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# THE EXISTENCE OF LEFT AVERAGING FUNCTIONS THAT ARE NOT RIGHT AVERAGING

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ABSTRACT. Let $G$ be a locally compact group. We show that $G$ is amenable as a discrete group if and only if $\sum_{i=1}^{n} \lambda_i {}_{x_i}f \in \mathscr{A}_0$ for any $f_0 \in \mathscr{A}_0$, $x_i \in G$, and $\lambda_i > 0$ $(i = 1, 2, \dots, n)$ with $\sum_{i=1}^{n} \lambda_i = 1$, where $\mathscr{A}_0$ is the set of functions that left average to 0. We also confirm a conjecture of Rosenblatt and Yang that there is a left averaging function that is not right averaging if $G$ is not amenable.

Let $G$ be a locally compact group with a fixed left Haar measure $\lambda$. Let $L^p(G)$ be the associated real Lebesgue spaces $(1 \le p \le \infty)$. For each $f \in L^\infty(G)$ and $x \in G$, ${}_xf \in L^\infty(G)$ is defined by ${}_xf(y) = f(xy)$, $y \in G$. Similarly, we can define $f_x \in L^\infty(G)$ by $f_x(y) = f(yx)$, $y \in G$. For $f \in L^\infty(G)$ and a constant $c$, we say that $f$ left averages to $c$ if $c \in \| \|_\infty$-closed convex hull of $\{{}_xf : x \in G\}$. Let $\mathscr{A}$ denote the set of all functions that left average to some constant, and let $\mathscr{A}_0$ denote the set of functions that left average to 0. Similarly, we say that $f$ right averages to $c$ if $c \in \| \|_\infty$-closed convex hull of $\{f_x : x \in G\}$, and we denote the set of all the right averaging functions by $\mathscr{A}_R$. A positive linear functional of norm 1 on $L^\infty(G)$ is called a mean. A mean $m$ is said to be left invariant if $m({}_xf) = m(f)$ for any $x \in G$ and $f \in L^\infty(G)$. We denote the set of all left invariant means by $\mathrm{LIM}(G)$. $G$ is said to be amenable if $\mathrm{LIM}(G) \neq \varnothing$. It is well known that $G$ is amenable if $G$ is amenable as a discrete group, but the converse may fail (see [1]).

It is natural to ask if every left averaging function is right averaging. This problem was first studied by Rosenblatt and Yang [3]. They showed that when $G$ is amenable as a discrete group, $\mathscr{A} = \mathscr{A}_R$ if and only if $\mathrm{LIM}(G) = \mathrm{RIM}(G)$, where $\mathrm{RIM}(G)$ is the set of all right invariant means i.e. the mean $m$ with $m(f_x) = m(f)$ for any $x \in G$ and $f \in L^\infty(G)$. They conjectured that when $G$ is not amenable, there is a left averaging function that is not right averaging.

In this note we prove that $G$ is amenable as a discrete group if and only if $\sum_{i=1}^{n} \lambda_i x_i f \in \mathscr{A}_0$ for any $f \in \mathscr{A}$, $x_i \in G$, and $\lambda_i > 0$ with $\sum_{i=1}^{n} \lambda_i = 1$.

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This improves the main result of Miao [2]. Then we apply this theorem to confirm the conjecture of Rosenblatt and Yang. To prove our result, we need the following lemmas.

**Lemma A.** (a) *For* $f \in L^\infty(G)$, *let* $\check{f} \in L^\infty(G)$ *be defined by* $\check{f}(x) = f(x^{-1})$ $(x \in G)$. *Then* $f \in \mathscr{A}$ *if and only if* $\check{f} \in \mathscr{A}_R$.

(b) $\mathscr{A} \subseteq A_R$ *if and only if* $\mathscr{A} = \mathscr{A}_R$.

