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PRESENTATIONS FOR 3-DIMENSIONAL SPECIAL LINEAR GROUPS OVER INTEGER RINGS

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ABSTRACT. The following 2-generator 6-relator presentation is obtained for the 3-dimensional special linear group $\mathrm{SL}(3, \mathbb{Z}_k)$ for each odd integer $k > 1$:

$$\begin{aligned}\mathrm{SL}(3, \mathbb{Z}_k) = \langle x, y | x^3 = y^3 &= (xy)^6 = (x^{-1}yx^{-1}y^{-1}xy)^2 \\ &= (xy^{-1}xyxy^{-1}x^{-1}y^{-1})^k \\ &= ((xy^{-1}xyxy^{-1}x^{-1}y^{-1})^{(k-1)/2}xy)^4 = 1\rangle .\end{aligned}$$

Alternative presentations for these groups and other groups associated with them are also given.

1. INTRODUCTION

The n-dimensional special linear group $\mathrm{SL}(n, \mathbb{Z})$ is the multiplicative group of all $n \times n$ matrices with integer entries having determinant 1. It is well known that $\mathrm{SL}(n, \mathbb{Z})$ is generated by its transvections, that is, by the matrices T_{ij} (for $1 \leq i \neq j \leq n$) with 1's on the diagonal and in the (i, j)th position and 0's elsewhere. In fact if $n \geq 3$, then $\mathrm{SL}(n, \mathbb{Z})$ has a presentation in terms of these generators, with relations as follows (c.f. [6, Corollary 10.3]):

(a) $[T_{ij}, T_{km}] = 1$ whenever $j \neq k$ and $i \neq m$,
(b) $[T_{ij}, T_{jk}] = T_{ik}$ whenever i, j, k are distinct, and
(c) $(T_{12}T_{21}^{-1}T_{12})^4 = 1$.

In other words, the latter are defining relations for $\mathrm{SL}(n, \mathbb{Z})$ as an abstract group.

If m is any positive integer, the group $\mathrm{SL}(n, \mathbb{Z}_m)$ is similarly defined as the group of all $n \times n$ matrices with entries from the ring $\mathbb{Z}_m$ (of all integers modulo m) and determinant 1, under matrix multiplication (modulo m). This is clearly a factor group of $\mathrm{SL}(n, \mathbb{Z})$, obtainable as the image under the homomorphism induced by reduction of integers modulo m. The kernel $K_{n,m}$ consists of all those matrices in $\mathrm{SL}(n, \mathbb{Z})$ that are congruent (modulo m) to the identity matrix, and when $n \geq 3$, the solution of the congruence subgroup problem for $\mathrm{SL}(n, \mathbb{Z})$ given in [1, 5] implies that this normal subgroup $K_{n,m}$ is also the smallest normal subgroup of $\mathrm{SL}(n, \mathbb{Z})$ containing the mth power of any

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transvection T_{ij} $(1 \le i \ne j \le n)$. In particular, it follows that a presentation for $\mathrm{SL}(n, \mathbb{Z}_m)$ may be obtained by simply adding the single relation $T_{12}^m = 1$ to those given in (a)–(c) above.

Similar presentations (using elementary matrices and Steinberg relations) for these and other special linear groups are given also in [3, 4].

In this paper several new presentations are obtained in the case where $n = 3$, that is, for the groups $\mathrm{SL}(3, \mathbb{Z}_m)$. All use many fewer generators and relations than those previously known, the best being the following 2-generator 6-relator presentation for $\mathrm{SL}(3, \mathbb{Z}_m)$ when m is odd:

$$\begin{aligned}\mathrm{SL}(3, \mathbb{Z}_m) = \langle x, y \mid x^3 &= y^3 = (xy)^6 \\ &= (x^{-1}yx^{-1}y^{-1}xy)^2 = (xy^{-1}xyxy^{-1}x^{-1}y^{-1})^m \\ &= ((xy^{-1}xyxy^{-1}x^{-1}y^{-1})^{(m-1)/2}xy)^4 = 1\rangle;\end{aligned}$$

see Theorem 6. In fact the presentation given in Theorem 6 turns out to be symmetric, in the sense that its two generators can be interchanged by an automorphism; but more importantly, its deficiency (the number of relations minus the number of generators) does not increase with m, in contrast to known presentations for other, similar families of finite groups. Also, as a bonus, compact presentations are obtained for finite direct products $\prod_i \mathrm{SL}(3, p_i)$, where the p_i are distinct primes. Note: $\mathrm{SL}(3, p)$ is $\mathrm{SL}(3, \mathbb{Z}_p)$ when p is prime.

