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RELATIONS BETWEEN CHAIN RECURRENT POINTS AND TURNING POINTS ON THE INTERVAL

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ABSTRACT. If a point is in the ω-limit set and the α-limit set of the same point, then we call it a γ-limit point. Then a γ-limit point is an ω-limit point and thus a nonwandering point. In this paper, we prove that, on the interval, a nonwandering point which is not a γ-limit point is in the closure of the set of forward images of turning points, and such points are not always the forward images of turning points. But a nonwandering point which is not an ω-limit point forward image of some turning point. Two examples are given. One shows that a chain recurrent point which is not nonwandering, a γ-limit point which is not recurrent and a recurrent point which is not periodic need not be in the closure of forward images of turning points. The other shows that an ω-limit point which is not a γ-limit point can be a limit point of forward images of turning points but not a forward image nor an ω-limit point of any turning point.

Let $I = [0, 1]$ be a compact interval on the real line. Let $f: I \to I$ be a continuous map. For each $x \in I$, we let $\mathrm{Orb}(x) = \{y \in I: y = f^i(x),\ i > 0\}$. Let $T = \{x \in I: f$ is not a local homeomorphism at $x\}$ denote the set of turning points of f and $\mathrm{Orb}(T) = \{y: y = f^i(x),\ i > 0,\ x \in T\}$ denote the set of forward images of turning points of f. A point x is called a periodic point if $f^n(x) = x$ for some $n > 0$. A point $x \in I$ is called an ω-limit (α-limit) point if there is a $y \in I$ such that x is an accumulation point of $\{f^n(y)\}$ (there is a sequence of integers $\{n_i\}$ with $n_i \to \infty$ and a sequence of points $\{y_{n_i}\}$ with $f^{n_i}(y_{n_i}) = y$ such that $x = \lim_{i\to\infty} y_{n_i}$). Denote the set of ω-limit points (α-limit points) of x by $\omega(x)$ $(\alpha(x))$. If $x \in \omega(y) \cap \alpha(y)$ for some y, then x is called a γ-limit point. A point x is called a chain recurrent point if for any number $\varepsilon > 0$ there exists a sequence of points, $x_0 = x, x_1, \dots, x_{n-1}$, $x_n = x$, such that $|f(x_i) - x_{i+1}| < \varepsilon$ for $i = 0, \dots, n-1$. $CR(f)$, $\Omega(f)$, $\Lambda(f)$, $\Gamma(f)$, $R(f)$, $P(f)$ represent the collection of the chain recurrent points, nonwandering points, ω-limit points, γ-limit points, recurrent points, periodic points, respectively. Without causing confusion, we write CR, Ω, Λ, Γ, R, P simply. Since Xiong has proved that $\Lambda^2(f) = \Gamma(f)$ for any continuous map on the interval in [X1], where $\Lambda^2(f) = \Lambda(f|_\Lambda)$, we will abuse $\Lambda^2(f)$ and $\Gamma(f)$

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in this paper. Sarkovskii [S] has shown that Λ is closed. Thus $\overline{P} \subset \Lambda$. In [X1], Xiong shows that $R \subset \Gamma \subset \overline{P}$. It is easy to see that $\Lambda \subset \Omega \subset CR$. Therefore

$$(*) \qquad P \subset R \subset \Gamma = \Lambda^2 \subset \overline{P} \subset \Lambda \subset \Omega \subset CR.$$

In this paper we will determine which classes of chain recurrent points (as determined by $(*)$) must be contained in $\overline{\mathrm{Orb}(T)}$. By Example 2, we can see that points in P and $CR \setminus \Omega$ may not be contained in $\overline{\mathrm{Orb}(T)}$. Thus we need only consider $\Omega \setminus P$.

For any continuous mapping $f: I \to I$ we have the following theorems.

Theorem 1. $\Omega(f) \setminus \Gamma(f) \subset \overline{\mathrm{Orb}(T)}$.

Theorem 2. $\Omega(f) \setminus \Lambda(f) \subset \mathrm{Orb}(T)$.

