THE DISTRIBUTION OF SMOOTH NUMBERS IN ARITHMETIC PROGRESSIONS

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Abstract. We estimate the number of integers $n$ up to $x$ in the arithmetic progression $a \pmod{q}$ with $n$ free of prime factors exceeding $y$. For a wide range of the variables $x$, $y$, $q$, and $a$ we show that this number is about $x/(qu^u)$, where $u = \log x / \log y$.

1. Introduction

We say a natural number $n$ is $y$-smooth if every prime factor $p$ of $n$ satisfies $p \leq y$. Let $\psi(x, y)$ denote the number of $y$-smooth integers up to $x$. Thanks to many researchers in this century, we now know a great deal about the function $\psi(x, y)$. Less studied is the function $\psi(x, y; q, a)$ which denotes the number of $y$-smooth integers $n \leq x$ with $n \equiv a \pmod{q}$. It is the object of this paper to get reasonable estimate for $\psi(x, y; q, a)$ for a wide range of the four variables.

Before stating the precise result we obtain, we briefly review some prior work. It is known that

\begin{equation}
\psi(x, y) \sim \rho(u)x, \quad u = \log x / \log y,
\end{equation}

uniformly in a large portion of the $x$, $y$ plane. Here, $\rho$ is the continuous solution to the differential difference equation $u\rho'(u) = -\rho(u - 1)$ for $u > 1$, with the initial condition $\rho(u) = 1$ for $u \leq 1$. Results of this nature are due to Dickman, Buchstab and de Bruijn. The largest $x$, $y$ region for which we know (1.1) is due to Hildebrand [Hi]:

\begin{equation}
y \geq \exp\{ (\log \log x)^{5/3 + \epsilon} \}
\end{equation}

for any fixed $\epsilon > 0$. For (1.1) to be useful, it would be good to know estimates for $\rho(u)$. This is provided by de Bruijn [de B]:

\begin{equation}
\rho(u) = \exp\{ -u(\log u + \log \log(u + 1) + O(1)) \}
\end{equation}

for $u \geq 1$. (In the cited paper, a more precise estimate of $\rho(u)$ is given.)

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What do we expect for $\psi(x, y; q, a)$? For this quantity to be nonzero it is necessary that no prime factor of $(a, q)$ exceed $y$. If this occurs, then

$$\psi(x, y; q, a) = \psi \left( \frac{x}{(a, q)}, y; \frac{q}{(a, q)}, \frac{a}{(a, q)} \right).$$

Thus we may as well assume that $(a, q) = 1$. Let $\psi_q(x, y)$ denote the number of $y$-smooth integers $n \leq x$ with $(n, q) = 1$. Then one might expect these $\psi_q(x, y)$ integers $n$ to be roughly equally distributed in the $\varphi(q)$ residue classes $a \pmod{q}$ with $(a, q) = 1$. Thus a natural function with which to compare $\psi(x, y; q, a)$ is $\frac{1}{\varphi(q)} \psi_q(x, y)$. Recently, Fouvry and Tenenbaum [F-T] have shown that

$$\psi(x, y; q, a) = \frac{1}{\varphi(q)} \psi_q(x, y) \left( 1 + O_A \left( e^{-c_1 (\log y)^{1/2}} \right) \right)$$

uniformly for

$$x \geq 3, \quad \exp \{ c_2 (\log \log x)^2 \} \leq y \leq x, \quad 1 \leq q \leq (\log x)^4, \quad (a, q) = 1$$

where $c_1, c_2$ are positive, absolute constants and $A > 0$ is arbitrary. (They also have a similar result when $q \leq e^{c_1 (\log y)^{1/2}}$.) Further, they show that

$$\psi_q(x, y) = \frac{\varphi(q)}{q} \psi(x, y) \left( 1 + O_\varepsilon \left( \frac{\log \log(qy) \log \log x}{\log y} \right) \right)$$

if

$$x \geq x_0(\varepsilon), \quad \exp \left\{ (\log \log x)^{5/3 + \varepsilon} \right\} \leq y \leq x,$$

and

$$\log \log (q + 2) \leq \left( \frac{\log y}{\log(u + 1)} \right)^{1-\varepsilon}.$$

In particular, (1.5) holds under hypothesis (1.4).

