let $A = \{x \in X,\ f(x) \ge 1\}$ and assume that $\alpha = \mu(A)$ is in $(0, 1)$, otherwise both inequalities are obviously satisfied. The inequality (2.19) is equivalent to

$$\int_A (f-1)\, d\mu \ge \exp\left(\int_{X\setminus A} \log f\, d\mu\right)\left[\exp\left(\int_A \log f\, d\mu\right) - 1\right]. \tag{2.21}$$

We have

$$\begin{aligned} &\exp\left(\int_{X\setminus A} \log f\, d\mu\right)\left[\exp\left(\int_A \log f\, d\mu\right) - 1\right] \\ &\quad \le \left(\exp \frac{1}{\alpha}\int_A \log f\, d\mu\right)^{\alpha} - 1 \le \left(\frac{1}{\alpha}\int_A f\, d\mu\right)^{\alpha} - 1, \end{aligned} \tag{2.22}$$

by Jensen's inequality. Now it is easy to verify by differentiation that for every $t \ge 0$ and $0 < \beta < 1$,

$$t - \beta \ge (t/\beta)^{\beta} - 1, \tag{2.23}$$

and (2.19) follows letting $t = \int_A f\, d\mu$ and $\beta = \alpha$.

The second inequality is very similar. Indeed, we have as above that (2.20) is equivalent to

$$\int_{X\setminus A} (f-1)\, d\mu \ge \exp\left(\int_A \log f\, d\mu\right)\left[\exp\left(\int_{X\setminus A} \log f\, d\mu\right) - 1\right], \tag{2.24}$$

and that the right side of (2.24) is less or equal to

$$\left(\exp \frac{1}{1-\alpha}\int_{X\setminus A} \log f\, d\mu\right)^{1-\alpha} - 1 \le \left(\frac{1}{1-\alpha}\int_{X\setminus A} f\, d\mu\right)^{1-\alpha} - 1, \tag{2.25}$$

by Jensen's inequality. Then (2.20) follows from (2.23) with $t = \int_{X\setminus A} f\, d\mu$ and $\beta = 1 - \alpha$.

Proof of Theorem 2.6. Let $f \in H_w$, $f \in IF$ with I inner and F outer. An application of (2.19) with $X = [0, 2\pi]$, $d\mu = (1/2\pi)P_z\, d\theta$, yields

$$\begin{aligned} &\frac{1}{2\pi}\int_0^{2\pi} P_z(\theta)|f\phi_f(e^{i\theta})|^2\, d\theta - |f\phi_f(z)|^2 \\ &\quad \le \frac{1}{2\pi}\int_0^{2\pi} P_z(\theta)|F(e^{i\theta})|^2\, d\theta - |I(z)|^2|F(z)|^2 \\ &\quad = \frac{1}{2\pi}\int_0^{2\pi} P_z(\theta)|f(e^{i\theta})|^2\, d\theta - |f(z)|^2, \end{aligned} \tag{2.26}$$

and the second inequality in (2.17) follows by Proposition 2.3. Further, we have $|1/\phi_f(e^{i\theta})| = \max\{1, |f(e^{i\theta})|\}$, a.e. on $[0, 2\pi]$ and for $z \in U$,

$$|1/\phi_f(z)| = \frac{1}{2\pi}\int_0^{2\pi} P_z(\theta) \log\max\{1, |f(e^{i\theta})|^2\} d\theta. \tag{2.27}$$

We apply (2.20) to obtain

$$
\begin{aligned}
&\frac{1}{2\pi}\int_0^{2\pi} P_z(\theta)|1/\phi_f(e^{i\theta})|^2\,d\theta - |1/\phi_f(z)|^2 \\
&\quad = E(\max\{1\,,\,|f|^2\}) \le E(|f|^2) \\
&\quad \ge \frac{1}{2\pi}\int_0^{2\pi} P_z(\theta)|f(e^{i\theta})|^2\,d\theta - |f(z)|^2\,,
\end{aligned}
\tag{2.28}
$$

for all $z \in U$. Thus by Proposition 2.3, $1/\phi_f \in H_w$ and

$$\|1/\phi_f\|_w^2 - |1/\phi_f(0)|^2 \le \|f\|_w^2 - |f(0)|^2\,. \tag{2.29}$$

Finally, we have $\phi_f' = -\phi_f^2(1/\phi_f)'$ and $|\phi_f| \le 1$ in U, which proves the other inequality in (2.17).

