THE EXISTENCE OF LEFT AVERAGING FUNCTIONS THAT ARE NOT RIGHT AVERAGING

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Abstract. Let $G$ be a locally compact group. We show that $G$ is amenable as a discrete group if and only if $\sum_{i=1}^{n} \lambda_i x_i f \in \mathcal{A}_0$ for any $f \in \mathcal{A}_0$, $x_i \in G$, and $\lambda_i > 0$ ($i = 1, 2, \ldots, n$) with $\sum_{i=1}^{n} \lambda_i = 1$, where $\mathcal{A}_0$ is the set of functions that left average to 0. We also confirm a conjecture of Rosenblatt and Yang that there is a left averaging function that is not right averaging if $G$ is not amenable.

Let $G$ be a locally compact group with a fixed left Haar measure $\lambda$. Let $L^p(G)$ be the associated real Lebesgue spaces ($1 \leq p \leq \infty$). For each $f \in L^\infty(G)$ and $x \in G$, $xf \in L^\infty(G)$ is defined by $xf(y) = f(xy)$, $y \in G$. Similarly, we can define $f_x \in L^\infty(G)$ by $f_x(y) = f(yx)$, $y \in G$. For $f \in L^\infty(G)$ and a constant $c$, we say that $f$ left averages to $c$ if $c \in || \cdot ||_\infty$-closed convex of $\{xf : x \in G\}$. Let $\mathcal{A}$ denote the set of all functions that left average to some constant, and let $\mathcal{A}_0$ denote the set of functions that left average to 0. Similarly, we say that $f$ right averages to $c$ if $c \in || \cdot ||_\infty$-closed convex of $\{f_x : x \in G\}$, and we denote the set of all the right averaging functions by $\mathcal{B}$. A positive linear functional of norm 1 on $L^\infty(G)$ is called a mean. A mean $m$ is said to be left invariant if $m(xf) = m(f)$ for any $x \in G$ and $f \in L^\infty(G)$. We denote the set of all left invariant means by $\text{LIM}(G)$. $G$ is said to be amenable if $\text{LIM}(G) \neq \emptyset$. It is well known that $G$ is amenable if $G$ is amenable as a discrete group, but the converse may fail (see [1]).

It is natural to ask if every left averaging function is right averaging. This problem was first studied by Rosenblatt and Yang [3]. They showed that when $G$ is amenable as a discrete group, $\mathcal{A} = \mathcal{B}$ if and only if $\text{LIM}(G) = \text{RIM}(G)$, where $\text{RIM}(G)$ is the set of all right invariant means i.e. the mean $m$ with $m(f_x) = m(f)$ for any $x \in G$ and $f \in L^\infty(G)$. They conjectured that when $G$ is not amenable, there is a left averaging function that is not right averaging.

In this note we prove that $G$ is amenable as a discrete group if and only if $\sum_{i=1}^{n} \lambda_i x_i f \in \mathcal{A}_0$ for any $f \in \mathcal{A}$, $x_i \in G$, and $\lambda_i > 0$ with $\sum_{i=1}^{n} \lambda_i = 1$. 

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This improves the main result of Miao [2]. Then we apply this theorem to confirm the conjecture of Rosenblatt and Yang. To prove our result, we need the following lemmas.

**Lemma A.** (a) For \( f \in L^\infty(G) \), let \( \tilde{f} \in L^\infty(G) \) be defined by \( \tilde{f}(x) = f(x^{-1}) \) \((x \in G)\). Then \( f \in \mathcal{A} \) if and only if \( \tilde{f} \in \mathcal{A}_R \).

(b) \( \mathcal{A} \subseteq \mathcal{A}_R \) if and only if \( \mathcal{A} = \mathcal{A}_R \).

**Proof.** (a) For any \( x \in G \) and \( t \in G \),

\[
(x\tilde{f})(t) = f(x^{-1}t) = \tilde{f}(tx^{-1}) = \tilde{f}_{x^{-1}}(t).
\]

Hence \((x\tilde{f}) = \tilde{f}_{x^{-1}}\). It is easy to see that the following are equivalent: (i) \( f \in \mathcal{A} \); (ii) there is a constant \( c \) such that \( c \in \text{convex} \|x\|_\infty \{xf : x \in G\} \); (iii) \( c \in \text{convex} \|x\|_\infty \{(x\tilde{f}) : x \in G\} = \text{convex} \|x\|_\infty \{\tilde{f}_{x^{-1}} : x \in G\} \); and (iv) \( \tilde{f} \in \mathcal{A}_R \).

(b) Let \( \mathcal{A} \subseteq \mathcal{A}_R \). If \( f \in \mathcal{A}_R \) then \( \tilde{f} \in \mathcal{A} \) by (a) since \( \tilde{f} = f \). Hence \( \tilde{f} = f \in \mathcal{A} \subseteq \mathcal{A}_R \) and \( f \in \mathcal{A} \) by (a) again. Therefore \( \mathcal{A} = \mathcal{A}_R \). □

**Lemma B.** \( \mathcal{A}_0 \) is a subspace if and only if for any \( f \in \mathcal{A}_0 \), \( \sum_{i=1}^n \lambda_i x_i f \in \mathcal{A}_0 \) for any \( x_i \in G \), \( \lambda_i > 0 \) \((i = 1, 2, \ldots, n)\) with \( \sum_{i=1}^n \lambda_i = 1 \).

**Proof.** Suppose that \( \mathcal{A}_0 \) is a subspace. If \( f \in \mathcal{A}_0 \), then \( xf \in \mathcal{A}_0 \) for any \( x \in G \). Let \( x_i \in G \) and \( \lambda_i > 0 \) \((i = 1, 2, \ldots, n)\), then \( \lambda_i x_i f \in \mathcal{A}_0 \) \((i = 1, 2, \ldots, n)\). So \( \sum_{i=1}^n \lambda_i x_i f \in \mathcal{A}_0 \) because \( \mathcal{A}_0 \) is a subspace.