In another recent paper, Granville [G] has obtained (1.3) with a weaker error term, but in a range much wider than (1.4). In particular, he shows

$$\psi(x, y; q, a) = \frac{1}{\varphi(q)} \psi_q(x, y) \left( 1 + O \left( \frac{\log q}{\log y} \right) \right)$$

uniformly holds in the range

$$2 \leq y \leq x, \quad 1 \leq q \leq \min \{ x, y^\alpha \}, \quad (a, q) = 1.$$ 

Here, $\alpha$ is any fixed positive quantity. Thus (1.7) contains the asymptotic result

$$\psi(x, y; q, a) \sim \frac{1}{\varphi(q)} \psi_q(x, y)$$

as $x \to \infty$, $\log q / \log y \to 0$.

In this paper, we do not get an asymptotic estimate for $\psi(x, y; q, a)$, but rather, upper and lower bounds. Note that by combining (1.1)–(1.3) and (1.5), one has

$$\psi(x, y; q, a) = \frac{x}{q} \exp \{ -u(\log u + \log \log(u + 1) + O(1)) \}$$

in the range (1.4). By (1.7), this result continues to hold for values of $q$ satisfying (1.6) and $q \leq y^\varepsilon$. We are able to show that the double inequality of (1.8) holds for a still larger range.
Theorem. For each \( \varepsilon > 0 \) and all \( x, y, q, a \) satisfying

\[
\begin{align*}
   (1.9) \quad x & \geq 2, \quad \exp\{(\log \log x)^2\} \leq y \leq x^{2/3-\varepsilon}, \\
   1 & \leq q \leq \min\{y^{4/3-\varepsilon}, (x/y)^{4/3-\varepsilon}\}, \quad (a,q) = 1,
\end{align*}
\]

we have

\[
(1.10) \quad \psi(x, y; q, a) = \frac{x}{q} \exp\{-u(\log u + \log \log u + O(1))\},
\]

where \( u = \log x/\log y \). The constants implied by the \( O \) notation depend at most on the choice of \( \varepsilon \).

The proof falls naturally into two cases: the upper bound implicit in (1.10) and the lower bound. The upper bound result rests strongly on [F2] in which a stronger result is proved for a somewhat narrower range. Our proof of the lower bound implicit in (1.10) bears some similarity to the method in [F1]. In particular, we too show that the case \( u \geq 2 \) follows from the case \( u < 2 \). However, our proof also introduces new elements including the Weyl-Hooley estimation of incomplete Kloosterman sums, the lower bound in the fundamental lemma of the sieve and a combinatorial argument reminiscent of [CEP].

Finally, we mention that in [G], Granville, using the theorem above, has shown that

\[
\frac{y}{\psi(x, y; q, a)} \sim \frac{y}{q(x, y)}
\]

for \( x, y, q, \) and \( a \) satisfying (1.9) and, in fact, in the expanded region where the lower bound on \( y \) is replaced with \( y \geq 2 \).

2. THE UPPER BOUND

In this section we establish the upper bound implicit in the theorem. We begin by stating the following result from Friedlander [F2].

Lemma 2.1. If \( (a, q) = 1, \) \( x \geq q^2y^5, \) and \( y \geq \exp\{(\log x)^{4/5}\}, \) then

\[
\psi(x, y; q, a) \ll \frac{x}{q} \rho \left( u - 4 - \frac{\log q}{\log y} \right),
\]

where \( u = \log x/\log y \).

Using (1.2) we have that

\[
\rho \left( u - 4 - \frac{\log q}{\log y} \right) = \exp\{-u(\log u + \log \log u + O(1))\}
\]

for \( q, y \) satisfying (1.9). Thus Lemma 2.1 gives us the upper bound in the theorem for

\[
\exp\{(\log x)^{4/5}\} \leq y \leq x^{1/8}.
\]

Our task in the remainder of this section is to expand this interval to \( \exp\{(\log \log x)^2\} \leq y \leq x^{2/3-\varepsilon} \).

The range \( x^{1/8} \leq y \leq x^{2/3-\varepsilon} \) is basically trivial since \( \psi(x, y; q, a) \) is majorized by the number of integers \( n \leq x \) in the residue class \( a \mod q \). Thus

\[
\psi(x, y; q, a) \leq \frac{x}{q} + 1,
\]
which gives the upper bound in the theorem when \( u \) is bounded from infinity and from 1.

