κ-TOPOLOGIES FOR RIGHT TOPOLOGICAL SEMIGROUPS

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Abstract. Given a cardinal κ and a right topological semigroup S with topology τ, we consider the new topology obtained by declaring any intersection of at most κ members of τ to be open. Under appropriate hypotheses, we show that this process turns S into a topological semigroup. We also show that under these hypotheses the points of any subsemigroup T with card T ≤ κ can be replaced by (new) open sets that algebraically behave like T. Examples are given to demonstrate the nontriviality of these results.

Let κ be a cardinal number. We call a κ-topology any topology for which the intersection of any family of open sets with no more than κ members is again open. Such topologies are easy to come by. If X is any topological space, the sets V of the form V = \bigcap_{i \in I} U_i where \( U_i: i \in I \) is any family of sets open in X with card I ≤ κ provide a base of open sets for a κ-topology on X. We call this the κ-topology on X, we denote it by κ-X, we call its members κ-open sets, and we call κ-X the κ-coreflection of X.

A semigroup S with a completely regular topology is called right topological if all the maps s → st are continuous for t ∈ S. The topological center of S is

\[ A(S) = \{ s \in S: t \mapsto st \text{ is continuous} \}. \]

One of our main results is that if A(S) contains a subset of cardinal κ that is dense in S then κ-S is a topological semigroup (that is, multiplication is jointly continuous). This theorem allows us to conclude that if T ⊆ S is a subsemigroup, card T ≤ κ and U is κ-open with T ⊆ U, then there is a κ-open semigroup T_0 with T ⊆ T_0 ⊆ U. These results hold in particular for Stone-Čech compactifications of discrete semigroups, the most important of which is βN, where N is the semigroup of positive integers with addition. In the latter case we shall see that the semigroups T_0 are, in one sense, large.

In the terminology of [3, §2], a space with a κ-topology is a P_κ-space. When κ = ℵ_0 (as in the case of βN), κ-topological spaces are more familiar as P-spaces (see [9, §1.65]). The space κ-X is then known as the P-space coreflection of X [9, Exercise 10B]). It is easy to see that in general κ-X is the κ-coreflection

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of $X$ determined by the following categorical property: the $\kappa$-topology is the finest topology $\tau$ on $X$ such that whenever $Y$ has a $\kappa$-topology and $f: Y \to X$ is continuous then $f: Y \to (X, \tau)$ is continuous. In particular, every set open (or closed) in $X$ is also $\kappa$-open (or $\kappa$-closed). The open sets in $\kappa_0X$ are precisely the unions of $G_\delta$'s in $X$. Obviously, if every point of a Hausdorff space $X$ has a basis consisting of not more than $\kappa$ neighborhoods then $\kappa$-$X$ is discrete. We also remark that if $\kappa$ is finite the $\kappa$-topology is just the original topology.

We shall restrict ourselves to completely regular (Hausdorff) spaces. For such spaces $X$ and each infinite cardinal $\kappa$, every point has a base of open neighborhoods in $\kappa$-$X$ that are closed sets in $X$ (and so also closed in $\kappa$-$X$). For let $G = \bigcap_{i \in I} U_i$ be any basic open $\kappa$-neighborhood of $x$, with $U_i$ open in $X$ and $\text{card} I \leq \kappa$. For each $i$, choose inductively a sequence $(W^n_i)$ of neighborhoods of $x$ open in $X$ with $W^n_i \subseteq U_i$ and $\text{cl} W^{n+1}_i \subseteq W^n_i$ for $n \geq 0$. Then $\bigcap_{i,n} W^n_i = \bigcap_{i,n} \text{cl} W^n_i$ is an open $\kappa$-neighborhood of $x$ (since $\text{card}(I \times N) \leq \kappa$) contained in $G$ and is obviously closed in $X$.

We now show that the $\kappa$-topology on a suitable right topological semigroup $S$ has a strong continuity property.

**Lemma 1.** Let $S$ be a right topological semigroup and let $\kappa$ be infinite. Suppose there is $K \subseteq \Lambda(S)$ such that $K$ is dense in $S$ and $\text{card} K \leq \kappa$. Then multiplication is continuous from $S \times \kappa$-$S$ to $S$.