For piecewise monotone maps, Nitecki [N] has proved that $\Lambda \subset \overline{P}$. Then $\Lambda = \overline{P}$. Xiong [X2] has furthermore proved that for any point $x \in \overline{P}$ there is a point $y \in \overline{P}$ such that $x \in \omega(y)$. That is, $\overline{P} = \Lambda(f|_{\overline{P}})$. In [X1], Xiong proves that $\Gamma = \Lambda(f|_{\overline{P}}) = \Lambda(f|_{\Lambda}) = \Lambda^2$ for any continuous map f. Thus $\Lambda = \overline{P} = \Gamma = \Lambda^2$ for piecewise monotone maps, and $\Omega \setminus \Gamma \subset \mathrm{Orb}(T)$. But this fact cannot be generalized to any continuous maps. We will point out in Example 1 that the Example 1 in [BC] is a counterexample; in fact, there is a point $x \in \Lambda \setminus \Gamma$ with $x \notin \mathrm{Orb}(T)$. We also point out that the point $x \notin \bigcup_{t \in T} \omega(t)$.

In Example 2, we will find out that $\Gamma \setminus P$ need not be a subset of $\overline{\mathrm{Orb}(T)}$. Xiong [X2] has proved that if f has no periodic points of periods not a power of 2 and f is piecewise monotone, then any point in $R \setminus P$ is an ω-limit point of some turning point. If we check the proof of this property in [X2] carefully, we can see that $R \setminus P \subset \overline{\mathrm{Orb}(T)}$ for any continuous map with no periodic points of period not the power of 2. This fact is not true if f has a periodic point with period not the power of 2. Example 2 is a counterexample.

Without mention, f is always a continuous mapping of I to itself.

1. Proof of Theorem 1

The following lemmas are known.

Lemma 1 ([CN]). *$x \in \Omega$ if and only if $x \in \alpha(x)$.*

Lemma 2. *If $(f(x) - x)(f(y) - y) < 0$, then there exists a fixed point between x and y.*

It is easy to prove the following lemma.

Lemma 3. $T(f^n) \subset \bigcup_{i=0}^{n-1} f^{-i}(T(f))$ *(therefore, if $q \in T(f^n)$, then there is a point $q' \in T(f)$ such that $f^n(q) \in \mathrm{Orb}(q')$).*

Proof of Theorem 1. Suppose that $x \in \Omega \setminus \overline{P}$ and C is the connected component of $I \setminus \overline{P}$ which contains x. Obviously, $x \notin R$ since $\overline{P} = \overline{R} \supset R$ [CH, Y]. Then, there exist two points a, $b \in C$ such that $(a, b) \cap \mathrm{Orb}(x) = \varnothing$.

By Lemma 1, without loss of generality, there exist a sequence of points $y_i \in (a, x)$ and a sequence of positive integers $n_i \to \infty$ such that $f^{n_i}(y_i) = x$ and $y_i \to x$ as $i \to \infty$. Obviously, we could suppose that $y_i < y_{i+1}$ and $n_i < n_{i+1}$.

Since $f^{n_i}(y_i) = x > y_i$ and $(a, b) \cap P = \varnothing$, it follows that $f^{n_i}(y_{i-1}) > y_{i-1}$ by Lemma 2. Also $f^{n_i}(y_{i-1}) = f^{n_i - n_{i-1}}(f^{n_{i-1}}(y_{i-1})) = f^{n_i - n_{i-1}}(x)$. Since $(a, b) \cap \mathrm{Orb}(x) = \varnothing$, it follows that $f^{n_i}(y_{i-1}) \geq b$. By the same reason, $f^{n_i}(x) \geq b$.

Since $f^{n_i}(y_i) = x < b$, there must be a turning point $q \in (y_{i-1}, x)$ of f^{n_i} with $q < f^{n_i}(q) \geq x$. This is to say that $f^{n_i}(q) \in (y_{i-1}, x]$. By Lemma 3, there exists a point $q' \in T(f)$ such that $f^{n_i}(q) \in \mathrm{Orb}(q')$.

Note that q is a function of i. Since $f^{n_i}(q) \to x$ as $i \to \infty$ it follows that $x \in \overline{\mathrm{Orb}(T(f))}$.

Thus $\Omega \setminus \overline{P} \subset \overline{\mathrm{Orb}(T(f))}$. Note that the proof of Theorem 1 in [X2] implies that $\overline{P} \setminus \Gamma \subset \overline{\mathrm{Orb}(T(f))}$. Therefore we have proved the theorem. □

2. Proof of Theorem 2

Lemma 4. *If n_i are natural numbers and K_i $(i = 1, 2, \ldots)$ are closed intervals such that $K_{i+1} \subset K_i$ and $f^{n_i}(K_i) \supset K_{i+1}$ for all $i \geq 1$, and $\bigcap_{i=1}^{\infty} K_i$ consists of a single point x, then there exists a point $y \in K_1$ such that $f^{l_j}(y) \in K_{j+1}$ and $f^{l_j}(y) \to x$ as $j \to \infty$ with $l_j = \sum_{i=1}^{j} n_i$.*