We now consider the interval \( \exp\{(\log \log x)^2\} \leq y \leq \exp\{(\log x)^4/5\} \). We shall use an argument similar to that of Friedlander's for Lemma 2.1. Our proof, in fact, works for \( y \) up to \( x^{1/\log \log x} \), but not beyond. So we still need the more precise Lemma 2.1, at least for the range \( x^{1/\log \log x} \leq y \leq x^{1/8} \).

Let \( P(n) \) denote the largest prime factor of \( n \). If \( x/y^3 < n \leq x \) and \( P(n) \leq y \), then \( n \) has at least one factorization as \( ml \) where \( x/y^3 < m \leq x/y^2 \). Thus

\[
\psi(x, y; q, a) \leq \psi\left(\frac{x}{y^3}, y; q, a\right) + \sum_{\frac{x}{y^3} < m \leq \frac{x}{y^2}} \sum_{\substack{l \leq x/m \atop P(m) \leq y \atop ml \equiv a \pmod{q} \atop (m, q) = 1}} 1
\]

\[
\leq \psi\left(\frac{x}{y^3}, y\right) + \sum_{x/y^3 < m \leq x/y^2} \left(\frac{x}{mq} + 1\right).
\]

Now

\[
\sum_{x/y^3 < m \leq x/y^2} \left(\frac{x}{mq} + 1\right) \leq 2\frac{x}{q} \sum_{x/y^3 < m \leq x/y^2} \frac{1}{m}
\]

\[
\leq 2\frac{x}{q} \left\{ \frac{\psi(x/y^2, y)}{x/y^2} + \int_{x/y^3}^{x/y^2} \frac{1}{t^2} \psi(t, y) \, dt \right\}.
\]

Using the fact that (1.2) and the Hildebrand region of validity of (1.1) imply

\[
\psi(t, y) = t \cdot \exp\{-u(\log u + \log \log u + O(1))\}
\]

uniformly for \( x/y^3 \leq t \leq x/y^2 \), \( \exp\{(\log \log x)^2\} \leq y \leq x^{1/4} \) (where \( u = \log x/\log y \)), we have from (2.1) and (2.2) that

\[
\psi(x, y; q, a) \leq \left(\frac{x}{y^3} + 2\frac{x}{q} + 2\frac{x}{q} \int_{x/y^3}^{x/y^2} \frac{dt}{t} \right) \exp\{-u(\log u + \log \log u + O(1))\}
\]

\[
= \frac{x \log y}{q} \exp\{-u(\log u + \log \log u + O(1))\}.
\]

This inequality gives the upper bound we are looking for provided \( \log y = \exp\{O(u)\} \), which holds if \( y \leq x^{1/\log \log x} \).

3. The Lower Bound

We may suppose \( \epsilon \) in the theorem satisfies \( 0 < \epsilon < 1/10 \). Fix \( \delta \) as \( \epsilon/4 \) and assume \( y^{2-\delta} \leq x \). Consider a number \( ml \leq x \) where

\[
\frac{x}{y^{2-\delta}} \leq m \leq \frac{x}{y^{2-2\delta}}
\]

(3.1)

and every prime \( p|m \) satisfies \( y^{\delta/2} \leq p \leq y \), \( p \nmid q \).

The number of representations of \( n \leq x \) in the form \( ml \), where \( m \) satisfies (3.1), is at most

\[
\left(1 + \frac{2u}{\delta}\right)^{4/\delta} \leq u^c
\]

(3.2)
for some constant $c_5$. Indeed the factor $l$ of $n$ has all of the prime factors of $n$ below $y^{\delta/2}$, so it is determined by its prime factors exceeding $y^{\delta/2}$. But $n$ has at most $2u/\delta$ such prime factors and $l$ can have at most $4/\delta$ of them, since $l \leq x/m < y^2$.