*Proof.* (a) For any $x \in G$ and $t \in G$,

$$({}_x\check{f})(t) = f(xt^{-1}) = \check{f}(tx^{-1}) = \check{f}_{x^{-1}}(t).$$

Hence $({}_x\check{f}) = \check{f}_{x^{-1}}$. It is easy to see that the following are equivalent: (i) $f \in \mathscr{A}$; (ii) there is a constant $c$ such that $c \in \overline{\text{convex}}^{\|\cdot\|_\infty}\{{}_xf : x \in G\}$; (iii) $c \in \overline{\text{convex}}^{\|\cdot\|_\infty}\{({}_x\check{f}) : x \in G\} = \overline{\text{convex}}^{\|\cdot\|_\infty}\{\check{f}_{x^{-1}} : x \in G\}$; and (iv) $\check{f} \in \mathscr{A}_R$.

(b) Let $\mathscr{A} \subseteq \mathscr{A}_R$. If $f \in \mathscr{A}_R$ then $\check{f} \in \mathscr{A}$ by (a) since $(\check{\check{f}}) = f$. Hence $\check{f} \in \mathscr{A} \subseteq \mathscr{A}_R$ and $f \in \mathscr{A}$ by (a) again. Therefore $\mathscr{A} = \mathscr{A}_R$. □

**Lemma B.** $\mathscr{A}_0$ *is a subspace if and only if for any* $f \in \mathscr{A}_0$, $\sum_{i=1}^n \lambda_{ix_i} f \in \mathscr{A}_0$ *for any* $x_i \in G$, $\lambda_i > 0$ $(i = 1, 2, \dots, n)$ *with* $\sum_{i=1}^n \lambda_i = 1$.

*Proof.* Suppose that $\mathscr{A}_0$ is a subspace. If $f \in \mathscr{A}_0$, then ${}_xf \in \mathscr{A}_0$ for any $x \in G$. Let $x_i \in G$ and $\lambda_i > 0$ $(i = 1, 2, \dots, n)$, then $\lambda_{ix_i} f \in \mathscr{A}_0$ $(i = 1, 2, \dots, n)$. So $\sum_{i=1}^n \lambda_{ix_i} f \in \mathscr{A}_0$ because $\mathscr{A}_0$ is a subspace.

Conversely, let $\sum_{i=1}^n \lambda_{ix_i} f \in \mathscr{A}_0$ for any $f \in \mathscr{A}_0$, $x_i \in G$, and $\lambda_i > 0$ $(i = 1, 2, \dots, n)$ with $\sum_{i=1}^n \lambda_i = 1$. It suffices to show that $\mathscr{A}_0 \supseteq \text{span}\{{}_xf - f : f \in L^\infty(G),\ x \in G\}$ by Lemma 2.1 in [2]. For $f_1 \in L^\infty(G)$ and $x_1 \in G$, ${}_{x_1}f - f \in \mathscr{A}_0$. Let $F = \sum_{i=1}^{n-1} a_i({}_{x_i}f_i - f_i) \in \mathscr{A}_0$ for any constants $a_i$, $x_i \in G$ and $f_i \in L^\infty(G)$ $(i = 1, 2, \dots, n-1)$. If $x_n \in G$ and $f_n \in L^\infty(G)$, then $\sum_{i=1}^n a_i({}_{x_i}f_i - f_i) \in \mathscr{A}_0$ where $a_n$ is a nonzero constant. Indeed, let $\varepsilon > 0$. Since ${}_{x_n}f_n - f_n \in \mathscr{A}_0$, there are $\lambda_k > 0$, $y_k \in G$ $(k = 1, 2, \dots, N)$ such that $\sum_{k=1}^N \lambda_k = 1$ and

$$\left\| \sum_{k=1}^N \lambda_{ky_k}({}_{x_n}f_n - f_n) \right\|_\infty < \frac{\varepsilon}{2|a_n|}.$$

Since $\sum_{k=1}^N \lambda_{ky_k} F \in \mathscr{A}_0$, there are $w_l > 0$ and $z_l \in G$ $(l = 1, 2, \dots, L)$ such that $\sum_{l=1}^L w_l = 1$ and

$$\left\| \sum_{l=1}^L w_{lz_l} \left( \sum_{k=1}^N \lambda_{ky_k} F \right) \right\|_\infty < \frac{\varepsilon}{2}.$$