Since $|\phi_f|\,,\ |f\phi_f| \le 1$ in U and $f = f\phi_f/\phi_f$, from Theorem 2.6 we obtain:

2.7. **Corollary.** *Every function in H_w is the quotient of two bounded functions in H_w.*

3. Invariant subspaces

The present section contains some applications of Theorem 2.6 concerning the invariant subspaces of the multiplication operator on H_w defined by (1.4). A closed subspace $\mathscr{M}$ of H_w is called invariant if $M_z\mathscr{M} \subset \mathscr{M}$. For a function $f \in H_w$ we denote by $[f]$ the smallest invariant subspace containing f; that is, the closure of the polynomial multiples of f in H_w. In order to prove the main result of this section we use the same method as in [3]. Let H^∞ be the algebra of bounded analytic functions in U with the norm $\|g\|_\infty = \sup_{z\in U}|g(z)|$, $g \in H^\infty$. We have

3.1. **Lemma.** *If $f \in H_w$ and $g \in H^\infty$ such that $gf \in H_w$, then $gf \in [f]$.*

The proof of the lemma is based on some simple properties of the linear operators T_t, $0 \le t < 1$, defined on the set $H(U)$ of analytic functions in U by

$$(T_t h)(z) = \frac{1}{1-t}\int_t^1 h(sz)\,ds \qquad z \in U\,,\ h \in H(U)\,. \tag{3.1}$$

We obviously have $\lim_{t\to 1}(T_t h)(z) = h(z)$ for all $z \in U$ and $h \in H(U)$. Some other properties are summarized below. For $h \in H(U)$ and $t \in [0\,,\,1)$, we denote by h_t the function given by $h_t(z) = h(tz)$, $z \in U$.

3.2. **Lemma.** *For every $t \in [0\,,\,1)$, we have*

(i) $(M_z T_t h)' = \frac{h - th_t}{1-t}$, $h \in H(U)$.

(ii) *If $h \in H_w$, then $T_t h \in H_w$ and $\|T_t h\|_w \le \|h\|_w$.*

(iii) *If $h \in H^\infty$, then $T_t h \in H_w$ and $fT_t h \in H_w$ whenever $f \in H_w$. Furthermore in this case, $fT_t h \in [f]$.*

Proof. (i) For every $\zeta \in U$ and $h \in H(U)$,

$$(M_z T_t h)(\zeta) = \frac{1}{1-t}\int_{t\zeta}^{\zeta} h(\lambda)\,d\lambda\,. \tag{3.2}$$

(ii) If $h(z) = \sum_{n\geq 0} a_n z^n$, $z \in U$, then

$$(T_t h)(z) = \sum_{n\geq 0} \left(\frac{1-t^{n+1}}{1-t}\right) \frac{a_n}{n+1} z^n, \qquad z \in U, \tag{3.3}$$

which shows that $T_t h \in H_w$ whenever $h \in H_w$ and $\|T_t h\|_w \leq \|h\|_w$.

(iii) From (i) we obtain that if $h \in H^\infty$ then $T_t h$ and $(T_t h)'$ are both in H^∞, hence $T_t h \in H_w$ and is also a multiplier. Moreover, there exists a sequence of polynomials $\{p_n\}$ with $\sup_n \|p_n'\|_\infty < \infty$, converging pointwise to $T_t h$ on U. It follows that $\sup_n \|p_n f\|_w < \infty$, and that at least a subsequence of $\{p_n f\}$ converges weakly in H_w. Also, its limit must be $fT_t h$ because the point evaluations are bounded linear functionals on H_w. Thus, $fT_t h \in [f]$.