All these presentations are derived rather indirectly from the Steinberg relations (c.f. [6], taking $n = 3$), and although there seems to be no obvious way to obtain them directly, that of course does not preclude the possibility that similar presentations can be achieved for $\mathrm{SL}(n, \mathbb{Z}_m)$ for more general n and m—and indeed this remains an open question.

The impetus for this work comes from a recent discovery made by the first author [2] in the course of tackling a problem to do with trivalent symmetric graphs: the group with presentation

$$\begin{aligned}&\langle h, p, q, r, a \mid h^3 = a^2 = p^2 = q^2 = r^2 = 1,\ pq = qp,\ pr = rp,\ rq = pqr, \\ &\quad h^{-1}ph = q,\ h^{-1}qh = pq,\ (rh)^2 = 1,\ a^{-1}pa = p,\ a^{-1}qa = r,\ (ha)^{12} = 1\rangle\end{aligned}$$

is (somewhat surprisingly) isomorphic to the semidirect product $\mathrm{SL}(3, \mathbb{Z}) \cdot \langle \theta \rangle$, where θ is the inverse-transpose automorphism of $\mathrm{SL}(3, \mathbb{Z})$. In fact there is an isomorphism taking h and p to the matrices

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ -1 & -1 & -1 \end{pmatrix}$$

respectively, and a to $c\theta$ where c is the matrix

$$\begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix},$$

with the images of q and r determined by the relations $h^{-1}ph = q$ and $a^{-1}qa = r$. Under the given mapping, the transvection T_{12} is the inverse of the image of the element $(hap)^4$, and so immediately we have the following:

Theorem 1. *For every positive integer* m *the semidirect product* $\mathrm{SL}(3, \mathbb{Z}_m) \cdot \langle\theta\rangle$ *of* $\mathrm{SL}(3, \mathbb{Z}_m)$ *by its inverse-transpose automorphism* θ *has the presentation*

$$\langle h, p, q, r, a \mid h^3 = a^2 = p^2 = q^2 = r^2 = 1,\ pq = qp,\ pr = rp,\ rq = pqr,\ h^{-1}ph = q,\ h^{-1}qh = pq,\ (rh)^2 = 1,\ a^{-1}pa = p,\ a^{-1}qa = r,\ (ha)^{12} = (hap)^{4m} = 1\rangle.$$

This presentation is refined in the next section.

2. Presentations for $\mathrm{SL}(3, \mathbb{Z})$ and $\mathrm{SL}(3, \mathbb{Z}_m)$

In the presentation given in the preceding theorem, the generators q and r are obviously redundant, and also many of the relations can be replaced by simpler ones or eliminated.

For example, the relation $rq = pqr$ can be seen to be a consequence of the others, as follows: $pqr = ph^{-1}phr = ph^{-1}prh^{-1} = ph^{-1}rph^{-1} = prhph^{-1} = prh^{-2}ph^2 = prpq = rq$. Similarly the relation $h^{-1}qh = pq$ gives $(ph)^3 = ph^{-2}ph^2h^{-1}ph = p(pq)q = p^2q^2 = 1$, and then both $h^{-1}qh = pq$ and $pq = qp$ can be replaced by the single relation $(ph)^3 = 1$, for this relation implies $h^{-1}qh = h^{-2}ph^2 = hphh = ph^{-1}ph = pq$, and also $qpq = h^{-1}phph^{-1}ph = h(hphph)hph = hphph = p$. The consequent elimination of the redundant generators q and r via $q = h^{-1}ph$ and $r = a^{-1}qa = a^{-1}h^{-1}pha$ gives $(ah^{-1}phah)^2 = 1$ in place of $(rh)^2 = 1$, and $(pah^{-1}pha)^2 = 1$ in place of $pr = rp$, but then since $(pah^{-1}pha)^2 = (a^{-1}ph^{-1}pha)^2 = (a^{-1}hph^{-1}a)^2 = a^{-1}hp^2h^{-1}a$, the second of these new relations is redundant as well. Thus we obtain the following presentation for $\mathrm{SL}(3, \mathbb{Z}) \cdot \langle\theta\rangle$:

$$\langle h, p, a \mid h^3 = a^2 = p^2 = (ph)^3 = (ah^{-1}phah)^2 = (ap)^2 = (ha)^{12} = 1\rangle,$$

and correspondingly, the following presentation for $\mathrm{SL}(3, \mathbb{Z}_m) \cdot \langle\theta\rangle$ whenever $m > 0$:

$$\langle h, p, a \mid h^3 = a^2 = p^2 = (ph)^3 = (ah^{-1}phah)^2 = (ap)^2 = (ha)^{12} = (hap)^{4m} = 1\rangle.$$