From (3.2) we have

\begin{equation}
\psi(x, y; q, a) \geq u^{-c_5} \sum_{m} \psi\left( \frac{x}{m}, y, q, a\v q \right),
\end{equation}

where $m$ satisfies (3.1) and $m\v q \equiv 1 \pmod{q}$. For any fixed $m$ let $w = x/m$, $b = a\v q$. Note that (3.1) implies $y^{2-2\delta} \leq w \leq y^{2-\delta}$. Thus

\[ y \leq w^{(2-2\delta)^{-1}} < w^{(2-2\delta)^{-1}} < w^{2/3-\v}, \]

and

\[ (w/y)^{(4/3)(1-\delta)} \geq (y^{1-2\delta})(4/3)(1-\delta) \geq y^{4/3-4\delta} = y^{4/3-\v} \geq q. \]

From these inequalities we see that $w$, $y$, $q$, and $b$ satisfy the hypothesis of the theorem (with $\v$ replaced with $\v' = \v/4$) and $y^{2-\delta} \geq w$. But if $\v' = \v/4$, we have $y^{2-\v'} > y^{2-\delta} \geq w$.

Suppose the theorem holds when $y^{2-\delta} \geq x$. In this case the theorem asserts that $\psi(x, y; q, a) \gg x/q$. The remaining case $y^{2-\delta} \leq x$ would thus follow from using this estimate on the right side of (3.3) and a lower bound for $\sum 1/m$. That is, we need only show two things: the case $y^{2-\delta} \geq x$ and the following lemma.

**Lemma 3.1.** If $0 < \delta < 1$, $\exp\{(\log \log x)^2\} \leq y$, $y^{2-\delta} \leq x$, then

\[ \sum_{m} \frac{1}{m} \geq \exp\{-u(\log u + \log \log u + O_\delta(1))\}, \]

where $m$ satisfies (3.1) and $u = \log x/\log y$.

### 4. Proof of Lemma 3.1

We divide the proof into two cases: $2 - \delta \leq u \leq e^{3/\delta}$, $u > e^{3/\delta}$. In the first case, let

\[ k := \left\lceil \frac{2(u - 2)}{\delta} \right\rceil + 3 \geq 1. \]

Suppose $m$ is a product of $k$ not necessarily distinct primes $p$ with $y^{\delta/2} < p \leq y^{(1+1/k)\delta/2}$ and $p \nmid q$. Then

\[ m > y^{k\delta/2} > y^{u-2+\delta} = \frac{x}{y^{2-\delta}}, \]

\[ m \leq y^{(k+1)\delta/2} \leq y^{u-2+2\delta} = \frac{x}{y^{2-2\delta}}. \]

That is, such a number $m$ satisfies (3.1). Thus

\begin{equation}
\sum 1/m \geq S^k/k!,
\end{equation}

where $S$ is the sum $\sum 1/p$ where $p$ runs over the primes in $(y^{\delta/2}, y^{(1+1/k)\delta/2}]$ which do not divide $q$. Since $u \leq e^{3/\delta}$ implies $k = O_\delta(1)$, we have $S \gg \delta 1$, so that (4.1) gives

\[ \sum 1/m \gg \delta 1 \gg \exp\{-u(\log u + \log \log u + O(1))\}. \]
We now assume $u > e^{3/\delta}$. Consider integers $m = m_1m_2$ where $m_1$ is composed of $v_1 := [u - 2]$ not necessarily distinct primes $p$ with

$$y^{1-1/\log u} < p \leq y, \quad p \nmid q,$$

and $m_2$ is composed of $v_2$ not necessarily distinct primes $p$ with

$$y^{\delta/2} < p \leq y^{(1+1/u)\delta/2}, \quad p \nmid q,$$

where $v_2 = v_2(m_1)$ is that integer such that

$$y^{(v_2-1)\delta/2} < x < y^{v_2\delta/2}. \quad (4.4)$$

We have to check that such integers $m = m_1m_2$ satisfy (cf. (3.1))

$$\frac{x}{y^{2-\delta}} < m \leq \frac{x}{y^{2-2\delta}}. \quad (4.5)$$

We first note that

$$2 \leq v_2 < 3 + \frac{2}{\delta} + \frac{2u}{\delta \log u} < u. \quad (4.5)$$

To see this note that $m_1 \leq y^{u-2} = x/y^2$ by (4.2), so that $v_2 \geq 2$ follows at once from (4.4). On the other hand,

$$m_1 > y^{v_1(1-1/\log u)} > y^{(u-3)(1-1/\log u)} > \frac{x}{y^{3+u/\log u}},$$

so that (4.4) implies

$$v_2 < 1 + \frac{\log(x/m_1 y^{2-\delta})}{\log(y^{\delta/2})} < 1 + 1 + \frac{1 + u/\log u}{\delta/2}$$

$$= 3 + \frac{2}{\delta} + \frac{2u}{\delta \log u} < 3 + \frac{2}{\delta} + \frac{2u}{3} < u,$$

using $0 < \delta < 1$, $u > e^{3/\delta}$. Thus we have (4.5).