Proof of Lemma 3.1. Let f, g be as in the statement. For every $t \in [0, 1)$ we have $fT_t g \in [f]$ by Lemma 3.2 and $\lim_{t\to 1}(fT_t g)(z) = f(z)g(z)$ for all $z \in U$. We are going to show that the norms $\|fT_t g\|_w$ remain bounded when t tends to 1. Note first that by the definition of H_w, it follows easily that the operator M_z is injective and has closed range on H_w, hence there exists a positive constant c such that $\|fT_t g\| \leq c\|M_z f T_t g\|_w$, $t \in [0, 1)$. Further,

$$\begin{aligned} \|M_z f T_t g\|_w^2 &\leq 2\int_U |f(M_z T_t g)'|^2 w\, dm + 2\int_U |f' M_z T_t g|^2 w\, dm \\ &\leq 2\int_U |f(M_z T_t g)'|^2 w\, dm + 2\|g\|_\infty^2 \|f\|_w^2, \end{aligned} \tag{3.4}$$

and by Lemma 3.2(i),

$$\begin{aligned} |f(M_z T_t g)'| &= \left| f\frac{g - tg_t}{1-t}\right| \leq \left|\frac{fg - tf_t g_t}{1-t}\right| + \left| g_t \frac{f - tf_t}{1-t}\right| + |fg_t| \\ &= |(M_z T_t fg)'| + |g_t (M_z T_t f)'| + |fg_t|. \end{aligned} \tag{3.5}$$

We obtain

$$\begin{aligned} \int_U |f(M_z T_t g)'|^2 w\, dm &\leq 3\|M_z T_t fg\|_w^2 + 3\|g\|_\infty^2 \|M_z T_t f\|_w^2 \\ &\quad + 3\|g\|_\infty^2 \int_U |f|^2 w\, dm. \end{aligned} \tag{3.6}$$

This leads to an estimation of the form

$$\|fT_t g\|_w^2 \leq c_1 \|fg\|_w^2 + c_2 \|g\|_\infty^2 \|f\|_w^2, \tag{3.7}$$

where c_1, c_2 are positive constants independent of t. As in the proof of Lemma 3.2(iii), there exists a sequence $\{t_n\}$ in $[0, 1)$, tending to 1 such that $fT_{t_n} g$ tends weakly to fg in H_w, hence $gf \in [f]$.

A function $f \in H_w$ is called a cyclic vector for M_z if $[f] = H_w$. From Lemma 3.1 we obtain

3.3. **Corollary.** *If f, $g \in H_w$, g is a cyclic vector for M_z and $|f(z)| \geq |g(z)|$ for all $z \in U$, then f is also cyclic.*

Proof. If $h = g/f$, then $h \in H^\infty$ and $hf = g$, i.e. $hf \in H_w$. Then by Lemma 3.1, $g \in [f]$, which shows that f is cyclic.

Remark. The above result remains true if H_w is replaced by the Dirichlet space D, because Lemma 3.1 holds for this space as well, with the same proof. This problem was raised for general Banach spaces of analytic functions by L. Brown and A. Shields [1, Question 3].

The main result of this section is

3.4. **Theorem.** *Let* $\mathscr{M} \neq \{0\}$, $\mathscr{N} \neq \{0\}$ *be invariant subspaces for the operator* M_z *on* H_w. *Then* (i) $\mathscr{M} \cap H^\infty \neq \{0\}$ *and* (ii) $\mathscr{M} \cap \mathscr{N} \neq \{0\}$.

Proof. (i) Let $f \in \mathscr{M}$, $f \neq 0$. By Theorem 2.6 there exist functions g, $h \in H^\infty \cap H_w$ such that $f = g/h$ and, by Lemma 3.1, $g = hf \in [f] \subset \mathscr{M}$. (ii) If $g \in \mathscr{M}$, $h \in \mathscr{N}$, g, $h \neq 0$ are bounded then $gh \in [g] \cap [h] \subset \mathscr{M} \cap \mathscr{N}$.

Theorem 3.4(ii) states that the operator M_z on H_w is cellular indecomposable and answers affirmatively Conjecture 1 of [3]. As it was pointed out in [3] this implies the fact that each nontrivial invariant subspace $\mathscr{M}$ of M_z has the codimension one property that is, $(z-\lambda)\mathscr{M}$ is a closed subspace of $\mathscr{M}$ having codimension 1 in $\mathscr{M}$, for every $\lambda \in U$. This follows from results obtained by S. Richter in [2] and Theorem 3.4. Indeed, if $\mathscr{M}$ is such a subspace and f, $g \in \mathscr{M} \setminus \{0\}$, then $[f] \cap [g] \neq \{0\}$ by Theorem 3.3 and by [2, Corollaries 3.12 and 3.15] $\mathscr{M}$ has the codimension one property.

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