Note that the inverse-transpose automorphism θ is effectively the same for both $\mathrm{SL}(3, \mathbb{Z})$ and $\mathrm{SL}(3, \mathbb{Z}_m)$, having order 2 and normalizing the special linear group in each case.

Next let $x = h$ and $y = aha$, and (just for aesthetics) also let $z = p$. The subgroup N generated by these three elements has index 2 in each of the above finitely-presented groups, with $Nh = N$, $Np = N$, $(Na)h = (Naha)a = Na$, and $(Na)p = (Np)a = Na$. Clearly $\{1, a\}$ is a Schreier transversal for N, and then x, y, and z can be taken as Schreier generators for N, with the Reidemeister relations obtained as the conjugates of the appropriate original relations by the element a in each case. Specifically, we now have the following:

Theorem 2. *The group* $\mathrm{SL}(3, \mathbb{Z})$ *has presentation*

$$\langle x, y, z \mid x^3 = y^3 = z^2 = (xz)^3 = (yz)^3 = (x^{-1}zxy)^2 = (y^{-1}zyx)^2 = (xy)^6 = 1\rangle.$$

Theorem 3. *For every positive integer* m, *the group* $\mathrm{SL}(3, \mathbb{Z}_m)$ *has presentation*

$$\langle x, y, z \mid x^3 = y^3 = z^2 = (xz)^3 = (yz)^3 = (x^{-1}zxy)^2(y^{-1}zyx)^2 = (xy)^6 = (xzyz)^{2m} = 1\rangle.$$

(Note that $(hap)^2 = haphap = hpahap = xzyz$, and then also $a^{-1}(hap)^2a = yzxz$, which is conjugate to $xzyz$ within the subgroup N.)

The 3-generator 9-relator presentation given in Theorem 2 is a refinement of the one given in Corollary 1(b) in [2], and is perhaps the best possible that can be obtained for $\mathrm{SL}(3, \mathbb{Z})$ using this approach. On the other hand, Theorem 3 can be further improved, as we shall see.

One way to continue is to examine the relation $(xzyz)^{2m} = 1$ more closely.

First, the relations we have imply $(xzyz)^4 = xyx^{-1}y^{-1}x^{-1}yx^{-1}y$. One nice way to see this is to use the following consequences of the original relations as a set of rewriting rules:

$$\begin{array}{lll} pa = ap, & qa = ar, & ra = aq, \\ ph = hq, & qh = hpq, & rh = h^{-1}r, \\ ph^{-1} = h^{-1}pq, & qh^{-1} = h^{-1}p, & rh^{-1} = hr; \end{array}$$

for then

$$\begin{aligned}(xzyz)^4 &= (hpahap)^4 = haphap(hap)^6 = hahqap(hap)^6 = haharphap(hap)^5 \\ &= hahah^{-1}rqap(hap)^5 = hahah^{-1}aqrphap(hap)^4, \quad \text{and so on},\end{aligned}$$

eventually giving

$$(xzyz)^4 = hahah^{-1}ah^{-1}ah^{-1}ahah^{-1}aha = xyx^{-1}y^{-1}x^{-1}yx^{-1}y.$$

The same sort of calculation also shows that

$$(xzyz)^2 = (hpahap)^2 = hahaph^{-1}aha = xyzx^{-1}y,$$

although this can also be obtained directly as follows:

$$\begin{aligned}(xzyz)^2 &= (xy^{-1}zy^{-1})^2 = xy^2zy^{-1}(xy^{-1}zy)y = xy^2zy^{-1}(y^{-1}zyx^{-1})y \\ &= xy(yzyzy)x^{-1}y = xyzx^{-1}y,\end{aligned}$$

using only the relations given in the presentation of Theorem 3.