Hence

$$\begin{aligned}
&\left\| \sum_{l=1}^L w_{lz_l} \left[ \sum_{k=1}^N \lambda_{ky_k} \left( \sum_{i=1}^n a_i({}_{x_i}f_i - f_i) \right) \right] \right\|_\infty \\
&\quad \le \left\| \sum_{l=1}^L w_{lz_l} \left( \sum_{k=1}^N \lambda_{ky_k} F \right) \right\|_\infty + |a_n| \sum_{l=1}^L w_l \left\| \sum_{k=1}^N \lambda_{ky_k}({}_{x_n}f_n - f_n) \right\|_\infty \\
&\quad < \varepsilon/2 + \varepsilon/2 = \varepsilon.
\end{aligned}$$

By induction, $\mathscr{A} \supseteq \text{span}\{{}_xf - f : x \in G,\ f \in L^\infty(G)\}$. □

**Theorem C.** *For a locally compact group $G$, $G$ is amenable as a discrete group if and only if $\sum_{i=1}^{n} \lambda_i x_i f \in \mathscr{A}_0$ for any $f \in \mathscr{A}_0$, $\lambda_i > 0$, and $x_i \in G$ with $\sum_{i=1}^{n} \lambda_i = 1$.*

*Proof.* This is a direct consequence of Theorem 2.3 of [2] and Lemma B. □

The following theorem confirms the conjecture of Rosenblatt and Yang [3]:

**Theorem D.** *If $\mathscr{A} \subseteq \mathscr{A}_R$, then $G$ is amenable as a discrete group.*

*Proof.* By Theorem C, it suffices to show that $\sum_{i=1}^{n} \lambda_i x_i f \in \mathscr{A}_0$ for any $f \in \mathscr{A}_0$, $x_i \in G$, and $\lambda_i > 0$ $(i = 1, 2, \dots, n)$ with $\sum_{i=1}^{n} \lambda_i = 1$. Since $\mathscr{A}_0 \subseteq \mathscr{A} \subseteq \mathscr{A}_R$, $f \in \mathscr{A}_R$ and it right averages to 0 by Corollary 1.4 of [3]. Hence for any $\varepsilon > 0$, there are $\beta_k > 0$, $y_k \in G$ $(k = 1, 2, \dots, N)$ such that $\sum_{k=1}^{N} \beta_k = 1$ and $\| \sum_{k=1}^{N} \beta_k f_{y_k} \|_\infty < \varepsilon$. Not that for each $i = 1, 2, \dots, n$,

$$\left\| {}_{x_i}\left( \sum_{k=1}^{N} \beta_k f_{y_k} \right) \right\|_\infty = \left\| \sum_{k=1}^{N} \beta_k (x_i f)_{y_k} \right\|_\infty < \varepsilon .$$

Hence

$$\begin{aligned} \left\| \sum_{k=1}^{N} \beta_k \left( \sum_{i=1}^{n} \lambda_i x_i f \right)_{y_k} \right\|_\infty &= \left\| \sum_{i=1}^{n} \lambda_i \left( \sum_{k=1}^{N} \beta_k (x_i f)_{y_k} \right) \right\|_\infty \\ &\le \sum_{i=1}^{n} \lambda_i \left\| \sum_{k=1}^{N} \beta_k (x_i f)_{y_k} \right\|_\infty < \varepsilon , \end{aligned}$$

that is $\sum_{i=1}^{n} \lambda_i x_i f$ right averages to 0. By Lemma A, $\sum_{i=1}^{n} \lambda_i x_i f \in \mathscr{A}$. Hence $\sum_{i=1}^{n} \lambda_i x_i f$ left averages to 0 by Corollary 1.4 of [3] again. Therefore, $\sum_{i=1}^{n} \lambda_i x_i f \in \mathscr{A}_0$. □

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