**Proof.** Take $a, b \in S$. Let $U$ be any neighborhood of $ab$. Let $U_0$ be an open neighborhood of $ab$ with $\text{cl} U_0 \subseteq U$. Using the fact that $S$ is right topological, find an open neighborhood $V$ of $a$ with $Vb \subseteq U_0$. Since $K \subseteq \Lambda(S)$, for each $k \in K \cap V$ we can find an open neighborhood $W_k$ of $b$ with $kW_k \subseteq U_0$. Then $G = \bigcap\{W_k: k \in K \cap V\}$ is a $\kappa$-open neighborhood of $b$. Since $K$ is dense in $S$, it follows that $K \cap V$ is dense in $V$. Therefore if $v \in V$ and $g \in G$, we see that $vg \in \text{cl}(K \cap V) \cdot g \subseteq \text{cl} U_0 \subseteq U$, again using the fact that $S$ is right topological. Thus $VG \subseteq U$, as required. $\square$

Our remaining results are corollaries of Lemma 1.

**Theorem 1.** Under the hypotheses of Lemma 1, $\kappa$-$S$ is a topological semigroup.

**Proof.** Let $a, b \in S$, and let $E$ be a $\kappa$-neighborhood of $ab$, say $E = \bigcap_{i \in I} U_i$ with each $U_i$ open in $S$ and $\text{card} I \leq \kappa$. For each $i \in I$ use Lemma 1 to find an open neighborhood $V_i$ of $a$ and a $\kappa$-neighborhood $G_i$ of $b$ with $V_iG_i \subseteq U_i$. Then $F = \bigcap_i V_i$, $G = \bigcap_i G_i$ are $\kappa$-neighborhoods of $a, b$ respectively with $FG \subseteq E$. $\square$

We can now prove our theorem about expanding semigroups. It says that, for subsemigroups $T$ that are small enough, the points of $T$ can be replaced by a family of $\kappa$-open sets that algebraically behave like $T$.

**Theorem 2.** Let $S$ be as in Lemma 1. Let $T \subseteq S$ be a subsemigroup with $\text{card} T \leq \kappa$. Let $E$ be a $\kappa$-open set with $T \subseteq E$. Then there is a disjoint family $\{T(t): t \in T\}$ of closed $\kappa$-open subsets of $S$ such that $t \in T(t)$ and $T(s)T(t) \subseteq T(st)$ for all $s, t \in T$.

**Proof.** First we produce a disjoint family $\{E_0(t): t \in T\}$ of $\kappa$-open sets with $t \in E_0(t) \subseteq E$ for each $t \in T$. For each pair $s, t$ of distinct points of $T$
find disjoint open neighborhoods $U_r(s), U_s(t)$ of $s$ and $t$ respectively. Then $E_0(t) = E \cap \bigcap \{U_r(s): s \in T, s \neq t\}$ satisfies our requirements.

The proof is now by induction. For each $n > 0$ we find closed (in $S$) $\kappa$-open neighborhoods $E_n(t)$ of $t$ for each $t \in T$. If $\{E_n(t): t \in T\}$ has been determined, we use Theorem 1 to find $\kappa$-open neighborhoods $F_{n+1}^i(s), G_{n+1}^i(t)$ with $F_{n+1}^i(s)G_{n+1}^i(t) \subseteq E_n(st)$; since there is a base for the $\kappa$-topology consisting of closed sets, we may (and do) presume that $F_{n+1}^i(s)$ and $G_{n+1}^i(t)$ are closed in $S$. Then $E_{n+1}(t) = \bigcap_{s \in T}(F_{n+1}^i(s) \cap G_{n+1}^i(t))$ is a closed $\kappa$-open set containing $t$, and the sets $\{E_{n+1}(t): t \in T\}$ satisfy $E_{n+1}(s)E_{n+1}(t) \subseteq E_n(st)$ for all $s, t \in T$. Put $T(t) = \bigcap_{n=1}^{\infty} E_n(t)$. The family $\{T(t): t \in T\}$ of closed, $\kappa$-open sets is disjoint and satisfies $T(s)T(t) \subseteq T(st)$ for all $s, t$.

From Theorem 2 we see immediately that $T_0 = \bigcup_{t \in T} T(t)$ is a semigroup. If we put $\lambda = \text{card } T$ then, being the union of $\lambda$ closed sets, $T_0$ is $\lambda$-closed. This establishes the following corollary.