Proof. By assumption, we have $f^{l_i}(K_1) \supset K_{i+1}$ for all $i \geq 1$. We can form a sequence of closed intervals $\{K'_j\}$ such that $K'_1 \subset K_1$, $K'_{j+1} \subset K'_j$, and $f^{l_j}(K'_j) = K_{j+1}$ for each positive integer j. It follows that $f^{l_i}(K'_j) \subset K_{i+1}$ for all $i < j$. Let $y \in \bigcap_{j=1}^{\infty} K'_j$. Then $f^{l_j}(y) \in K_{j+1}$ for all $j \geq 1$ and furthermore $f^{l_j}(y) \to x$. □

If C is a component of $I \setminus \overline{P}$, it has been shown [X4, C] that for any x, $y \in C$,

$$(f^n(x) - x)(f^m(y) - y) > 0$$

for all integers n, $m > 0$ such that $f^n(x)$, $f^m(y) \in C$.

If $f^n(x) > x$ whenever $f^n(x) \in C$ (or $f^n(x) < x$), we say that C is of increasing type (or decreasing type).

Lemma 5. *If $x \notin \Lambda$ is in a component C of $I \setminus \overline{P}$ of increasing type, then there is an interval $[z, x]$ such that, for any $y \in [z, x]$ and any positive integer n, $f^n(y) \notin [z, x)$.*

Proof. Suppose it is not true. Then there is an increasing sequence of points $y_i \in C$ with $y_i \neq x$ and $y_i \to x$, and a sequence of integers $n_i > 0$ such that $y_i < f^{n_i}(y_i) < y_{i+1} < x$. Since $f^{n_i}(y_i) > y_i$ and there is no periodic point in $[y_i, x]$, we know that $f^{n_i}(x) > x$ by Lemma 2.

Denote $K_i = [y_i, x]$. Then

$$f^{n_i}(K_i) \supset [f^{n_i}(y_i), f^{n_i}(x)] \supset K_{i+1}.$$

Also, $\bigcap_{i=1}^{\infty} K_i$ consists of a single point x. By Lemma 4, there is a point $y \in K_1$, so that $f^{l_j}(y) \to x$ where $l_j = \sum_{i=1}^{j} n_i$. It follows that $x \in \omega(y) \subset \Lambda$. This contradicts the assumption that $x \notin \Lambda$. □

Proof of Theorem 2. Let $x \in \Omega \setminus \Lambda$. Recall that Λ is closed. Since $x \notin \Lambda$ and $\Lambda \supset R$, we have $x \notin R$. Then there is an interval (a, b) containing no points of Λ such that $x \in (a, b)$ and $(a, b) \cap \mathrm{Orb}(x) = \varnothing$. Also, as in the proof of Theorem 1, for any $x \in \Omega \setminus \Lambda$ we may assume without loss of generality that there exist a sequence of points $y_i < x$ in (a, b) and an increasing sequence of positive integers $n_i \to \infty$ such that $f^{n_i}(y_i) = x$, for all $i = 1, 2, \ldots$ and $y_i \to x$. Then (a, b) is of increasing type. By Lemma 5, there is an interval $[z, x]$ so that, for any $y \in [z, x]$ and any integer $n > 0$, we have $f^n(y) \notin [z, x)$. Without loss of generality, we can assume that $y_1, y_2, \ldots, y_i, \ldots$ are all in $[z, x]$. Then $f^n(y_i) \notin [z, x)$.

By Lemma 2, for any point w in (y_1, y_3), we have $f^{n_2}(w) \geq x$ since $f^{n_2}(y_2) = x$. Thus y_2 is a turning point of f^{n_2}. By Lemma 3, $x = f^{n_2}(y_2) \in \mathrm{Orb}(T(f))$. □

3. Examples

Proposition 6 [X3]. *$\overline{R(f)} = R(f)$ implies that f has only periodic points of periods 2^n, $n \geq 0$.*

In Example 1 we show that $\Lambda \setminus \overline{P}$ is not always contained in $\mathrm{Orb}(T)$.

Example 1 [BC]. Let $f\colon [-1, 2] \to [-1, 2]$ be the map constructed by Block and Coven in Example 1 of [BC]. The following properties are either already observed in [BC] or easy to be seen.