Now if m is odd, say $m = 2j+1$, the relation $(xzyz)^{2m} = 1$ becomes $(xzyz)^{4j+2} = 1$ and thereby

$$1 = ((xzyz)^4)^j(xzyz)^2 = (xyx^{-1}y^{-1}x^{-1}yx^{-1}y)^jxyzx^{-1}y,$$

which in turn gives

$$z = z^{-1} = x^{-1}y(xyx^{-1}y^{-1}x^{-1}yx^{-1}y)^jxy = (x^{-1}yxyx^{-1}y^{-1}x^{-1}y)^jx^{-1}yxy.$$

In other words, when m is odd the generator z is redundant, and therefore, can be eliminated. Thus we have (ignoring the trivial case where $m = 1$):

Theorem 4. *For every odd integer $k > 1$ the group $\mathrm{SL}(3, \mathbb{Z}_k)$ has presentation*

$$\begin{aligned}\langle x, y \mid x^3 = y^3 = (xy)^6 &= ((x^{-1}yxyx^{-1}y^{-1}x^{-1}y)^{(k-1)/2}x^{-1}yxy)^2 \\ &= ((x^{-1}yxyx^{-1}y^{-1}x^{-1}y)^{(k-1)/2}x^{-1}yxyx)^3 \\ &= ((x^{-1}yxyx^{-1}y^{-1}x^{-1}y)^{(k-1)/2}x^{-1}yxy^{-1})^3 \\ &= ((x^{-1}yxyx^{-1}y^{-1}x^{-1}y)^{(k-1)/2}x^{-1}yxyxyx^{-1})^2 \\ &= ((x^{-1}yxyx^{-1}y^{-1}x^{-1}y)^{(k-1)/2}x^{-1}yxy^{-1}xy^{-1})^2 = 1\rangle.\end{aligned}$$

This 2-generator 8-relator presentation (obtainable using Tietze transformations from the one in Theorem 3) is rather complicated; a much improved version is given later.

3. Direct products

In this section we recall the solution to the congruence subgroup problem for $\mathrm{SL}(n, \mathbb{Z})$ when $n \geq 3$ (see [1] or [5]): if m is any positive integer, then the subgroup $K_{n,m}$ of all matrices that are congruent modulo m to the identity matrix I_n is also the smallest normal subgroup of $\mathrm{SL}(n, \mathbb{Z})$ containing the mth power of any transvection T_{ij} $(1 \leq i \neq j \leq n)$.

Now if k and m are coprime positive integers, clearly $K_{n,k} \cap K_{n,m} = K_{n,km}$; while on the other hand, as $1 = ku + mv$ for some $u, v \in \mathbb{Z}$, the product $K_{n,k}K_{n,m}$ contains every transvection: $T_{ij} = T_{ij}^{ku+mv} = (T_{ij}^{k})^u (T_{ij}^{m})^v \in K_{n,k}K_{n,m}$ whenever $1 \leq i \neq j \leq n$; and therefore, $K_{n,k}K_{n,m} = \mathrm{SL}(n, \mathbb{Z})$. In particular, this means the factor group $\mathrm{SL}(n, \mathbb{Z})/K_{n,km}$ is isomorphic to the direct product of its subgroups $K_{n,k}/K_{n,km}$ and $K_{n,m}/K_{n,km}$; but then because $K_{n,k}/K_{n,km} = K_{n,k}/(K_{n,k} \cap K_{n,m}) \cong (K_{n,k}K_{n,m})/K_{n,m} = \mathrm{SL}(n, \mathbb{Z})/K_{n,m}$ and similarly $K_{n,m}/K_{n,km} \cong \mathrm{SL}(n, \mathbb{Z})/K_{n,k}$, the quotient $\mathrm{SL}(n, \mathbb{Z})/K_{n,km}$ is isomorphic to the direct product of the individual factors $\mathrm{SL}(n, \mathbb{Z})/K_{n,k}$ and $\mathrm{SL}(n, \mathbb{Z})/K_{n,m}$.

The following consequence is almost immediate:

Theorem 5. *If $k_1, k_2, \ldots, k_s$ are pairwise coprime positive integers whose product is m, then the direct product $\prod_{1 \leq i \leq s} \mathrm{SL}(3, \mathbb{Z}_{k_i})$ has presentation*

$$\begin{aligned} \langle x, y, z \mid x^3 = y^3 = z^2 = (xz)^3 = (yz)^3 = (x^{-1}zxy)^2 \\ = (y^{-1}zyx)^2 = (xy)^6 = (xzyz)^{2m} = 1 \rangle . \end{aligned}$$

This can be proved by extending the above argument to show $\mathrm{SL}(n, \mathbb{Z})/K_{n,m}$ is isomorphic to the direct product of the individual factors $\mathrm{SL}(n, \mathbb{Z})/K_{n,k_i}$ for $1 \leq i \leq s$ (and any $n \geq 3$), and then applying the results of §2.