We are now in a position to confirm that $m = m_1m_2$ satisfies (3.1). Indeed, from (4.3) and (4.4) we have

$$m > m_1 y^{v_2\delta/2} \geq \frac{x}{y^{2-\delta}},$$

while from (4.3)–(4.5) we have

$$m \leq m_1 y^{v_2(1+1/u)\delta/2} = m_1 y^{(v_2-1)\delta/2} y^{\delta/2} y^{v_2/2}$$

$$< \frac{x}{y^{2-\delta}} y^{\delta/2} y^{\delta/2} = \frac{x}{y^{2-2\delta}}.$$  

Since $u > e^{3/\delta} > 10$, the intervals (4.2) and (4.3) are disjoint. Thus

$$\sum \frac{1}{m} \geq \sum \frac{1}{m_1 m_2} = \sum \frac{1}{m_1} \sum \frac{1}{m_2}$$

where $m_1, m_2$ run over the numbers described above. Let $S_2$ denote the sum $\sum 1/p$ where $p$ satisfies (4.3). Then

$$S_2 \asymp 1/u,$$
using \( \exp\{\log \log x^2\} \leq y \) and \( u > e^{3/\delta} \). Indeed,
\[
S_2 = \log \log (y^{(1+1/u)\delta/2}) - \log \log (y^{\delta/2}) + O(\exp\{-\log y^{3/5-\epsilon}\}) + O\left(\frac{1}{\delta y^{\delta/2}}\right)
\]
\[
= \log \left(1 + \frac{1}{u}\right) + O(\exp\{-\log \log x^{6/5-2\epsilon}\}) + O\left(\frac{1}{\delta y^{\delta/2}}\right).
\]

We conclude from (4.5) that uniformly for each \( m_1 \) we have
\[
(4.7) \quad \sum \frac{1}{m_2} \geq \frac{S_{m_2}}{v_2!} = \exp\{-O(u/\delta)\} = \exp\{-O_\delta(u)\}.
\]

Further, if \( S_1 \) is the sum \( \sum 1/p \) where \( p \) satisfies (4.2), we have
\[
S_1 \sim \frac{1}{\log u}
\]
uniformly for \( \exp\{\log \log x^2\} \leq y \). Thus
\[
(4.8) \quad \sum \frac{1}{m_1} \geq \frac{S_{m_1}}{v_1!} = \exp\{-u(\log u + \log \log u + O(1))\}.
\]

Combining (4.6)-(4.8), we have our result.

5. The case \( y^{2-\delta} \geq x \)

We begin by stating three technical results.

**Lemma 5.1** (the lower bound in the fundamental lemma of the sieve). Let \( D, z \geq 1 \) and let \( q \) be a positive integer. Let
\[
P_q = \prod_{p < z, p \nmid q} p, \quad s = \frac{\log D}{\log z}.
\]

Then there is a sequence \( \{\lambda_d\}_{d=1}^{\infty} \) of "sieving weights" such that for each positive integer \( l \),
\[
\sum_{d \mid l} \lambda_d = 1 \quad \text{if } (l, P_q) = 1, \quad \sum_{d \mid l} \lambda_d \leq 0 \quad \text{if } (l, P_q) > 1,
\]
\[
|\lambda_d| \leq 1 \quad \text{for all } d, \quad \lambda_d = 0 \quad \text{for all } d > D \text{ and for all } d \nmid P_q,
\]
\[
\sum_{d \mid l} \lambda_d = \left(\prod_{p \nmid P_q} (1 - p^{-1})\right) \left(1 + O(e^{-s})\right).
\]

For example, see Lemma 5 in [F-I]. Actually this result is implicit in [H-R] with the better error term \( O(s^{-s}) \).