(1) The Cantor middle third set C in $[0, 1]$ is a minimal set under f.
(2) $\omega(-\frac{2}{3}) = C$.
(3) $-\frac{2}{3} \in \Lambda \setminus \overline{P}$.
(4) Any point $t \in [-\frac{2}{3}, 0]$ is a turning point and $f(t) = \frac{2}{3} \in C$. 1 is a turning point and $1 \in C$. Let $s_n = \frac{3^n - 1}{3^n}$ for integers $n \geq 1$. Then each s_n is a turning point and $s_n \in C$.
(5) Except for those turning points described in (4), all other turning points can be represented by $t_n = 1 - \frac{1}{2 \times 3^n}$ for $n \geq 0$. Moreover $f^{m_n}(t_n) = -\frac{2}{3} - \frac{1}{3^{n+2}}$ for some integer $m_n > 0$ and $f^2(-\frac{2}{3} - \frac{1}{3^n}) = \frac{2}{9} + \frac{1}{3^{n+1}} \in C$.

By (5) we know that $-\frac{2}{3} \in \overline{\mathrm{Orb}(T)}$. $-\frac{2}{3}$ can never be the image of a turning point in (4) since $f(C) \subset C$ by (1). For the t_n's in (5), we know that for $0 < j < m_n$ we have $f^j(t_n) \in [0, 2]$. Also, $f^{m_n}(t_n) = -\frac{2}{3} - \frac{1}{3^{n+2}}$, $f^{m_n+1}(t_n) \in [\frac{1}{3}, \frac{2}{3}]$, and $f^{m_n+2}(t_n) \in C$. Thus $-\frac{2}{3} \notin \mathrm{Orb}(t_n)$. Therefore $-\frac{2}{3} \notin \mathrm{Orb}(T)$. By (3), $-\frac{2}{3} \in (\Lambda \setminus \overline{P}) \cap (\overline{\mathrm{Orb}(T)} \setminus \mathrm{Orb}(T))$.

Moreover by (1) and the above argument it is easy to see that $\omega(t) \subset C$ for any turning point $t \in T$. Thus $-\frac{2}{3} \notin \bigcup_{t \in T} \omega(t)$.

Example 2. (See Figure 1) Let $f\colon [0, 3] \to [0, 3]$ be defined as follows.

$$f(x) = \begin{cases} \sqrt{x}, & \text{when } x \in [0, 1], \\ 4x - 3, & \text{when } x \in [1, \frac{3}{2}], \\ -2x + 6, & \text{when } x \in \frac{3}{2}, 3]. \end{cases}$$

For any point $x \in (0, 1)$ we have that $1 \in \alpha(x) \cap \omega(x)$. Note that 1 is a fixed point. So $x \in CR$. But $x \notin \Omega$. Thus $x \in CR \setminus \Omega$. Obviously $x \notin \overline{\mathrm{Orb}(T)}$.

Next we show that $\Gamma \setminus R$ may not be contained in $\overline{\mathrm{Orb}(T)}$.

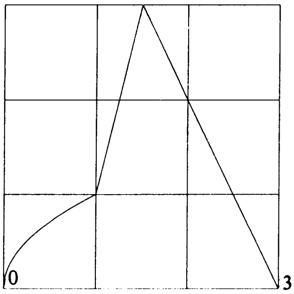

FIGURE 1. $P \not\subset \overline{\mathrm{Orb}(T)}$; $CR \setminus \Omega \not\subset \overline{\mathrm{Orb}(T)}$; $(\Gamma \setminus R) \cap \overline{\mathrm{Orb}(T)} = \varnothing$.

The turning point of f is $\frac{3}{2}$. $\mathrm{Orb}(\{\frac{3}{2}\}) = \{0, \frac{3}{2}, 3\}$. Since $\frac{3}{2}$ and 3 are wandering points and 0 is a fixed point, we have $\Omega = \Gamma$ by Theorem 1. Moreover, 0, $\frac{3}{2}$, and 3 are not in the set $\Gamma \setminus R$, i.e., $(\Gamma \setminus R) \cap \overline{\mathrm{Orb}(T)} = \varnothing$. It is easy to see that f has a periodic point with period 3. Since $\overline{R} \subset \Omega = \Gamma$ it follows from Proposition 6 that $\Gamma \setminus R \neq \varnothing$.

Moreover $R \neq P$. For otherwise, $\overline{P} = P$ by [X5], a contradiction to $\Gamma \setminus R \neq \varnothing$. Evidently $\frac{3}{2}$ and 3 are not recurrent points. Thus $(R \setminus P) \cap \overline{\mathrm{Orb}(T)} = \varnothing$. Without mention, f is always a continuous mapping of I to itself.

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