4. A compact symmetric presentation for $\mathrm{SL}(3, \mathbb{Z}_k)$

If k is an odd integer greater than 1, Theorem 5 gives this 3-generator 9-relator presentation for the direct product $\mathrm{SL}(3, 2) \times \mathrm{SL}(3, \mathbb{Z}_k)$:

$$\begin{aligned} \langle x, y, z \mid x^3 = y^3 = z^2 = (xz)^3 = (yz)^3 = (x^{-1}zxy)^2 \\ = (y^{-1}zyx)^2 = (xy)^6 = (xzyz)^{4k} = 1 \rangle . \end{aligned}$$

Again $(xzyz)^4 = xyx^{-1}y^{-1}x^{-1}yx^{-1}y$ is implied by the other relations, so the final relation can be replaced by $(xyx^{-1}y^{-1}x^{-1}yx^{-1}y)^k = 1$ if necessary. On the other hand, the third generator z cannot be eliminated, for the subgroup L generated by x and y in this group has index 8: its right cosets may be taken as L, Lz, Lzx, Lzy, $Lzxz$, $Lzyz$, $Lzxzy$, and $Lzyzx$, or (in terms of our earlier generators) as L, Lp, Lq, Lr, Lpq, Lpr, Lqr, and Lrq respectively. These cosets are permuted (under right multiplication) by the generators x, y, and z as follows:

x induces the permutation $(Lz, Lzx, Lzxz)(Lzyz, Lzyzx, Lzxzy)$,
y induces the permutation $(Lz, Lzy, Lzyz)(Lzxz, Lzxzy, Lzyzx)$,
and
z induces the permutation $(L, Lz)(Lzx, Lzxz)(Lzy, Lzyz)(Lzxzy, Lzyzx)$;

for example, the relations $x^3 = y^3 = z^2 = (xz)^3 = (yz)^3 = (x^{-1}zxy)^2 = (y^{-1}zyx)^2 = 1$ give

$$\begin{aligned}(Lzxzy)x &= Lzx(zyxy^{-1})y = Lzx(yx^{-1}y^{-1}z)y \\ &= L(zxyx^{-1})y^{-1}zy = L(xy^{-1}x^{-1}z)y^{-1}zy \\ &= (Lxy^{-1}x^{-1})zy^{-1}zy = Lzy^{-1}zy = Lyzyy = Ly^{-1}zy^{-1} = Lzyz,\end{aligned}$$

although this calculation is much easier using the earlier generators.

These permutations can also be checked using the Todd–Coxeter enumeration process, and the interested reader may even wish to verify by computer (using CAYLEY for instance) that they generate a group of order 168, isomorphic to $\mathrm{SL}(3, 2)$.

More importantly, this permutation representation of $\mathrm{SL}(3, 2) \times \mathrm{SL}(3, \mathbb{Z}_k)$ on cosets of the subgroup L can now be used to obtain a much neater presentation of $\mathrm{SL}(3, \mathbb{Z}_k)$.

First, using $\{1, z, zx, zy, zxz, zyz, zxzy, zyzx\}$ as transversal, the Reidemeister–Schreier process gives the following presentation for the subgroup L:

$$\langle x, y | x^3 = y^3 = (xy)^6 = (x^{-1}yx^{-1}y^{-1}xy)^2 = (xy^{-1}xyxy^{-1}x^{-1}y^{-1})^k = 1\rangle.$$

Note that all the Schreier generators are easily expressible as words in x and y; for instance,

$$\begin{aligned}(zyzx)x(zxzy)^{-1} &= zyz(x^{-1}y^{-1}z)x^{-1}z = zyz(y^{-1}zyxy^{-1})x^{-1}z \\ &= (zyz)y^{-1}zy(xy^{-1}x^{-1}z) \\ &= (y^{-1}zy^{-1})y^{-1}zy(zxyx^{-1}) = y^2zyzyzxyx^{-1} = yxyx^{-1}.\end{aligned}$$

Also many of the relators provided by the Reidemeister–Schreier process are conjugates or inverses of each other. For instance $(xyx^{-1}y^{-1}x^{-1}yx^{-1}y)^k$ is the inverse of $(y^{-1}xy^{-1}xyxy^{-1}x^{-1})^k$, which in turn is obviously a conjugate of $(xy^{-1}xyxy^{-1}x^{-1}y^{-1})^k$.