**Lemma 5.2** (the Erdös-Turán inequality). Let \( H \geq 1 \) and let \( \{a_n\}_{n=1}^{\infty}, \{x_n\}_{n=1}^{\infty} \) be real sequences with each \( a_n \geq 0 \) and \( \sum a_n < \infty \). Then
\[
\left|\sum_n a_n \psi(x_n)\right| \leq \frac{1}{H} \sum_n a_n + \sum_{h \leq H} \frac{1}{h} \left|\sum_n a_n e(hx_n)\right|
\]
holds uniformly, where \( \psi(\theta) := \theta - [\theta] - \frac{1}{2} \) and \( e(\theta) := e^{2\pi i\theta} \).

For example, see Iwaniec [I, Lemma 6] or Laborde [L].
**Lemma 5.3** (incomplete Kloosterman sums). If $A < B$ and $q$ is a positive integer, then
\[
\sum_{\substack{A < m \leq B \\ (m, q) = 1}} e\left(\frac{bm}{q}\right) \ll \eta (b, q)^{1/2} q^{1/2+\eta} + (b, q) \frac{B - A}{q}
\]
for any $\eta > 0$, where $\overline{m}$ denotes the inverse of $m \pmod{q}$.

For example, see Hooley [Ho, Lemma 1].

Finally, we need the following simple inequalities:

\begin{equation}
\sum_{h \leq H} (h, q) \leq \sum_{d \mid q} d \leq H = \tau(q) H,
\end{equation}

\begin{equation}
\sum_{h \leq H} \frac{(h, q)}{h} \leq \tau(q) + \int_1^H \frac{\tau(q)}{t} \, dt = \tau(q)(1 + \log H),
\end{equation}

where $\tau(q)$ denotes the number of natural divisors of $q$.

We can now proceed with the proof of the lower bound in the theorem when $y^2 - \delta \geq x$ (where $\delta = \varepsilon/4$). Let $s_0 \geq 1$ be an absolute constant so large that the factor $1 + O(e^{-s_0})$ in Lemma 5.1 is at least $1/2$. Let $P_q$ denote the product of the primes $p < y^{\delta/s_0}$, $p \mid q$. Consider integers $n = ml < x$, where
\begin{equation}
x/y < m \leq y, \quad (m, P_q) = 1, \quad ml \equiv a \pmod{q}.
\end{equation}

Note that these conditions imply $P(n) \leq y$. The number of representations of $n \leq x$ as $ml$ in this fashion is at most the number of divisors of $n$ free of prime factors below $y^{\delta/s_0}$. But $n \leq x < y^2$, so that $n$ has at most $2s_0^2/\delta$ prime factors that can possibly be used to make up $m$. Thus $n$ has at most $2^{2s_0^2/\delta} = O(1)$ representations as $ml$. Thus
\begin{equation}
\psi(x, y; q, a) \gg_{\delta} \sum_{ml \leq x}^\ast 1
\end{equation}

where the star indicates the sum is over pairs $m, l$ satisfying (5.3). By Lemma 5.1, with $D = y^{\delta/s_0}$ and $z = y^{\delta/s_0}$, we have
\[
\sum_{ml \leq x}^\ast 1 \geq \sum_{x/y < m \leq y} \sum_{d \mid m} \lambda_d = \sum_{(d, q) = 1} \lambda_d \sum_{k l \leq x/d} 1
\]
\[
= \sum_{(d, q) = 1} \lambda_d \sum_{x/(dy) < k \leq y/d} \left(\left[ \frac{x}{dkq} - \frac{a \overline{d} \overline{k}}{q} \right] - \left[ -\frac{a \overline{d} \overline{k}}{q} \right] \right)
\]
\begin{equation}
= \frac{x}{q} \sum_{(d, q) = 1} \lambda_d \frac{d}{k} \sum_{k \mid d} \frac{1}{k}
\end{equation}
\[
- \sum_{(d, q) = 1} \lambda_d \sum_{k} \left\{ \psi \left( \frac{x}{dkq} - \frac{a \overline{d} \overline{k}}{q} \right) - \psi \left( -\frac{a \overline{d} \overline{k}}{q} \right) \right\}
\]
\[
= M - R, \quad \text{say,}
\]
where a bar over a number denotes the inverse modulo $q$ and $\sum^{(d)}_k$ denotes a sum over integers $k$ satisfying $x/(dy) < k \leq y/d$ and $(k, q) = 1$. We view $M$ as the main term in (5.5) and $R$ as the error term.