**Corollary 1.** Let $S, T, E$ be as in Theorem 2 and put $\lambda = \text{card } T$. Then there is a $\lambda$-closed $\kappa$-open semigroup $T_0$ with $T \subseteq T_0 \subseteq E$.

It is worth drawing attention to two special cases of Theorem 2.

**Corollary 2.** (i) Let $S$ be as in Lemma 1. If $e \in S$ is idempotent and $E$ is a $\kappa$-neighborhood of $e$, there is a closed $\kappa$-open subsemigroup $E_0$ with $e \in E_0 \subseteq E$.

(ii) Let $S$ be as in Lemma 1 and in addition compact. Let $T \subseteq S$ be a finite subsemigroup. Then there is a disjoint family $\{T(t): t \in T\}$ of compact $\kappa$-open subsets of $S$ with $t \in T(t)$ and $T(s)T(t) \subseteq T(st)$ for all $s, t \in T$.

Corollary 2 (ii) was part of the original inspiration for this paper. It was discovered about 30 years ago that $\beta N$ is naturally a compact right topological semigroup with an operation $+$ that extends addition in $N$ and this semigroup has proved to be an invaluable tool in Ramsey Theory (see the surveys [6, 7]). It was clear from the beginning that $N \subseteq \Lambda(\beta N)$ (and in fact $N = \Lambda(\beta N)$, see [4]) and so the conclusions of Theorems 1 and 2 hold for $\beta N$ with $\kappa = \aleph_0$. As remarked above, the $\aleph_0$-topology is the $P$-space topology.

**Corollary 3.** (i) $\beta N$ is jointly continuous in its $P$-space coreflection topology.

(ii) If $T$ is a countable subsemigroup of $\beta N$ and $E$ is a $G_\delta$ with $T \subseteq E$ then there is a $G_\delta$ subsemigroup $T_0$ with $T \subseteq T_0 \subseteq E$. If $T$ is finite, $T_0$ can be chosen to be a compact $G_\delta$.

In the case of $\beta N$ we can add a little more. Each $G_\delta$ in $\beta N$ is large in the sense that it contains a set open in the subspace $N^* = \beta N \setminus N$ [9, Corollary 3.27]. Thus the semigroup $T_0$ is large in $N^*$. In particular, if $G$ is a countable semigroup in $N^*$, then each element $t$ of $G$ can be ‘expanded’ to a compact set $G(t)$ with nonempty interior in $N^*$ in such a way that $\{G(t): t \in G\}$ is disjoint and satisfies $G(s) + G(t) \subseteq G(s + t)$ for all $s, t \in G$.

The question arises of whether we can arrange for $G(s) + G(t) = G(s + t)$ in this situation. The answer is that equality is never achieved. The reason is that $G(s) + G(t)$ is nowhere dense in $N^*$ [4, Theorem 8.1] but $G(s + t)$ has nonempty interior.

If $G$ is a finite group in $\beta N$ then $G_0 = \bigcup_{t \in G} G(t)$ is compact subsemigroup with nonempty interior. However, whether $\beta N$ contains nontrivial finite
subgroups is unknown (and the question of the existence of such subgroups appears difficult).

The example $\beta N$ shows that our results do have nontrivial content in at least one interesting case. Obtaining significant examples with $\kappa > \aleph_0$ is more difficult.

**Example 1.** For each cardinal $\kappa$, there exist a compact right topological semigroup $S$ and a subsemigroup $T$ of $S$ for which $T(t)$ is nontrivial for each $t \in T$.

We begin with any infinite discrete semigroup $L$ with $\text{card } L = \kappa$. Put $S = \beta L$. Then $S$ can be made a right topological semigroup with $L \subseteq \Lambda(S)$ (see [6] or [7]). Let $U$ be the set of $\kappa$-uniform ultrafilters on $L$, that is $U = \{p \in S : \text{for all } A \subseteq S \text{ with } \text{card } A < \kappa, p \notin \text{cl}_S A\}$. By [3, Corollary 7.8(b)], $\text{card } U = 2^{2^\kappa}$. By [3, Corollary 7.8(a)], if $p \in U$ then each neighborhood base of $p$ has cardinal strictly greater than $\kappa$. So if $T$ is a subsemigroup of $S$ generated by a subset of $U$ of cardinal $\kappa$, then for each $t \in T \cap U$ the $\kappa$-open set $T(t)$ consists of more than one point. Now [5, Theorem 2.5] gives conditions under which $U$ is a semigroup, and this holds in particular if $L$ is cancellative (in fact, $U$ is then an ideal [5, Corollary 2.10]). Thus, for cancellative $L$, we have $T \subseteq U$ and our objective is achieved.