Conversely, let \( \sum_{i=1}^n \lambda_i x_i f \in \mathcal{A}_0 \) for any \( f \in \mathcal{A}_0 \), \( x_i \in G \), and \( \lambda_i > 0 \) \((i = 1, 2, \ldots, n)\) with \( \sum_{i=1}^n \lambda_i = 1 \). It suffices to show that \( \mathcal{A}_0 \supseteq \text{span}\{xf - f : f \in L^\infty(G), x \in G\} \) by Lemma 2.1 in [2]. For \( f_i \in L^\infty(G) \) and \( x_i \in G \), \( x_i f_i - f \in \mathcal{A}_0 \). Let \( F = \sum_{i=1}^{n-1} a_i(x_i f_i - f_i) \in \mathcal{A}_0 \) for any constants \( a_i \), \( x_i \in G \) and \( f_i \in L^\infty(G) \) \((i = 1, 2, \ldots, n-1)\). If \( x_n \in G \) and \( f_n \in L^\infty(G) \), then \( \sum_{i=1}^n a_i(x_i f_i - f_i) \in \mathcal{A}_0 \) where \( a_n \) is a nonzero constant. Indeed, let \( e > 0 \). Since \( x_i f_i - f_i \in \mathcal{A}_0 \), there are \( \lambda_k > 0 \), \( y_k \in G \) \((k = 1, 2, \ldots, N)\) such that \( \sum_{k=1}^N \lambda_k y_k = 1 \) and

\[
\left\| \sum_{k=1}^N \lambda_k y_k (x_n f_n - f_n) \right\|_\infty < \frac{e}{2|a_n|}.
\]

Since \( \sum_{k=1}^N \lambda_k y_k F \in \mathcal{A}_0 \), there are \( w_l > 0 \) and \( z_l \in G \) \((l = 1, 2, \ldots, L)\) such that \( \sum_{l=1}^L w_l = 1 \) and

\[
\left\| \sum_{l=1}^L \sum_{k=1}^N \lambda_k y_k (x_n f_n - f_n) \right\|_\infty < \frac{e}{2}.
\]

Hence

\[
\left\| \sum_{l=1}^L \sum_{k=1}^N \lambda_k y_k \left( \sum_{i=1}^n a_i(x_i f_i - f_i) \right) \right\|_\infty < \frac{e}{2} + \frac{e}{2} = e.
\]

By induction, \( \mathcal{A} \supseteq \text{span}\{xf - f : x \in G, f \in L^\infty(G)\} \). □
Theorem C. For a locally compact group \( G \), \( G \) is amenable as a discrete group if and only if \( \sum_{i=1}^{n} \lambda_{ix_i}f \in \mathcal{A}_0 \) for any \( f \in \mathcal{A}_0 \), \( \lambda_i > 0 \), and \( x_i \in G \) with \( \sum_{i=1}^{n} \lambda_i = 1 \).

Proof. This is a direct consequence of Theorem 2.3 of [2] and Lemma B. \( \square \)

The following theorem confirms the conjecture of Rosenblatt and Yang [3]:

Theorem D. If \( \mathcal{A} \subseteq \mathcal{A}_R \), then \( G \) is amenable as a discrete group.

Proof. By Theorem C, it suffices to show that \( \sum_{i=1}^{n} \lambda_{ix_i}f \in \mathcal{A}_0 \) for any \( f \in \mathcal{A}_0 \), \( x_i \in G \), and \( \lambda_i > 0 \) \((i = 1, 2, \ldots, n)\) with \( \sum_{i=1}^{n} \lambda_i = 1 \). Since \( \mathcal{A}_0 \subseteq \mathcal{A} \subseteq \mathcal{A}_R \), \( f \in \mathcal{A}_R \) and it right averages to 0 by Corollary 1.4 of [3]. Hence for any \( \varepsilon > 0 \), there are \( \beta_k > 0 \), \( y_k \in G \) \((k = 1, 2, \ldots, N)\) such that \( \sum_{k=1}^{N} \beta_k = 1 \) and \( \| \sum_{k=1}^{N} \beta_k f y_k \|_\infty < \varepsilon \). Not that for each \( i = 1, 2, \ldots, n \),

\[
\| x_i \left( \sum_{k=1}^{N} \beta_k f y_k \right) \|_\infty = \| \sum_{k=1}^{N} \beta_k (x_i f) y_k \|_\infty < \varepsilon .
\]

Hence

\[
\left\| \sum_{k=1}^{N} \beta_k \left( \sum_{i=1}^{n} \lambda_{ix_i} f \right) y_k \right\|_\infty = \left\| \sum_{i=1}^{n} \lambda_i \left( \sum_{k=1}^{N} \beta_k (x_i f) y_k \right) \right\|_\infty \\
\leq \sum_{i=1}^{n} \lambda_i \left\| \sum_{k=1}^{N} \beta_k (x_i f) y_k \right\|_\infty < \varepsilon ,
\]

that is \( \sum_{i=1}^{n} \lambda_{ix_i} f \) right averages to 0. By Lemma A, \( \sum_{i=1}^{n} \lambda_{ix_i} f \in \mathcal{A} \). Hence \( \sum_{i=1}^{n} \lambda_{ix_i} f \) left averages to 0 by Corollary 1.4 of [3] again. Therefore, \( \sum_{i=1}^{n} \lambda_{ix_i} f \in \mathcal{A}_0 \). \( \square \)

References


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