To estimate $\sum^{(d)}_k 1/k$ in the main term, note first that by sieving with the prime factors of $q$, we have

$$\sum_{k \leq t} 1 = \frac{\varphi(q)}{q} t + O(\tau(q)).$$

Thus by partial summation we have, uniformly for every $d \leq D$,

$$\sum_k^{(d)} \frac{1}{k} = \frac{\varphi(q)}{q} \log \frac{y^2}{x} + O \left( \frac{\tau(q) dy}{x} \right).$$

Using this together with Lemma 5.1 (and $\log D/\log z = s_0$), we get

$$M = \frac{x \varphi(P_q)}{q P_q} (1 + O(e^{-s_0})) \frac{\varphi(q)}{q} \log \frac{y^2}{x} + O \left( \frac{y \tau(q)}{q} \sum_d |\lambda_d| \right) \geq \frac{1}{2} \frac{x \varphi(q P_q)}{q P_q} \log \frac{y^2}{x} + O \left( \frac{y \tau(q)}{q} \right) \geq \frac{x \varphi(q P_q)}{q P_q} \log \frac{y^2}{x}.$$
The factor \( e(xh/dkq) \) is smooth in the variable \( k \) and so partial summation is appropriate for the first inner sum:

\[
\sum_{k}^{(d)} e \left( \frac{xh}{dkq} \right) e \left( -\frac{ad\bar{k}h}{q} \right) = e \left( \frac{xh}{yq} \right) \sum_{k}^{(d)} e \left( -\frac{ad\bar{k}h}{q} \right) + 2\pi i \int_{x/dy}^{y/d} \frac{xh}{dt^2q} e \left( \frac{xh}{dtq} \right) \sum_{k \leq t}^{(d)} e \left( -\frac{ad\bar{k}h}{q} \right) dt \]

\[
\ll \left( 1 + \frac{yh}{q} \right) \max_{t \leq y/d} \left| \sum_{k \leq t}^{(d)} e \left( -\frac{ad\bar{k}h}{q} \right) \right| \ll_{\eta} \left( 1 + \frac{yh}{q} \right) (h, q) \left( q^{1/2+\eta} + \frac{y}{dq} \right),
\]

for any \( \eta > 0 \), using Lemma 5.3 for the last step.

From (5.7) we thus have for any \( \eta > 0 \) (using Lemma 5.3 for the second inner sum),

\[
R \ll_{\eta} \sum_{d \leq D} \left\{ \frac{y}{H} + \sum_{h \leq H/d} \frac{1}{h} \left( 1 + \frac{yh}{q} \right) (h, q) \left( q^{1/2+\eta} + \frac{y}{dq} \right) \right\} \leq \frac{yD}{H} + \sum_{d \leq D} \left( 1 + \log \frac{H}{d} + \frac{yH}{dq} \right) \tau(q) \left( q^{1/2+\eta} + \frac{y}{dq} \right),
\]

by (5.1) and (5.2). Thus for any \( \eta > 0 \),

\[
R \ll_{\eta} \frac{yD}{H} + \tau(q) \left\{ Dq^{1/2+\eta} \log H + \frac{y}{q} \log D \log H + \frac{yH}{q^{1/2-\eta}} \log D + \frac{y^2H}{q^2} \right\}.
\]

Recall that \( D = y^{\delta/30} \leq y^{\delta} \). Using \( \tau(q) \log^2 y \ll_{\eta} y^\eta \) for any \( \eta > 0 \), we have

\[
R \ll_{\eta} \frac{y^{1+\delta}}{H} + y^{2\eta} \left\{ y^{\delta} q^{1/2} + \frac{y}{q} + \frac{yH}{q^{1/2}} + \frac{y^2H}{q^2} \right\} < y^{2\eta+\delta} \left\{ \frac{y}{H} + q^{1/2} + \frac{y}{q} + \frac{yH}{q^{1/2}} + \frac{y^2H}{q^2} \right\}.
\]

We now choose \( \eta = \delta/4 = \epsilon/16 \) and

\[
H = \begin{cases} 
q^{1/4}, & \text{if } q > y^{2/3}, \\
q^{y^{-1/2}}, & \text{if } y^{1/2} < q \leq y^{2/3}.
\end{cases}
\]

Recalling the hypotheses of the theorem, it is now easily seen that \( R = o(x/q) \). This result with (5.4)–(5.6) completes the proof.

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SMOOTH NUMBERS IN ARITHMETIC PROGRESSION

REFERENCES


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