For any semigroup $S$ with a Hausdorff topology there is always a cardinal $\kappa$ such that $\kappa$-$S$ is jointly continuous, for when $\kappa = \text{card } S$, $\kappa$-$S$ is discrete. This suggests that we might use the smallest cardinal with this property as a measure of how discontinuous the multiplication of $S$ is. Theorem 2 shows that sometimes a cardinal smaller than $\text{card } S$ will do. We now give two examples, one to show that even for semigroups satisfying the conditions of Lemma 1, $\text{card } S$ might be necessary. The other shows that a cardinal smaller than the $\kappa$ of Theorem 1 is sometimes sufficient. (Of course, if $S$ is jointly continuous to begin with, then $\kappa = 0$ is sufficient, but our example is not even separately continuous and is, we believe, more significant.)

**Example 2.** (i) Given regular cardinal $\kappa$, there is a semigroup $S$ with $\text{card } S = \kappa$ that satisfies the conditions of Lemma 1 but for which $\lambda$-$S$ is jointly continuous only if $\lambda \geq \kappa$.

Let $\kappa$ be a regular cardinal (which we regard as an ordinal), so that $\kappa = \text{cf } \kappa$. Write the elements of $\bigoplus_\kappa Z$, the direct sum of $\kappa$ copies of $Z$, as (transfinite) sequences $(z_\alpha)_{\alpha < \kappa}$ with $z_\alpha = 0$ for all but finitely many $\alpha$. Define a total order on $\bigoplus_\kappa Z$ by $(z_\alpha) < (w_\alpha)$ if and only if $z_\mu < w_\mu$ where $\mu = \max\{\alpha : z_\alpha \neq w_\alpha\}$ (this is a 'reverse' lexicographic order). Then $\bigoplus_\kappa Z$ is a totally ordered group (with the usual operation $+$. We obtain $S$ by adjoining to $\bigoplus_\kappa Z$ two further elements $\infty$ and $-\infty$. We extend the order to $S$ by writing $-\infty < x < \infty$ for all $x \in \bigoplus_\kappa Z$, and we extend $+$ by writing $(-\infty) + x = -\infty$ and $\infty + x = \infty$ for all $x \in \bigoplus_\kappa Z$, and $x + (-\infty) = -\infty$, and $x + \infty = \infty$ for all $x \in S$. We give $S$ a topology by declaring each $x \in \bigoplus_\kappa Z$ to be an isolated point and taking the intervals $[-\infty, x)$ to be basic neighborhoods of $-\infty$ and $(x, \infty]$ to be basic neighborhoods of $\infty$ for $x \in \bigoplus_\kappa Z$. Then $S$ is a right topological semigroup and $\Lambda(S) = \bigoplus_\kappa Z$ (if $x \searrow -\infty$ then $\infty + x = \infty \Rightarrow -\infty = \infty + (-\infty)$, so that $\infty \notin \Lambda(S)$; the other properties of $S$ are equally easy to see).

Now if $\lambda < \kappa$ (= $\text{cf } \kappa$ by hypothesis) the intersection of $\lambda$ intervals of the
form \((x, \infty)\) contains another of this same form. So we see that \(\lambda\cdot S = S\). In particular, \(\Lambda(\lambda\cdot S) \neq S\), so \(\lambda\cdot S\) does not have a continuous multiplication. (In this example, \(\kappa\cdot S\) is discrete.)

(ii) Given any uncountable cardinal \(\kappa\) there is a semigroup \(S\) that satisfies the conditions of Lemma 1 and that has the property that every dense subset of \(S\) has cardinal at least \(\kappa\), but for which \(\kappa_0\cdot S\) is jointly continuous.