On the other hand, the matrices given in §1 (when taken together with the subsequent definitions $x = h$ and $y = aha$), provide us with a concrete faithful matrix representation of the group L over the ring $\mathbb{Z}_{2k}$: as representatives of the generators x and y of L we may define the matrices

$$X = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

respectively, treating their entries as integers modulo $2k$. In particular, these matrices satisfy the relations of the presentation obtained for L above, with

$$XY^{-1}XYXY^{-1}X^{-1}Y^{-1} = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

the square of the transvection T_{13}.

Now consider the matrix $W = (XY^{-1}XYXY^{-1}X^{-1}Y^{-1})^{(k-1)/2}XY$. Clearly

$$W = \begin{pmatrix} 1 & 0 & k-1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} k-1 & -1 & k-2 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix},$$

and then, modulo $2k$ remember,

$$W^4 = \begin{pmatrix} k+1 & k & k \\ 0 & 1 & 0 \\ k & 0 & 1 \end{pmatrix},$$

which has order 3, and is also easily seen to be congruent to the identity modulo k. Next, W^4 and its conjugate $X^{-1}W^4X$ together generate a Frobenius group of order 21: for

$$(W^4)^{-1}X^{-1}W^4X = \begin{pmatrix} k+1 & k & 0 \\ k & k+1 & k \\ 0 & k & 1 \end{pmatrix},$$

which has order 7 and is conjugated to its 4th power by each of W^4 and $X^{-1}W^4X$. Moreover, this subgroup is normalized by each of X and Y: putting $U = (W^4)^{-1}X^{-1}W^4X$, we find by straightforward calculation that $X^{-1}(X^{-1}W^4X)X = U^{-1}W^4$ and $Y^{-1}W^4Y = U^3W^4$ and $Y^{-1}(X^{-1}W^4X)Y = W^4$; indeed also $X^{-1}UX = Y^{-1}UY = U^2$ and $X^{-1}W^4X = U^2W^4$.

Correspondingly, in L the element $((xy^{-1}xyxy^{-1}x^{-1}y^{-1})^{(k-1)/2}xy)^4$ and its conjugate by x generate a normal subgroup F of order 21, identifiable with any subgroup of index 8 in the group $\mathrm{SL}(3, 2)$, and then L is a direct product of F with an isomorphic copy of $\mathrm{SL}(3, \mathbb{Z}_k)$. In particular, adjoining the relation $((xy^{-1}xyxy^{-1}x^{-1}y^{-1})^{(k-1)/2}xy)^4 = 1$ to the presentation found earlier for L must give a presentation for $\mathrm{SL}(3, \mathbb{Z}_k)$ alone.

Thus we have:

Theorem 6. *If k is any odd integer greater than 1, the 3-dimensional special linear group $\mathrm{SL}(3, \mathbb{Z}_k)$ over the ring $\mathbb{Z}_k$ of integers modulo k has presentation*

$$\begin{aligned}\langle x, y \mid x^3 = y^3 = (xy)^6 = (x^{-1}yx^{-1}y^{-1}xy)^2 &= (xy^{-1}xyxy^{-1}x^{-1}y^{-1})^k \\ &= ((xy^{-1}xyxy^{-1}x^{-1}y^{-1})^{(k-1)/2}xy)^4 = 1\rangle.\end{aligned}$$

In particular, when $k = p$ for some odd prime p, the latter is a presentation of the group $\mathrm{SL}(3, p)$ and more generally if k is the product of distinct odd primes $p_1, p_2, \ldots, p_s$, it is a presentation of the direct product $\prod_{1\le i\le s}\mathrm{SL}(3, p_i)$.

Finally we note that this presentation is symmetric, in the sense that the generators x and y may be interchanged by an automorphism of the group. Equivalently, those relations that are obtained by interchanging occurrences of x and y in the given relations are also satisfied, and this is an immediate consequence of the fact that the matrix representatives X and Y considered above are conjugates of each other by $c\theta$, where θ is the inverse-transpose automorphism and c is the matrix given in the introduction.

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