Let \(\kappa\) be an uncountable cardinal. We start with a semigroup \(T\) that has an identity 1 and satisfies (a) \(\Lambda(T)\) contains a countable subset \(Z\) dense in \(T\) (and notice that we may take \(1 \in Z\)) and (b) every point of \(T\) has a countable neighborhood base. There do exist compact right topological semigroups with these properties that are not topological (for example, the semigroup \(S\) used in [1, Example 1], but with the first copy \(T = \{e^{i\theta} : 0 \leq \theta < 2\pi\}\) of the circle group replaced by \(\{e^{in} : n \in \mathbb{Z}\} = \mathbb{Z}_0\) (say) to give \(\mathbb{Z}_0 \cup T_1 \cup T_2\)).

We consider the direct product \(T^\kappa\) with the direct product topology. Then \(\Lambda(T^\kappa) = \Lambda(T)^\kappa \supseteq Z^\kappa\). However, we can find a smaller subset of \(\Lambda(T^\kappa)\) that is dense in \(T^\kappa\); this is the direct sum of \(\kappa\) copies of \(Z\), a subset dense in the direct product, and it has cardinal exactly \(\kappa\). Theorem 1 tells us that \(\kappa\cdot T^\kappa\) has continuous multiplication, but as in (i) the \(\kappa\)-topology is uninteresting since it is discrete. Moreover, no subset of \(T^\kappa\) with fewer than \(\kappa\) elements is dense.

We shall determine the \(\kappa_0\)-topology on \(T^\kappa\). Let \((t_\alpha)_{\alpha < \kappa}\) be an element of \(T^\kappa\). For each \(\alpha\), let \(\{U_n(t_\alpha) : n = 1, 2, \ldots\}\) be a neighborhood base of \(t_\alpha\) in \(T\) with \(U_n(t_\alpha) \setminus \{t_\alpha\}\). For any finite subset \(F\) of \(\kappa\) with \(\text{card} F = r\), we write \(V(F) = \prod_{\alpha < \kappa} V_\alpha\) where \(V_\alpha = T\) for \(\alpha \notin F\), \(V_\alpha = U_n(t_\alpha)\) for \(\alpha \in F\). If \(E\) is any countable set of predecessors of \(\kappa\), we write

\[
W(E) = \bigcap\{V(F) : F \text{ is a finite subset of } E\}.
\]

Then \(W(E)\) is an \(\kappa_0\)-neighborhood of \((t_\alpha)\). It is easy to see that in fact \(W(E) = \prod_\alpha W_\alpha\) where \(W_\alpha = T\) if \(\alpha \notin E\), \(W_\alpha = \{t_\alpha\}\) if \(\alpha \in E\). It can now be seen that the \(\kappa_0\)-topology on \(T\) is determined by neighborhoods of the form \(W(E)\).

It is not difficult to check directly that this topology makes multiplication in \(T^\kappa\) continuous. It is perhaps more illuminating to observe that if \(T_d\) is \(T\) with the discrete topology then \(\kappa_0\cdot (T_d)^\kappa\) is the same as \(\kappa_0\cdot T^\kappa\). Since multiplication in \(T_d^\kappa\) is continuous, so is multiplication in \(\kappa_0\cdot (T_d)^\kappa\) (the argument is as in the proof of Theorem 1).

We conclude with a question. One of the difficult problems about compact right topological semigroups is to determine how the topological and algebraic structures interact. This is even true for minimal one-sided ideals though these have a simple algebraic structure. Thus, for a minimal left ideal \(L\), the set \(E(L)\) of idempotents in \(L\) is a left-zero semigroup \((ef = e\text{ for all }e, f)\), the semigroups \(eL\), for \(e \in E(L)\), are isomorphic groups, and algebraically \(L\) is isomorphic to the direct product \(E(L) \times (eL)\) [2, 1.3.11, 1.2.16]. Topologically \(L\) is compact. If \(S\) has a separately continuous multiplication, then \(L\) is isomorphic to the topological direct product of the compact subsemigroups \(E(L)\) and \((eL)\) [2, Theorem 1.5.1], but this may not be so in more general cases (see [8] for the semigroup \(\beta N\)). Theorem 1 tells us that \(\kappa\cdot S\) is jointly continuous for some \(\kappa\) (though it is not compact); is it true that \(\kappa\cdot L\) is isomorphic to a topological direct product of \(\kappa\cdot E(L)\) and \(\kappa\cdot (eL)\)?
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